Title: Mathematical Physics Core Lecture

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Collection: Mathematical Physics - Core 2023/24

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Manifolds

goal: formalise the idea of "n-dimentional smooth shape"

n-dimensional $\to$ locally looks like $\mathbb{R}^n$

Def: a <u>chart</u> on M ← set is a pair (U, φ) with $U \subseteq M$, $\varphi : U \to \mathbb{R}^n$

s.t. φ is injective and $\underline{\varphi(U) \text{ is open in } \mathbb{R}^n}$

$\varphi^{-1} : \varphi(U) \to U$

$\varphi(U)$ is open

it looks like $\mathbb{R}^n$

pe"

<u>Def</u> an n-atlas is a collection $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$

with $\varphi_i : U_i \subseteq M \to \mathbb{R}^n$

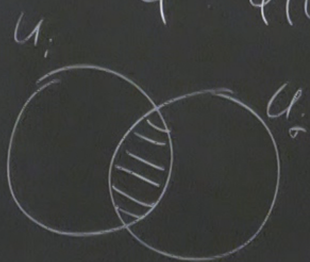
- $\bigcup_{i \in I} U_i = M$
- $\varphi_i(U_i \cap U_j)$ open for all i, j
- the transition maps $\varphi_i \circ \varphi_j^{-1}$ are $\mathcal{C}^\infty$

$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to \mathbb{R}^n$

Def: a <u>chart</u> on $M^{\swarrow\,\text{set}}$ is a pair (U, φ) with $U \subseteq M$, $\varphi : U \to \mathbb{R}^n$

s.t. φ is injective and <u>$\varphi(U)$ is open in $\mathbb{R}^n$</u>

$\varphi^{-1} : \varphi(U) \to U$



$\varphi(U)$ is open

it looks like $\mathbb{R}^n$

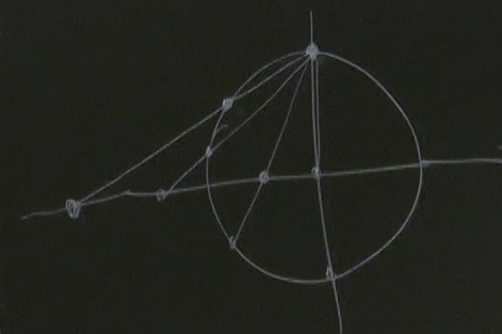
Def: a smooth manifold is a set M together with an n-atlas A_M

