Title: Mathematical Physics Core Lecture

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Today:

- Tangent space (cont.d)
- Cotangent space
- Tensors

basis for T_a

$$\left.\frac{\partial}{\partial x^i}\right|_a : f \mapsto \frac{\partial \hat{f}}{\partial x^i}($$

basis for $T_a M$ with chart (U, φ) $\varphi = (x^1, \ldots, x^n)$

$\left.\frac{\partial}{\partial x^i}\right|_a : f \mapsto \frac{\partial \hat{f}}{\partial x^i}(\hat{a})$

let (V, ψ), $\psi = (y^1, \ldots, y^n)$ be a chart with $a \in V$

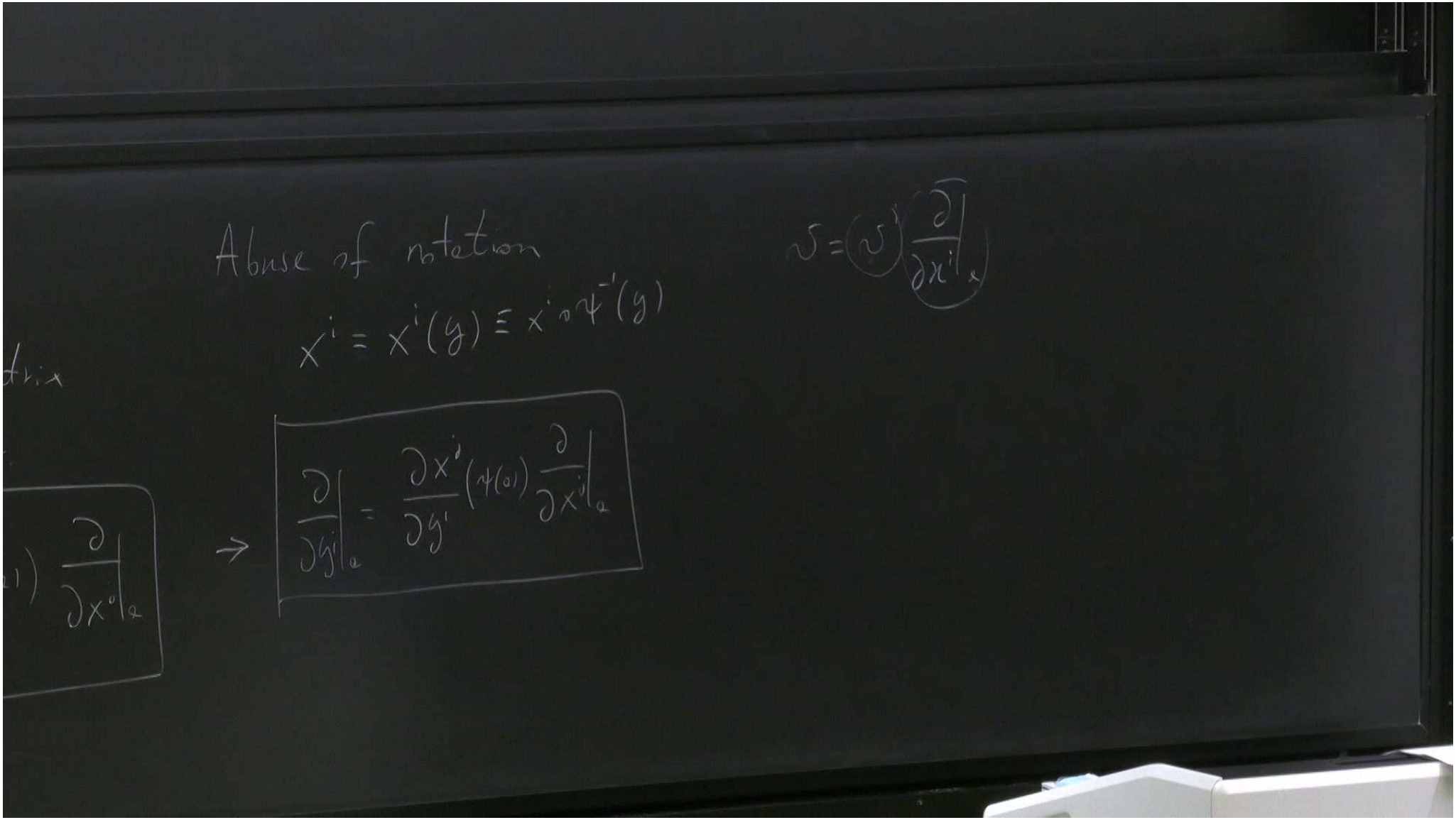
$\left.\frac{\partial}{\partial y^i}\right|_a f = \frac{\partial (f \circ \psi^{-1})}{\partial y^i}(\psi(a)) = D_i(f \circ \psi^{-1})(\psi(a))$

