Title: Mathematical Physics Core Lecture

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Tensors on manifolds

$$\mathcal{T}^{r,s}_e M = \mathcal{T}^{r,s}(T_e M)$$

(U, φ) coordinates $\varphi = (x^1, \ldots, x^n)$

basis $\left.\frac{\partial}{\partial x^{i_1}}\right|_e \otimes \left.\frac{\partial}{\partial x^{i_r}}\right|_e \otimes \cdots \otimes dx_e^{j_1} \otimes \cdots \otimes dx_e^{j_s}$

$$\dim\left(\mathcal{T}^{r,s}_e M\right) = \dim(M)^{r+s}$$

et

want

rts

at $e \in M$

want $e \in M \mapsto X_e \in T_e M$

smoothly

Today:

Bundles and fields

Tangent bundle

$(U,\varphi) \in \mathcal{A}_M$

$(\tilde{U},\tilde{\varphi}) \in \mathcal{A}_{TM}$

$$TM = \bigsqcup_{x\in M} T_x M = \bigcup_{x\in M} \{x\}\times T_x M$$

manifold?

$$= \{(x,v) \mid x\in M,\ v\in T_x M\}$$

$$\tilde{U} = \bigsqcup_{x\in U} T_x M = \{(x,v) \mid x\in U,\ v\in T_x M\}$$

$$\tilde{\varphi}\left(x, v^i \frac{\partial}{\partial x^i}\Big|_x\right) = \left(\varphi(x), v^1, v^2, \dots, v^n\right) \in \mathbb{R}^{2n}$$

TM manifold of dim $2\cdot \dim(M)$

natural projection $\pi : TM \to M$
$(x,v) \mapsto x$

vector field smooth $X : M \to TM$

such that $\pi \circ X = id_M$ $\left(\pi(X(a)) = a \right)$

$X(a) \in TM$

$X(a) = (b, v)$ $b \in M$, $v \in T_b M$

must be $b = a$

<u>Aside</u>: if M is set of all points that a particle can take

TM configuration space of Lagrangian mechanics

$$TM = \{(a,v) \mid a \in M,\ v \in T_a M\}$$

<u>Cotangent bundle</u> $\quad T^*M = \bigsqcup_{p\in M} T^*_p M \qquad \tilde{U} = \bigsqcup_{p\in U} T^*_p M \qquad \tilde{\varphi}(p,$

<u>Tensor bundle of type (r, s)</u> $\quad T^{r,s}M = \bigsqcup_{p\in M} T^{r,s}_p M \qquad \tilde{U} = \bigsqcup_{p\in U} T^{r,s}_p M$

$$\tilde{\varphi}\left(p, T^{i_1\cdots i_r}{}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes dx^{j_1}_p \otimes \cdots\right) = \left(\varphi(p), \left(T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)\right)$$

$$\tilde{\varphi}(x, \alpha_i dx^i) = (\varphi(x), \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{2n}$$

$$\dim(T^*M) = 2\dim(M)$$

$\dot{v}_1 \cdots v_s))$

$$\tilde{\varphi}(x, \alpha_i\, dx^i) = (\varphi(x), \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{2n}$$

$\dim(T^*M) = 2\dim(M)$

$\alpha : U \underset{\text{open}}{\subseteq} M \to T^*M, \quad \pi \circ \alpha = id_U$ local covector field (1-form)

$T : U \underset{\text{open}}{\subseteq} M \to T^{r,s}M \quad \underbrace{\pi \circ T = id_U}_{\downarrow}$

section of the bundle

- $X : (U \underset{\text{open}}{\subseteq} M) \to TM$ smooth, $\pi \circ X = \mathrm{id}_U$

local vector field

Def if $(U, \varphi) \in \mathcal{A}_M$. Then if $W \underset{\text{open}}{\subseteq} \varphi(U) \subseteq \mathbb{R}^n$

then $\varphi^{-1}(W)$ is open in M

Abuse of notation identify $X(a) = (a, X_a)$

with X_a

- $\emptyset \subseteq M$ is open (M open)
- if $A, B \underset{\text{open}}{\subseteq} M$ then $A \cap B$ open
- if A_i, $i \in I$ open, then $\bigcup_{i \in I} A_i$ open

Frame fields

$\Gamma(TM, U) = \{ X : U \subseteq M \to TM \text{ vector fields} \}$

$\Rightarrow$ vector space

$$(X+Y)(a) = (a, X_a + Y_a)$$

$$(\alpha X)(a) = (a, \alpha X_a)$$

if $f \in C^\infty(U, \mathbb{R})$ then $(fX)(a) = (a, f(a) X_a)$

$X(e) = (e, X_e)$ $Y(e) = (e, Y_e)$

"basis" of $\Gamma(TM, U)$?

frame

find n vector fields that generate the others

$f(a) X_a)$

Def a local frame on M is a set $\{E_i\}_{i=1}^{n}$, $E_i : a \in U \mapsto (a, E_i|_a) \in TM$ $\quad (\dim(M) = n)$

s.t. $\{E_i|_a\}_{i=1}^{n}$ is a basis for T_aM

dual co-frame $\quad \{\varepsilon^i : a \in U \mapsto (a, \varepsilon^i|_a) \in T^*M\}_{i=1}^{n}$ s.t. $\{\varepsilon^i|_a\}$ is the dual basis to $\{E_i|_a\}$

$X^i : a \in U \mapsto X^i_a \in \mathbb{R}$

$X(a) = (a, X_a) = (a, X^i_a E_i|_a)$

$X : U \subseteq M \to TM$ vector field

$TM \quad (\dim(M)=n)$

$X|_U = X^i E_i$

the dual basis to $\{E_i|_p\}$

$X^i : p \in U \mapsto X^i_p \in \mathbb{R}$

$\alpha \in \Gamma(T^*M)$

$\alpha|_U = \alpha_i \varepsilon^i \quad \alpha_i \in C^\infty(U,\mathbb{R})$

Coordinate frame $\quad (U, (x^1, \dots, x^n))$

$$\frac{\partial}{\partial x^i} : q \in \underset{\text{open}}{U \subseteq} M \mapsto \left(q, \left.\frac{\partial}{\partial x^i}\right|_q\right) \in TM \qquad \text{frame}$$

$$dx^i : q \in \underset{\text{open}}{U \subseteq} M \mapsto \left(q, dx^i_q\right) \in T^*M \qquad \text{dual co-frame}$$

local frames always exist

global frames generally don't

↳ existence of global frame $\Leftrightarrow$ TM trivial $\Leftrightarrow$ $TM \cong M \times \mathbb{R}^n$

local frames always exist

global frames generally don't

↳ existence of global frame $\Leftrightarrow$ TM trivial $\Leftrightarrow$ $TM \cong M \times \mathbb{R}^n$

S^0, S^1, S^3, S^7

$TS^1 \cong S^1 \times \mathbb{R}$