we say that $\dim(M) = n$

Examples

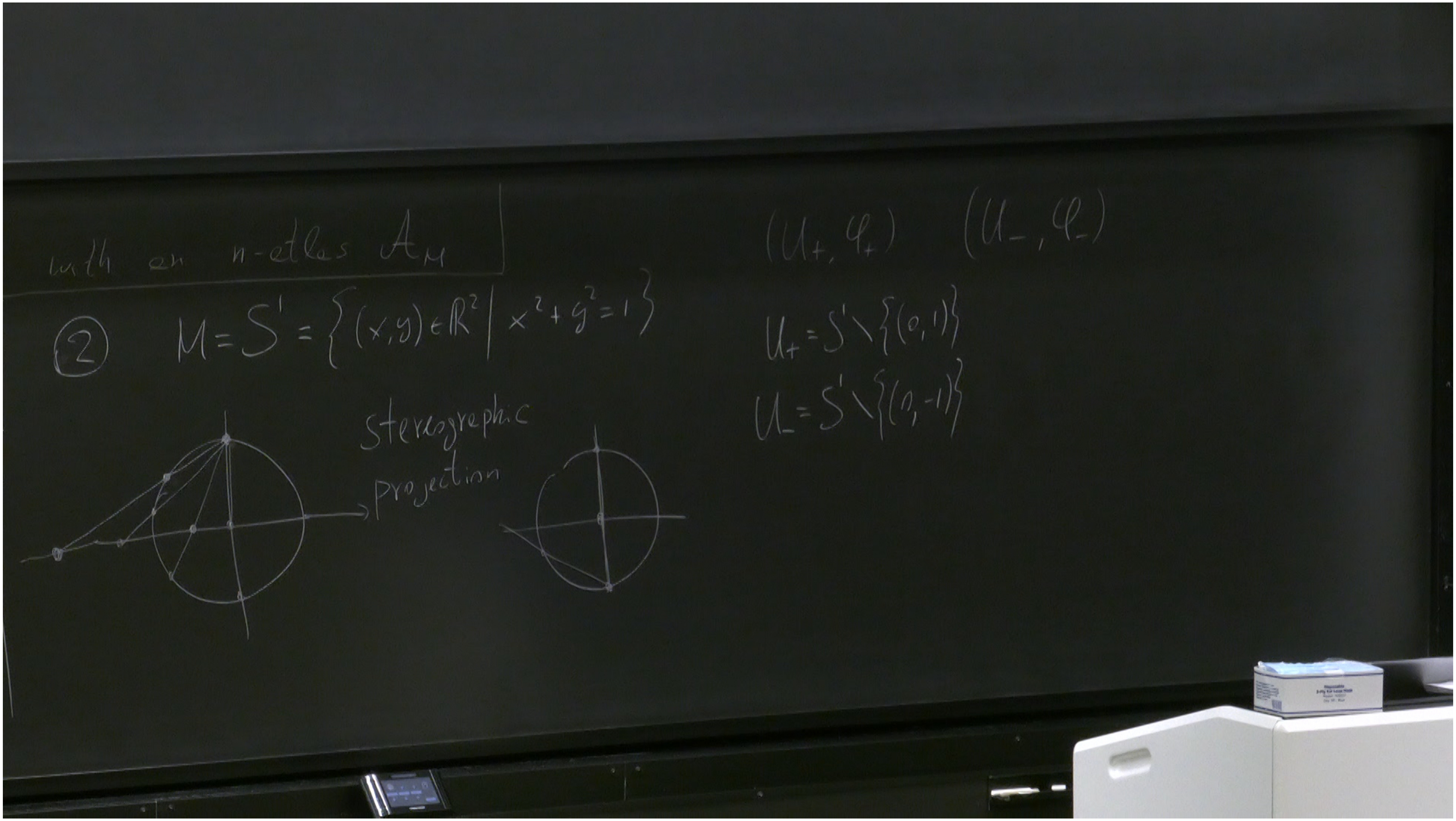
① $M = \mathbb{R}^n$ $\quad A = \{(\mathbb{R}^n, id)\}$

