Title: Mathematical Physics Core Lecture

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Today:

- Tangent space (cont.d)
- Cotangent space
- Tensors

basis for T

$$\left.\frac{\partial}{\partial x^i}\right|_e : f \mapsto \frac{\partial \hat{f}}{\partial x^i}($$

basis for $T_a M$ with chart (U, φ) $\varphi = (x^1, \dots, x^n)$

$$\frac{\partial}{\partial x^i}\Big|_a : f \mapsto \frac{\partial \hat{f}}{\partial x^i}(\hat{a})$$

let (V, ψ), $\psi = (y^1, \dots, y^n)$ be a chart with $a \in V$

$$\frac{\partial}{\partial y^i}\Big|_a f = \frac{\partial (f \circ \psi^{-1})}{\partial y^i}(\psi(a)) = D_i (f \circ \psi^{-1})(\psi(a))$$

$$= D_i\big(\underbrace{f\circ\varphi^{-1}}\circ\underbrace{\varphi\circ\psi^{-1}}\big)(\psi(q))$$

$$= \underbrace{D_j(f\circ\varphi^{-1})(\varphi(q))}\; D_i(\varphi\circ\psi^{-1})^j(\psi(q))$$

J_i^j Jacobian matrix of transition func.

$$\left(\frac{\partial}{\partial x^j}\Big|_q f\right) J_i^j = \frac{\partial}{\partial y^i}\Big|_q f \longrightarrow \boxed{\frac{\partial}{\partial y^i}\Big|_q = D_i(x^j\circ\psi^{-1})(\psi(q))\,\frac{\partial}{\partial x^j}\Big|_q}$$

trix

$$\left.\ldots)\ \frac{\partial}{\partial x^j}\right|_a \rightarrow$$

Abuse of notation

$$x^i = x^i(y) \equiv x^i \circ \psi^{-1}(y)$$

$$\boxed{\left.\frac{\partial}{\partial y^i}\right|_a = \frac{\partial x^j}{\partial y^i}(\psi(a)) \left.\frac{\partial}{\partial x^j}\right|_a}$$

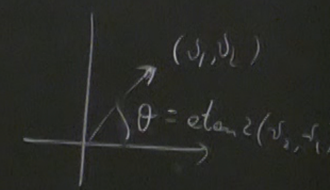
$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_a$$

$$s(v) = e^{v_3} v_1$$
$$t(v) = e^{v_3} v_2$$

(U, ψ) $\quad U = C \setminus \{(-1, 0, v_3) \mid v_3 \in R\}$

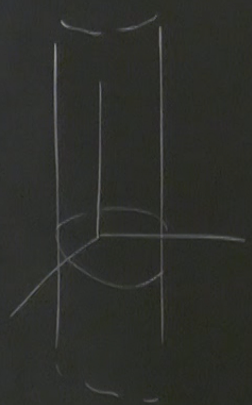
$\psi = (\theta, z)$ $\quad \theta(v) = \mathrm{atan2}(v_2, v_1) \in (-\pi, \pi)$

$z(v) = v_3 \in R$



ex

$$C = \{(v_1, v_2, v_3) \mid v_1^2 + v_2^2 = 1\}$$



$$v_0 = (v_1, v_2, v_3)$$

$$(C, \varphi = (s,t))$$

$$s(v) = e^{v_3} v_1$$
$$t(v) = e^{v_3} v_2$$

$$\varphi(C) = \mathbb{R}^2 \setminus \{0\}$$

$$\left.\frac{\partial}{\partial \theta}\right|_{v_0} = \frac{\partial s}{\partial \theta}(\theta(v_0), z(v_0)) \left.\frac{\partial}{\partial s}\right|_{v_0} + \frac{\partial t}{\partial \theta}(\theta(v_0), z(v_0)) \left.\frac{\partial}{\partial t}\right|_{v_0}$$
$$= -e^{v_3} v_2 \left.\frac{\partial}{\partial s}\right|_{v_0} + e^{v_3} v_1 \left.\frac{\partial}{\partial t}\right|_{v_0}$$

$$s(\theta, z) = e^z \cos\theta$$
$$t(\theta, z) = e^z \sin\theta$$

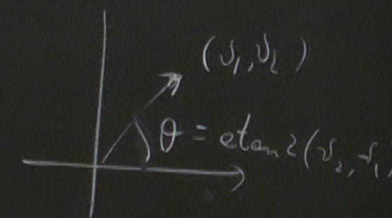
s_1

s_2

(U, ψ) $\quad U = C \setminus \{(-1, 0, s_3) \mid s_3 \in \mathbb{R}\}$

$\psi = (\theta, z)$ $\quad \theta(s) = \text{atan2}(s_2, s_1) \in (-\pi, \pi)$

$Z(s) = s_3 \in \mathbb{R}$



$\frac{\partial s}{\partial z} = s$ $\quad \frac{\partial s}{\partial \theta} = -t$

