Title: Mathematical Physics Core Lecture

Speakers: Giuseppe Sellaroli

Collection: Mathematical Physics - Core 2023/24

Date: January 10, 2024 - 11:30 AM

URL: https://pirsa.org/24010027

Today's plan

- smooth functions
- Tangent space
- Cotangent space?

$f : M \to N$ smooth?

(U, φ) chart in $\mathcal{A}_M$

(V, ψ) in $\mathcal{A}_N$

$f(U) \subseteq V$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, shift left, "\varphi"] & V \arrow[d, "\psi"] \\ \varphi(U) \subseteq \mathbb{R}^m \arrow[u, shift left, "\varphi^{-1}"] \arrow[r, "\psi \circ f \circ \varphi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

$f : M \to N$ smooth at $x \in M$

if for all charts (U, φ) in $\mathcal{A}_M$, (V, ψ) in $\mathcal{A}_N$

with $x \in U$, $f(x) \in V$

The map $\psi \circ f \circ \varphi^{-1}$ is smooth

it's enough to check 2 charts

$f: M \to N$ smooth

if it is smooth at each $a \in M$

Notation

$$\hat{f} = \psi \circ f \circ \varphi^{-1}$$

coordinate representation of f

$$\hat{a} = \varphi(a)$$

$$\hat{f}(\hat{a}) = \psi \circ f \circ \varphi^{-1}(\varphi(a))$$

Def $f: M \to N$ is a diffeomorphism if:

- invertible
- smooth
- f^{-1} smooth

...sm

Def: M is diffeomorphic to N
if there is a diffeomorphism $f: M \to N$

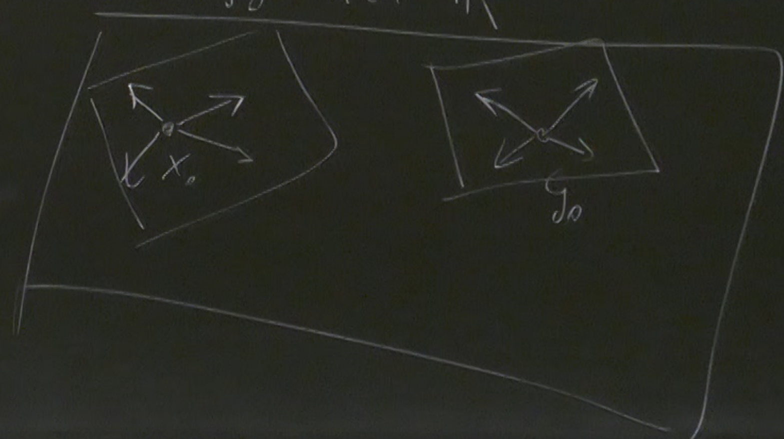
Goal: generalise the idea of tangent plane of surfaces

Approach: derivation

(vs curves)

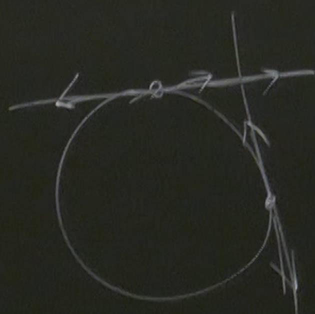
$M = \mathbb{R}^n$ tangent space at $x_0 \in \mathbb{R}^n$

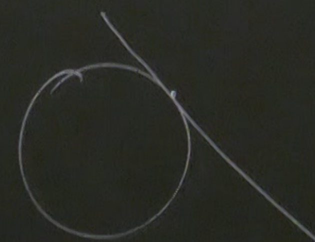
$\rightarrow$ identify with $\mathbb{R}^n$



rfaces $z = f(x,y)$ (graph of $f: \mathbb{R}^2 \to \mathbb{R}$)

ph $\hookrightarrow \{(x,y,z) \mid z = f(x,y)\}$





① to each $v \in \mathbb{R}^n$ we can associate the directional derivative evaluated at x_0

$$D_v : f \in C^\infty(\mathbb{R}^n; \mathbb{R}) \mapsto v \cdot \nabla f(x_0) = v^i \frac{\partial f}{\partial x^i}(x_0) \in \mathbb{R}$$

(linear operator $\begin{cases} (f+g) \\ (\alpha f)(x \end{cases}$

Def: if M is a space where $C^\infty(M, \mathbb{R})$ makes sense

a derivation at $a \in M$ is a linear operator $X : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ that directional derivative evaluated at x_0

$\frac{\partial f}{\partial x^i}(x_0) \in \mathbb{R}$ (linear operator $\begin{cases} (f+g)(x) = f(x) + g(x) \\ (\alpha f)(x) = \alpha f(x) \end{cases}$