② $M = S^1 = \{(x,y) \in$



stereo

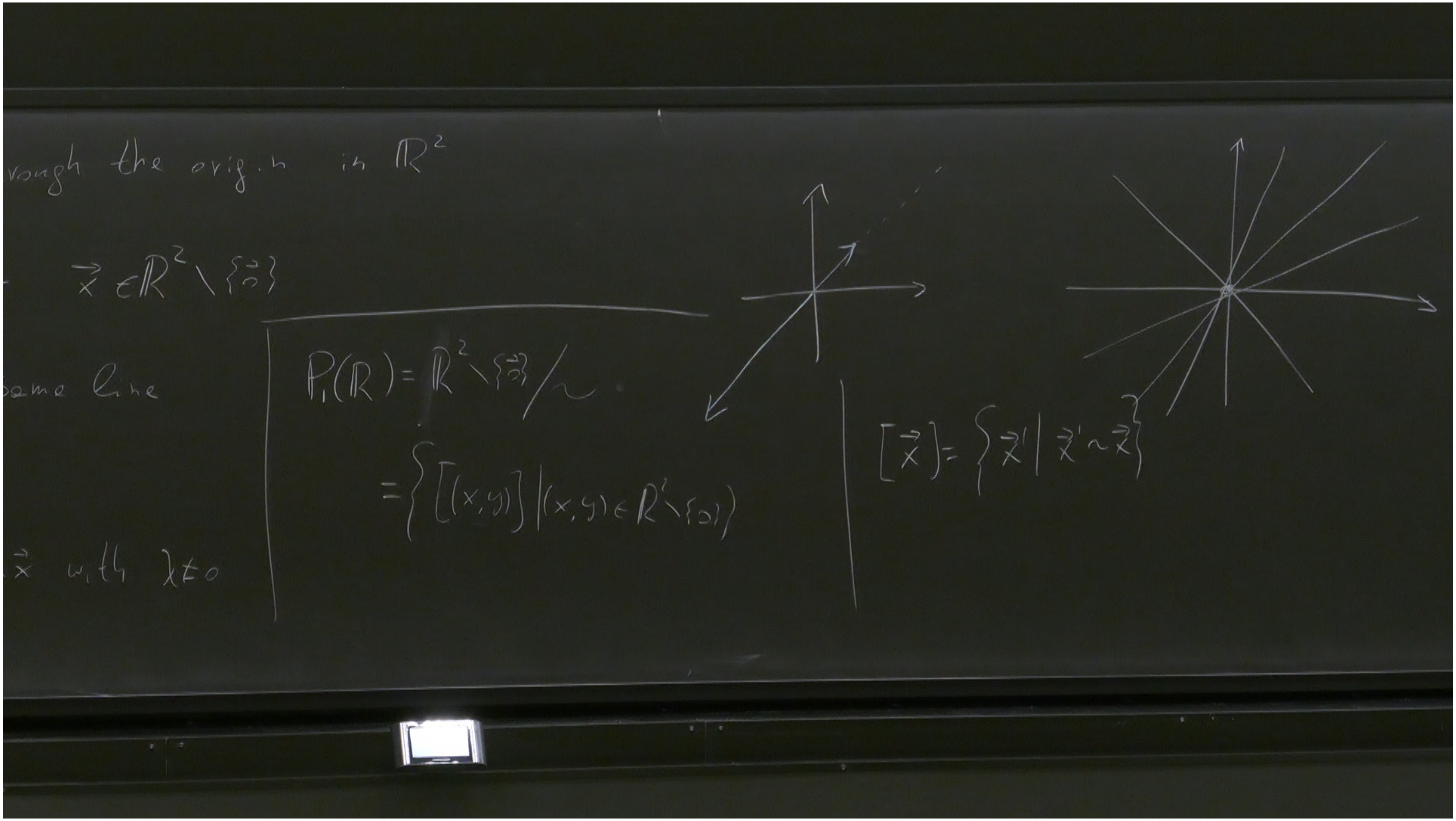
projec


with an n-atlas $\mathcal{A}_M$
(2) $M = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$
stereographic projection
(U_+, φ_+) (U_-, φ_-)
$U_+ = S^1 \setminus \{(0,1)\}$
$U_- = S^1 \setminus \{(0,-1)\}$

③ <u>projective line</u>: set of all lines through the origin in $\mathbb{R}^2$

- line is defined by non-zero direction vector $\vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}\}$
- if $\vec{x}' = \lambda \vec{x}$ with $\lambda \neq 0$ they give the same line

→ equivalence relation: $\vec{x}' \sim \vec{x}$ if $\vec{x}' = \lambda \vec{x}$ with $\lambda \neq 0$


rough the origin in $\mathbb{R}^2$
$\vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}\}$
same line
$\vec{x}$ with $\lambda \neq 0$
$P_1(\mathbb{R}) = \mathbb{R}^2 \setminus \{\vec{0}\} / \sim$
$= \{ [(x,y)] \mid (x,y) \in \mathbb{R}^2 \setminus \{0\} \}$
$[\vec{x}] = \{ \vec{x}' \mid \vec{x}' \sim \vec{x} \}$

(3) **projective line**: set of all lines through the origin in $\mathbb{R}^2$

- line is defined by non-zero direction vector $\vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}\}$
- if $\vec{x}' = \lambda \vec{x}$ with $\lambda \neq 0$ they give the same line

$\rightarrow$ equivalence relation: $\vec{x}' \sim \vec{x}$ if $\vec{x}' = \lambda \vec{x}$ with $\lambda \neq 0$

$P_1(\mathbb{R}) = \mathbb{R}^2 \setminus \{$

$= \{ [(x, y$

two charts (U_1, φ_1) (U_2, φ_2)

$U_1 = \{ [(x,y)] \mid x \neq 0 \}$

$U_2 = \{ [(x,y)] \mid y \neq 0 \}$

$\varphi_1 : [(x,y)] \mapsto y/x$

$\varphi_2 : ([x,y]) \mapsto x/y$

④ if F

(m <

then

Circle

$\rightarrow y = \sqrt{\ }$

④ if $F : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ smooth, Jacobian matrix has full rank $(=m)$