$s_1 = \cos(\theta)$ $\quad s_2 = \sin(\theta)$

$s_3 = z$

$\frac{\partial t}{\partial z} = t$ $\quad \frac{\partial t}{\partial \theta} = s$

Let V be a real, finite-dim. vector space

dual

Def dual of V is $V^* = \{\alpha : V \to \mathbb{R} \text{ linear}\}$

linear maps are specified by action on a basis

$V \cong V^*$

Choose a basis

$\{e_i\}_{i=1}^n \subseteq V$

Let V be a real, finite-dim vector space

Def dual of V is $V^* = \{\alpha : V \to \mathbb{R} \text{ linear}\}$

linear maps are specified by action on a basis

$V \cong V^*$

choose a basis $\{e_i\}_{i=1}^n \subseteq V$

dual basis $\{\varepsilon^j\}_{i=1}^n \subseteq V^*$ $\quad \varepsilon^j : V \to \mathbb{R}$

$\varepsilon^j(e_i) = \delta^j_i$

Cotangent space at $a \in M$

$T^*_a M = (T_a M)^*$

Coordinate basis

if (U,φ) $\quad \varphi = (x^1, \dots, x^n)$

$\{ dx^i_e \}_{i=1}^n$ dual basis of $\left\{ \frac{\partial}{\partial x^i}\Big|_e \right\}_{i=1}^n$

$dx^i_e : T_e M \to \mathbb{R}$ $\qquad dx^i_e\left(\frac{\partial}{\partial x^j}\Big|_e \right) = \delta^i_j$

$$\boxed{dg^i_e = \frac{\partial g^i}{\partial x^j}(\varphi(e)) \, dx^j_e}$$

Def: a tensor on V of type (r,s) is a multi-linear map $T: V^*$

Ex. $r=1, s=0$ → vectors (identified)

$s=1, r=0$ → covectors

$r=0, s=2$ → bilinear map (ex: scalar product)

-linear map

$$T: \underbrace{V^* \times V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times V \cdots \times V}_{s \text{ copies}} \to \mathbb{R}$$

Tensor products

Def if V, W real, fin. dim, vect. spaces, $\alpha: V \to$

the tensor product of α, β is the bilinear map

$T^{r,s}(V)$ = set of all (r,s)-tensors on V $\to$ vector space with pointwis spaces, $\alpha: V \to \mathbb{R}$, $\beta: W \to \mathbb{R}$

he bilinear map $\alpha \otimes \beta : (v, w) \in V \times W \mapsto \alpha(v)\,\beta(w)$

ce with pointwise operations $(\alpha T + \beta S)(\cdots) = \alpha\, T(\cdots) + \beta\, S(\cdots)$

$T \in \mathcal{T}^{r,s}(V)$ → fully specified by action on basis elements of V, V^*

$$T(\alpha_i \varepsilon^i, \ldots) = \alpha_i T(\varepsilon^i, \ldots) = e_i(\alpha_j \varepsilon^j)\, T(\varepsilon^i, \ldots)$$

$$T = \underbrace{T(\varepsilon^{i_1}, \varepsilon^{i_2}, \ldots, \varepsilon^{i_r}, e_{j_1}, \ldots, e_{j_s})}\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

$T^{i_1 i_2 \cdots i_r}{}_{j_1 \cdots j_s}$ components of T in basis

$T \in \mathcal{T}^{r,s}(V)$ → fully specified by action on basis elements of V, V^*

$\{e_i\}$ basis of V

$\{\varepsilon^j\}$ basis of V^* (dual)

$$T(\alpha_i \varepsilon^i, \ldots) = \alpha_i T(\varepsilon^i, \ldots) = e_i(\alpha_j \varepsilon^j)\, T(\varepsilon^i, \ldots)$$

$$T = \underbrace{T(\varepsilon^{i_1}, \varepsilon^{i_2}, \ldots, \varepsilon^{i_r}, e_{j_1}, \ldots, e_{j_s})}_{T^{i_1 i_2 \cdots i_r}{}_{j_1 \cdots j_s}}\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

components of T in basis