$$= D_i\left(f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}\right)(\psi(a))$$

$$= D_a\left(f \circ \varphi^{-1}\right)(\varphi(a)) \; D_i\left(\varphi \circ \psi^{-1}\right)^a(\psi(a))$$

J_i^a Jacobian matrix of transition func.

$$\left(\frac{\partial}{\partial x^a}\Big|_a f\right) J_i^a = \frac{\partial}{\partial y^i}\Big|_a f \quad \rightarrow \quad \boxed{\frac{\partial}{\partial y^i}\Big|_a = D_i\left(x^a \circ \psi^{-1}\right)(\psi(a)) \; \frac{\partial}{\partial x^a}\Big|_a}$$


Abuse of notation
$x^i = x^i(y) \equiv x^i \circ \varphi^{-1}(y)$
$v = v^i \frac{\partial}{\partial x^i}$
$\frac{\partial}{\partial y^i}\Big|_a = \frac{\partial x^j}{\partial y^i}(\varphi(a)) \frac{\partial}{\partial x^j}\Big|_a$

$s(v) = e^{v_3} v_1$

$t(v) = e^{v_3} v_2$

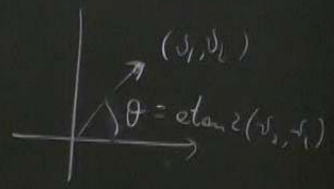
(U, Ψ)

$U = \mathbb{C} \setminus \{(-1, 0, v_3) \mid v_3 \in \mathbb{R}\}$

$\Psi = (\theta, z)$

$\theta(v) = \operatorname{atan2}(v_2, v_1) \in (-\pi, \pi)$

$z(v) = v_3 \in \mathbb{R}$



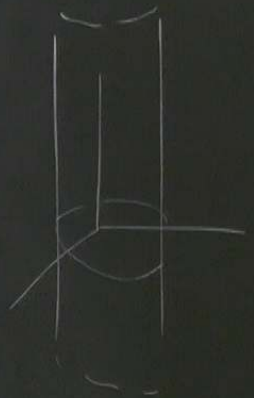
(ex) $C = \{(v_1, v_2, v_3) \mid v_1^2 + v_2^2 = 1\}$

$(C, \varphi(s,t))$

$\varphi(C) = \mathbb{R}^2 \setminus \{0\}$

$s(v) = e^{v_3} v_1$

$t(v) = e^{v_3} v_2$



$v_0 = (v_1, v_2, v_3)$

$$\frac{\partial}{\partial \theta}\Big|_{v_0} = \frac{\partial s}{\partial \theta}(\theta(v_0), z(v_0)) \frac{\partial}{\partial s}\Big|_{v_0} + \frac{\partial t}{\partial \theta}(\theta(v_0), z(v_0)) \frac{\partial}{\partial t}\Big|_{v_0}$$

$$= -e^{v_3} v_2 \frac{\partial}{\partial s}\Big|_{v_0} + e^{v_3} v_1 \frac{\partial}{\partial t}\Big|_{v_0}$$

$s(\theta, z) = e^z \cos\theta$

$t(\theta, z) = e^z \sin\theta$

(U, φ) $\quad U = C \setminus \{(-1, 0, v_3) \mid v_3 \in R\}$

$\varphi = (\theta, z)$ $\quad \theta(v) = \operatorname{atan2}(v_2, v_1) \in (-\pi, \pi)$