$(m<n)$

then $F^{-1}(\{\vec{0}\})$ is a smooth manifold

rank $J = 1$

$J = (2x, 2y)$

$F(x,y) = x^2+y^2-1$

Circle $\quad x^2+y^2-1=0$

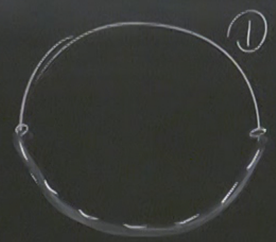
$U = \mathbb{R}^2 \setminus \{\vec{0}\}$

locally $\quad y = f(x)$

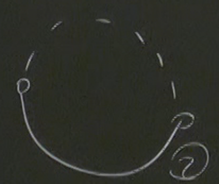
$\to y = \sqrt{1-x^2}$

$y = -\sqrt{1-x^2}$

$x = \sqrt{1-y^2}$



$y=\sqrt{1-x^2}$ $\quad U_1 \quad$ $\varphi_1(x,y)=x$ $\quad$ $\varphi_1^{-1}(x)=(x,\sqrt{1-x^2})$

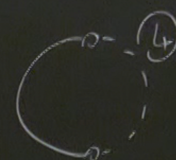


$y=-\sqrt{1-x^2}$ $\quad$ $\varphi_2(x,y)=x$ $\quad$ $\varphi_2^{-1}(x)=(x,-\sqrt{1-x^2})$

$\ldots y) = x$ $\qquad \varphi_1^{-1}(x) = (x, \sqrt{1-x^2})$

$\ldots) = (x, -\sqrt{1-x^2})$

$x = \sqrt{1-y^2}$ $\qquad \varphi_3(x,y) = y$ $\qquad \varphi_3^{-1}(y) = (\sqrt{1-y^2}, y)$

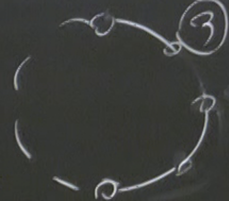


$x = -\sqrt{1-y^2}$ $\qquad \varphi_4(x,y) = y$ $\qquad \varphi_4^{-1}(y) = (-\sqrt{1-y^2}, y)$

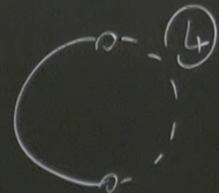
$\varphi_3 \circ \varphi_1^{-1}(t) = \varphi_3(t, \sqrt{1-t^2}) = \sqrt{1-t^2}$

$t \in (0,1)$

$x) = (x, \sqrt{1-x^2})$



$x = \sqrt{1-y^2}$ $\quad \varphi_3(x,y) = y$ $\quad \varphi_3^{-1}(y) = (\sqrt{1-y^2}, y)$



$x = -\sqrt{1-y^2}$ $\quad \varphi_4(x,y) = y$ $\quad \varphi_4^{-1}(y) = (-\sqrt{1-y^2}, y)$

$$\varphi_3 \circ \varphi_1^{-1}(t) = \varphi_3(t, \sqrt{1-t^2}) = \sqrt{1-t^2}$$

$$t \in (0,1)$$

}

n atlas

$,1] \cong_s (-1,1)$

<u>Def</u> a manifold is a set <u>M</u> <u>with</u> a <u>maximal atlas</u>

↓

atlas that cannot be extended

(it contains all equivalent atlases)

