

Title: Mathematical Physics Core Lecture

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Tensors on manifolds

$$\dim(T_e M) = \dim(M)^{r+s}$$

$$T_e M = T_e(T_e M)$$

(U, φ) coordinates $\varphi = (x^1, \dots, x^n)$

basis $\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \dots \otimes dx^i \otimes \dots \otimes dx^j$

$r+s$ | at $a \in M$
Want $x \in M \mapsto X_x \in T_x M$
smoothly

Today:

Bundles and fields

Tangent bundle

$$(U, \varphi) \in \mathcal{A}_M$$

$$(\tilde{U}, \tilde{\varphi}) \in \mathcal{A}_{TM}$$

$$TM = \bigsqcup_{q \in M} T_q M = \bigcup_{q \in M} \{q\} \times T_q M \quad \text{manifold?}$$

$$= \{(q, v) \mid q \in M, v \in T_q M\}$$

$$\tilde{U} = \bigsqcup_{q \in U} T_q M = \{(q, v) \mid q \in U, v \in T_q M\}$$

$$\tilde{\varphi}\left(q, s^i \frac{\partial}{\partial x^i} \Big|_q\right) = \left(\varphi(q), s^1, s^2, \dots, s^n\right) \in \mathbb{R}^{2n}$$

TM manifold of dim $2 \cdot \dim(M)$

natural projection

$$\pi: TM \rightarrow M$$

$$(q, v) \mapsto q$$

vector field smooth $X: M \rightarrow TM$

such that $\pi \circ X = \text{id}_M$ ($\pi(X(a)) = a$)

$$X(a) \in TM$$

$$X(a) = (b, v) \quad b \in M, v \in T_b M$$

must be $b = a$

Aside: if M is set of all points that a particle can take

TM configuration space of Lagrangian mechanics

$$\{(q, v) \mid q \in M, v \in T_q M\}$$

Cotangent bundle

$$T^*M = \bigsqcup_{q \in M} T_q^*M$$

$$\tilde{U} = \bigsqcup_{q \in U} T_q^*M$$

$\tilde{\varphi}(q)$

Tensor bundle of type (r, s)

$$T^{r,s}M = \bigsqcup_{q \in M} T_q^{r,s}M$$

$$\tilde{U} = \bigsqcup_{q \in U} T_q^{r,s}M$$

$$\tilde{\varphi}(q, T_{i_1 \dots i_r}^{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \Big|_q \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \Big|_q \otimes \dots \otimes dx^{j_1} \Big|_q \otimes \dots \otimes dx^{j_s} \Big|_q) = (\varphi(q), (T_{i_1 \dots i_r}^{j_1 \dots j_s}))$$

$$\tilde{\varphi}(a, \alpha; dx^i) = (\varphi(a), \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{2n}$$

$$\dim(T^*M) = 2 \dim(M)$$

$\left(\begin{array}{c} \dots \\ \alpha_1 \dots \alpha_n \end{array} \right)$

$$\tilde{\varphi}(\alpha, \alpha; dx^i) = (\varphi(e), \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{2n}$$

$$\dim(T^*M) = 2 \dim(M)$$

$$\alpha: U \underset{\text{open}}{\subseteq} M \rightarrow T^*M, \quad \pi \circ \alpha = \text{id}_U \quad \text{local covector field (1-form)}$$

$$T: U \underset{\text{open}}{\subseteq} M \rightarrow T^*M$$

$$\pi \circ T = \text{id}_U$$

↓
section of the bundle

$X: \underbrace{U \subseteq M}_{\text{open}} \rightarrow TM$ smooth, $\pi \circ X = \text{id}_U$

local vector field

Def if $(u, \varphi) \in A_M$. Then if $W \subseteq \varphi(U) \subseteq \mathbb{R}^n$
then $\varphi^{-1}(W)$ is open in M

Abuse of notation identifies $X(a) = (e, X_e)$
with X_e

- $\Phi \subseteq M$ is open (M open)
- if $A, B \subseteq M$ then $A \cap B$ open
- if $A_i, i \in I$ open, then $\bigcup_{i \in I} A_i$ open

Frame fields

$$\Gamma(TM, U) = \{ X: U \subseteq M \rightarrow TM \text{ vector fields} \}$$

→ vector space

$$(X+Y)(a) = (a, X_a + Y_a)$$

$$(\alpha X)(a) = (a, \alpha X_a)$$

$$\text{if } f \in C^\infty(U, \mathbb{R}) \text{ then } (fX)(a) = (a, f(a)X_a)$$

$$X(e) = (e, X_e) \quad Y(e) = (e, Y_e)$$

"basis" of $\Gamma(TM, U)$?

find n vector fields that generate the others

↙ frame

(e, X_e)

Def a local frame on M is a set $\{\bar{E}_i\}_{i=1}^n$, $E_i: \alpha \in U \mapsto (\alpha, E_i|_\alpha) \in TM$ ($\dim(M)=n$)

s.t. $\{\bar{E}_i|_\alpha\}_{i=1}^n$ is a basis for $T_\alpha M$

dual ω -frame $\{\bar{E}^i: \alpha \in U \mapsto (\alpha, \bar{E}^i|_\alpha) \in T^*M\}_{i=1}^n$ s.t. $\{\bar{E}^i|_\alpha\}$ is the dual basis to $\{\bar{E}_i|_\alpha\}$

$X: U \subseteq M \rightarrow TM$ vector field

$$X|_\alpha = (\alpha, X_\alpha) = (\alpha, X_i \bar{E}_i|_\alpha)$$

$$X_i: \alpha \in U \mapsto X_i \in \mathbb{R}$$

TM ($\dim(M)=n$)

$$X|_U = X^i E_i$$

the dual basis to $\{E_i\}$

$$X^i: \alpha \in U \mapsto X^i \in \mathbb{R}$$

$$\alpha \in T^*(TM)$$

$$\alpha|_U = \alpha_i E^i$$

$$\alpha_i \in C^\infty(U, \mathbb{R})$$

Coordinate frame $(U, (x^1, \dots, x^n))$

$$\frac{\partial}{\partial x^i} : q \in U \subseteq M \mapsto (q, \frac{\partial}{\partial x^i} \Big|_q) \in TM \quad \text{frame}$$

$$dx^i : q \in U \subseteq M \mapsto (q, dx^i|_q) \in T^*M \quad \text{dual co-frame}$$

local frames always exist

global frames generally don't

↳ ~~existence~~ existence of global frame \Leftrightarrow TM trivial \Leftrightarrow $TM \cong M \times \mathbb{R}^n$

local frames always exist

global frames generally don't



↳ ~~existence~~ existence of global frame \Leftrightarrow TM trivial \Leftrightarrow TM $\cong M \times \mathbb{R}^n$

S^0, S^1, S^3, S^7

$$TS^1 \cong S^1 \times \mathbb{R}$$