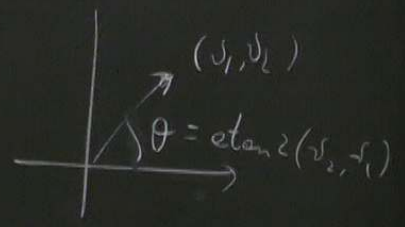
$z(v) = v_3 \in R$

$\frac{\partial s}{\partial z} = s \qquad \frac{\partial s}{\partial \theta} = -t$

$v_1 = \cos(\theta) \qquad v_2 = \sin(\theta)$

$v_3 = z$

$\frac{\partial t}{\partial z} = t \qquad \frac{\partial t}{\partial \theta} = s$



Let V be a real, finite-dim vector space

Def dual of V is $V^* = \{\alpha : V \to \mathbb{R} \text{ linear}\}$

linear maps are specified by action on a basis

$V \cong V^*$

Choose a basis $\{e_i\}_{i=1}^n \subset V$

dual

Let V be a real, finite-dim vector space

Def dual of V is $V^* = \{\alpha : V \to \mathbb{R} \text{ linear}\}$

linear maps are specified by action on a basis

$V \cong V^*$

choose a basis $\{e_i\}_{i=1}^n \subseteq V$

dual basis $\{\varepsilon^i\}_{i=1}^n \subseteq V^*$ $\varepsilon^i : V \to \mathbb{R}$

$\varepsilon^i(e_j) = \delta^i_j$

Cotangent space at $e \in M$

$T^*_e M = (T_e M)^*$

Coordinate basis

if (U, φ) $\varphi = (x^1, \dots, x^n)$

$\{dx^i_e\}_{i=1}^n$ dual basis of $\left\{\left.\frac{\partial}{\partial x^i}\right|_e\right\}_{i=1}^n$

$dx^i_e : T_eM \to \mathbb{R}$ $\quad dx^i_e\left(\left.\frac{\partial}{\partial x^j}\right|_e\right) = \delta^i_j$

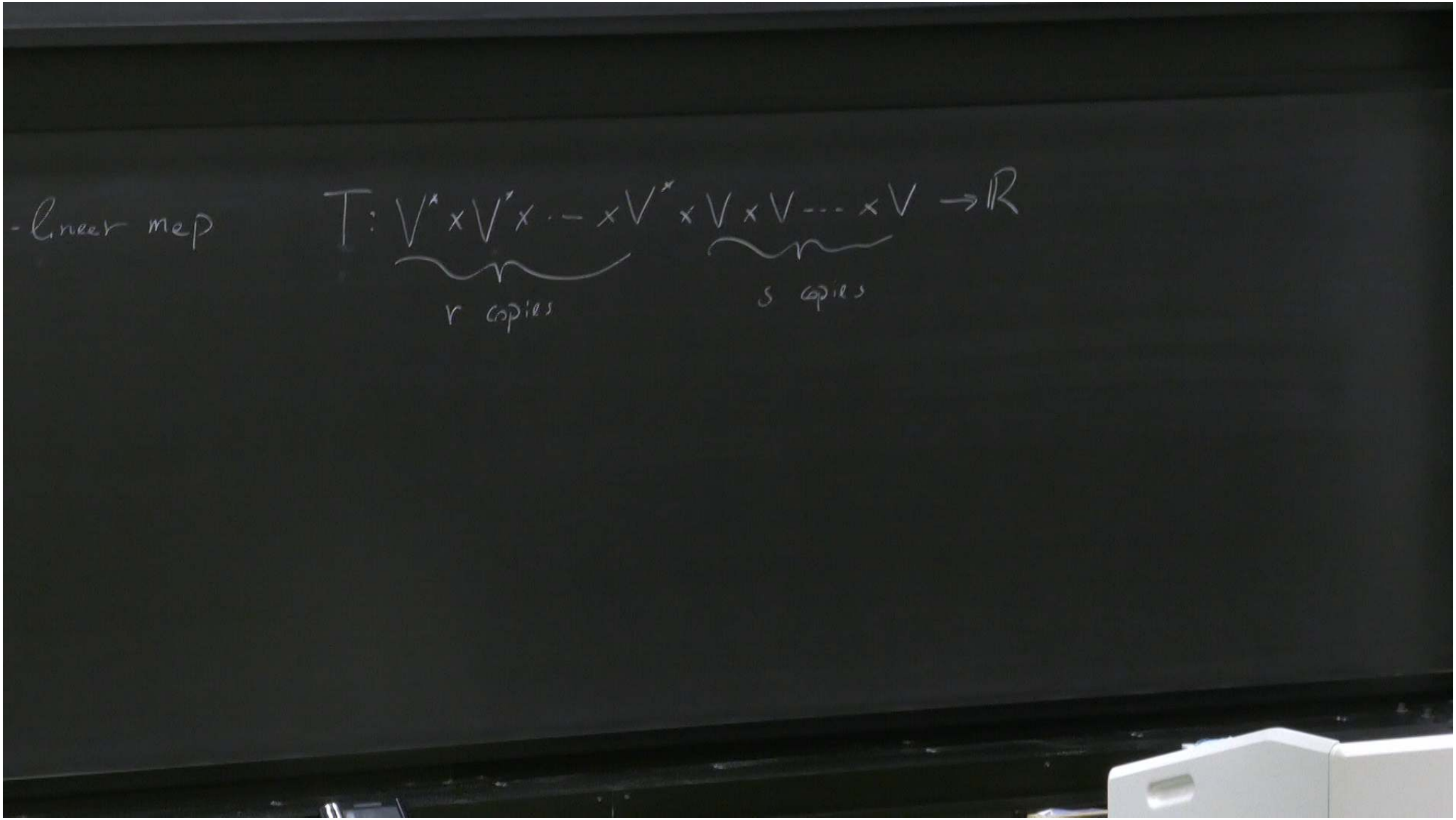
$$dg^i_e = \frac{\partial g^i}{\partial x^j}(\varphi(e))\, dx^j_e$$