$S^1 \cong \mathbb{R} \cong \mathbb{R}^2$

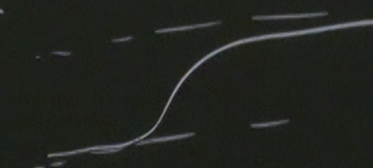
$F^{-1}(\{n\}) \subseteq \mathbb{R}^n$

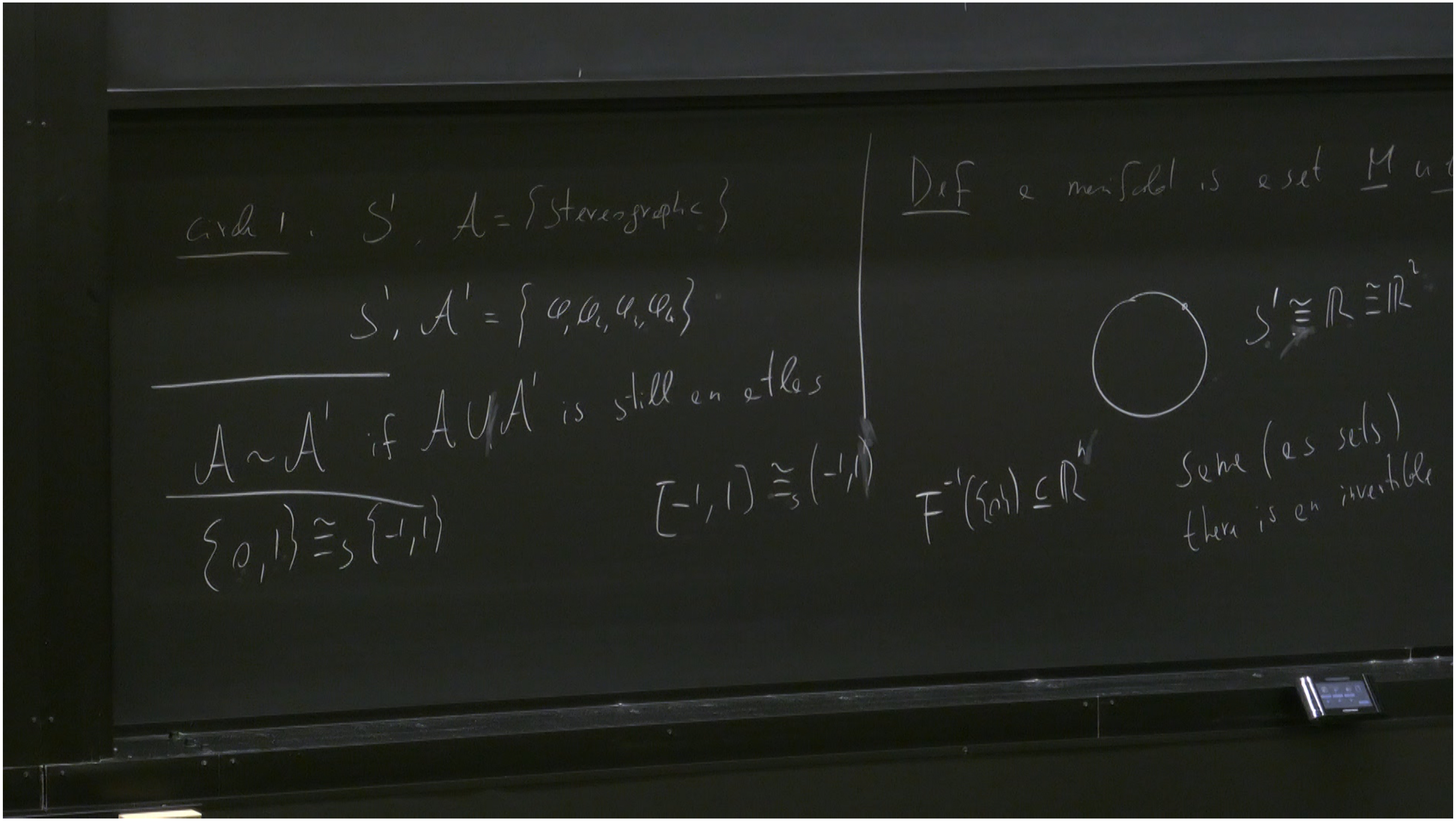
Same (as sets)

there is an invertible map

$(-1,1) \cong \mathbb{R}$

tanh




Circle 1. S^1, $A = \{$stereographic$\}$
S^1, $A' = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$
$A \sim A'$ if $A \cup A'$ is still an atlas
$\{0,1\} \cong_s \{-1,1\}$
$[-1,1) \cong_s (-1,1)$
Def a manifold is a set M
$S^1 \cong \mathbb{R} \cong \mathbb{R}^2$
$F^{-1}(\{0\}) \subseteq \mathbb{R}^n$
same (as sets)
there is an invertible