$(M, \mathbb{R})$ makes sense

erator $X : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ that satisfies the product rule

$X(fg) = f(a)X(g) + g(a)X(f)$

Prop. if X is a derivation at $x_0 \in \mathbb{R}^n$, Then $X = D_v$, $v \in \mathbb{R}^n$

① if f constant, $X(f) = 0$ for all X | $X(1^2) = 1 \cdot X(1) + 1$

② if $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, then $f(x) = f(x_0) + (x^i - x_0^i)\, g_i(x)$ | $g_i(x)$

$X = D_v \,, \quad v \in \mathbb{R}^n$

$$X(1^2) = 1 \cdot X(1) + 1 \cdot X(1) = 2X(1)$$

$(x^i - x_0^i)\, g_i(x) \qquad g_i(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}) \qquad g_i(x_0) = \frac{\partial f}{\partial x^i}(x_0)$

$$f(x) = f(x_0) + f(x) - f(x_0)$$

$$= f(x_0) + \int_0^1 \frac{d}{dt} f(x_0 + t(x - x_0))\Big|_{t=s} ds$$

$$= f(x_0) + \int_0^1 \frac{\partial f}{\partial x^i}(x_0 + s(x - x_0))(x^i - x_0^i)\, ds$$

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(x_0 + s(x - x_0))\, ds$$

X derivation

$$X(f) = \underset{X(f(x_0))}{\underset{\uparrow}{0}} + g_i(x_0) X(x^i - x$$

$$= \frac{\partial f}{\partial x^i}(x_0) X(x^i$$

X derivation

$$X(f) = 0 + g_i(x_0)\, X(x^i - x_0^i) + \cancel{(x^i - x_0^i)\big|_{x=x_0} X(g_i)}$$

$X(f(x_0))$

$$= \frac{\partial f}{\partial x^i}(x_0)\, X(x^i) = D_v f \qquad v^i = X(x^i)$$

$f(x) = f(x_0) + (x - x_0)\, g_i(x)$ | $g_i(x) \in C^\infty$

- $T_{x_0}\mathbb{R}^n$ space of all derivations $\rightarrow$ vector space $\big((X+Y)(f) = X(f) + Y(f) \big)$
- $T_{x_0}\mathbb{R}^n$ = set of all directional derivatives
- $v \in \mathbb{R}^n \longmapsto D_v \in T_{x_0}\mathbb{R}^n$ linear, injective, surjective

$(X+Y)(f) = X(f) + Y(f)\ )$

ive

<u>Def</u> if M is a manifold and $a \in M$

then tangent space to M at a is

$T_a M = \{\text{derivations at } a \in M\}$ Vector space

$$g_i(x) = \int_0 \frac{\partial f}{\partial x^i}\big(x_0 + s(x - x_0)\big)\, ds$$

let (U,φ) be a chart with $a \in U$ | identify $\varphi : M \to \mathbb{R}^n$ as a vector of scalar functions

define $\left.\frac{\partial}{\partial x^i}\right|_a : f \in C^\infty(M,\mathbb{R}) \mapsto \frac{\partial \hat{f}}{\partial x^i}(\hat{a}) \in \mathbb{R}$ → derivation

$$\frac{\partial \hat{f}}{\partial x^i}(\hat{a}) = \left(\frac{\partial}{\partial x^i} f \circ \varphi^{-1}\right)(\varphi(a))$$

$\left\{ \left.\frac{\partial}{\partial x^i}\right|_a \right\}_{i=1}^n$ is linearly independent

$\varphi : M \to \mathbb{R}^n$ as a vector of scalar functions $\quad \varphi = (x^1, x^2, \dots, x^n) \quad x^i : M \to \mathbb{R}$

$\to$ derivation

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^{n}$ is linearly independent

f constant $\Rightarrow X(f) = 0$

$b \in U \qquad f(b) = f(a) + (x^i(b) - x^i(a))\, g_i(b)$

$$g_i(b) = \int_0^1 \frac{\partial \hat{f}}{\partial x^i}\big(\varphi(a) + s(\varphi(b) - \varphi(a))\big)\, ds$$

$$X(f) = 0 + X(x^i) \cdot \frac{\partial \hat{f}}{\partial x^i}(\hat{a}) + 0$$

$$X = X(x^i) \left.\frac{\partial}{\partial x^i}\right|_a$$

$\Rightarrow T_a M \cong \mathbb{R}^n$

$\psi = (y^1, y^2, \dots, y^n)$

$$\left.\frac{\partial}{\partial y^i}\right|_a = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$