Def: a tensor on V of type (r,s) is a multi-linear map $T: V^{\ldots}$

Ex. $r=1, s=0 \rightarrow$ vectors (identified)

$s=1, r=0 \rightarrow$ covectors

$r=0, s=2 \rightarrow$ bilinear map (ex: scalar product)


- linear map
$T: V^* \times V^* \times \cdots \times V^* \times V \times V \cdots \times V \to \mathbb{R}$
r copies
s copies

Tensor products

Def if V, W real, fin. dim, vect. spaces, $\alpha: V \to$

the tensor product of α, β is the bilinear map

$T^{r,s}(V)$ = set of all (r,s)-tensors on V → vector space with pointwis

spaces, $\alpha : V \to \mathbb{R}$, $\beta : W \to \mathbb{R}$

ne bilinear map $\alpha \otimes \beta : (v, w) \in V \times W \mapsto \alpha(v)\,\beta(w)$

ce with pointwise operations $(\alpha T + \beta S)(\cdots) = \alpha T(\cdots) + \beta S(\cdots)$

$T \in \mathcal{T}^{r,s}(V)$ → fully specified by action on basis elements of V, V^*

$$T(\alpha^i \varepsilon_i, \dots) = \alpha_i T(\varepsilon^i, \dots) = e_i(\alpha_j \varepsilon^j)\, T(\varepsilon^i, \dots)$$

$$T = \underbrace{T(\varepsilon^{i_1}, \varepsilon^{i_2}, \dots, \varepsilon^{i_r}, e_{j_1}, \dots, e_{j_s})}\, e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \dots \otimes \varepsilon^{j_s}$$

$T^{i_1 i_2 \dots i_r}_{j_1 \dots j_s}$ components of T in basis

$T \in \mathcal{T}^{r,s}(V)$ → fully specified by action on basis elements of V, V^*

$\{e_i\}$ basis of V

$\{\varepsilon^j\}$ basis of V^* (dual)

$$T(\alpha_i \varepsilon^i, \dots) = \alpha_i T(\varepsilon^i, \dots) = e_i(\alpha_j \varepsilon^j)\, T(\varepsilon^i, \dots)$$

$$T = \underbrace{T(\varepsilon^{i_1}, \varepsilon^{i_2}, \dots, \varepsilon^{i_r}, e_{j_1}, \dots, e_{j_s})}\; e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \dots \otimes \varepsilon^{j_s}$$

$T^{i_1 i_2 \dots i_r}_{j_1 \dots j_s}$ components of T in basis