Title: Quantum Field Theory for Cosmology - Lecture 20240130

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: January 30, 2024 - 4:00 PM

URL: https://pirsa.org/24010014

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QFT for Cosmology, Achim Kempf, Lecture 7

The driven harmonic oscillator cont'd:

D. Energy eigenstates

* Recall $\hat{H}(t) = \omega\left(a^+(t)a(t)+\frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\left(a^+(t)+a(t)\right)J(t)$

$$= \begin{cases} \omega\left(a^+_{in}a_{in}+\frac{1}{2}\right) & \text{for } t<0 \\ \text{something} & \text{for } 0\le t\le T \\ \omega\left(a^+_{out}a_{out}+\frac{1}{2}\right) & \text{for } T<t \end{cases}$$

(Q: How come that $\hat{H}(t<0) \neq \hat{H}(t>T)$?
A: We use the Heisenberg picture!)

Here, $a_{in} := a(0)$, $a_{out} := a(T)$ and $a_{out} = a_{in} + J_0$

with: $J_0 := i\frac{1}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}dt'$

* For $t<0$, we diagonalized the Hamiltonian

$$\hat{H}(t) = \omega\left(a^+_{in}a_{in}+\tfrac{1}{2}\right) = \hat{H}_{t<0} = \text{const.}$$

by using $[a_{in}, a^+_{in}]=1$ to construct its eigenbasis:

$$\hat{H}_{t<0}|n_{in}\rangle = E_n^{(in)}|n_{in}\rangle$$

Namely:

$$E_n^{(in)} = \omega\left(n+\tfrac{1}{2}\right), \quad n = 0,1,2,3\ldots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}}(a^+_{in})^n|0_{in}\rangle$$

Note: The set $\{|n_{in}\rangle\}$ is a Hilbert basis of the Hilbert space $\mathcal{H}$.

* By $t>T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega\left(a^+_{out}a_{out}+\tfrac{1}{2}\right) = \hat{H}_{t>T} = \text{const.}$$

What are its eigenvectors $|n_{out}\rangle$ and eigenvalues E_n^{out}?

* There is a unique vector $|0_{out}\rangle \in \mathcal{H}$ obeying: (up to normalization and phase)

$$a_{out}|0_{out}\rangle = 0$$

* We define the set of vectors $\{|n_{out}\rangle\}$:

1/22

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we have:

$$[a_{out}, a^+_{out}] = 1$$

$\Rightarrow$ we can construct the eigenbasis of $H_{t>T}$ with the same method as the eigenbasis of $H_{t<0}$:

both describe a free oscillator of frequency ω.

$$\hat{H}_{t>T} |n_{out}\rangle = E_n^{(out)} |n_{out}\rangle \text{ with } E_n^{(out)} = \omega(n + \tfrac{1}{2}) = E_n^{(in)}$$

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space $\mathcal{H}$.

How are the two bases related?

* Recall: Both, $\{|n_{in}\rangle\}$ and $\{|n_{out}\rangle\}$ are ON bases of $\mathcal{H}$.

  $\Rightarrow$ Each basis vector $|n_{in}\rangle$ is a linear combination of the basis vectors $\{|n_{out}\rangle\}$ and vice versa.

* Therefore, in particular:

  There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \Lambda_n |n_{out}\rangle$$

"Bogolubov Transformation"

* Meaning of the Λ_n?

  - Recall: The system's state is frozen in state $|\psi\rangle = |0_{in}\rangle$.
  - Assume we measure at a time $t > T$ the energy, i.e., we measure
    $$\hat{H}(t) = \omega(a^+_{out} a_{out} + \tfrac{1}{2})$$
  - What is the probability amplitude for finding the energy eigenvalue E_n?
  - Clearly:
    $$\text{prob.amp.}(|n_{out}\rangle \text{ at } t>T) = \langle n_{out}|\psi\rangle$$
    i.e.:
    $$\text{prob.}(|n_{out}\rangle \text{ at } t>T) = |\langle n_{out}|\psi\rangle|^2$$

3/22

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There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \Lambda_n \, |n_{out}\rangle$$

$\llcorner$ "Bogolubov Transformation"

energy eigenvalues E_n:

- Clearly:

$$\text{prob.amp.}(|n_{out}\rangle \text{ at } t>T) = \langle n_{out} | \psi \rangle$$

i.e.: $$\text{prob.}(|n_{out}\rangle \text{ at } t>T) = |\langle n_{out} | \psi \rangle|^2$$

- Calculate:

$$\langle n_{out} | \psi \rangle = \langle n_{out} | 0_{in} \rangle$$
$$= \langle n_{out} | \sum_m \Lambda_m \, | m_{out} \rangle$$
$$= \Lambda_n$$

$\Rightarrow$ If the oscillator started in its ground state, then at time $t>T$ the probability for finding the oscillator in its n'th excited state is given by:

$$\text{prob.}(|n_{out}\rangle \text{ at } t>T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after charges and currents $J(x,t)$ excited the vacuum.

Calculation of Λ_n:

Proposition: $$\Lambda_n = e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n$$

Proof: The claim is that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n \, |n_{out}\rangle$.

We need to check that indeed: $a_{in} |0_{in}\rangle = 0$

Using $a_{out} = a_{in} + J_0$, we need to check: $(a_{out} - J_0) \, |0_{in}\rangle = 0$

Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n \, |n_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) \sum_n J_0^n \overbrace{\frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}}}^{= \frac{1}{n!}} (a_{out}^+)^n |0_{out}\rangle$$

5 / 22

5

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$$\text{prob.}(|n_{out}\rangle \text{ at } t>T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after charges and currents $J(x,t)$ excited the vacuum.

Indeed:

$$(a_{out} - J_0)\sum_n e^{-\frac{1}{2}|J_0|^2}\frac{1}{\sqrt{n!}}J_0^n|n_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}(a_{out} - J_0)\sum_n J_0^n \overbrace{\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n!}}}^{\frac{1}{n!}}(a^+_{out})^n|0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}(a_{out} - J_0)\, e^{J_0 a^+_{out}}|0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}\left(a_{out}e^{J_0 a^+_{out}} - J_0 e^{J_0 a^+_{out}}\right)|0_{out}\rangle$$

using $AB = [A,B] + BA$

$$= e^{-\frac{1}{2}|J_0|^2}\left([a_{out}, e^{J_0 a^+_{out}}] + e^{J_0 a^+_{out}}a_{out} - J_0 e^{J_0 a^+_{out}}\right)|0_{out}\rangle$$

$$\overset{(*)}{=} e^{-\frac{1}{2}|J_0|^2}\left((J_0 - J_0)e^{J_0 a^+_{out}} + e^{J_0 a^+_{out}}a_{out}\right)|0_{out}\rangle = 0 \quad ✓$$

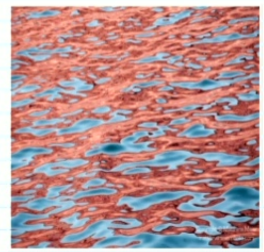
Note: In the last step, (*), we used that: $[a_{out}, e^{J_0 a^+_{out}}] = J_0 e^{J_0 a^+_{out}}$.

Exercise: Show that, more generally, $[a,a^+]=1$ implies $[a, f(a^+)] = f'(a^+)$ by induction.

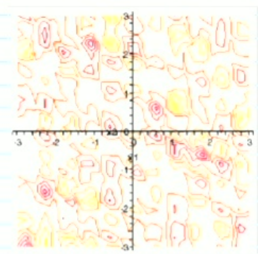
Hint: Show that: $[a,a^+]=1$, $[a,a^{+2}] = 2a^+$, $[a,a^{+3}] = 3a^{+2}$, $\dots$, $[a,a^{+n}] = n(a^+)^{n-1}$

Exercise: Verify that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2}\frac{1}{\sqrt{n!}}J_0^n|n_{out}\rangle$ obeys $\langle 0_{in}|0_{in}\rangle = 1$.

Apply this strategy to the mode oscillators in QFT:



Making waves...



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$$= \Lambda_n$$

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Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) \sum_n J_0^n \underbrace{\frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}}}_{=\frac{1}{n!}} (a^\dagger_{out})^n |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) e^{J_0 a^\dagger_{out}} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2} \left(a_{out} e^{J_0 a^\dagger_{out}} - J_0 e^{J_0 a^\dagger_{out}} \right) |0_{out}\rangle$$

using $AB = [A,B] + BA$

$$= e^{-\frac{1}{2}|J_0|^2} \left([a_{out}, e^{J_0 a^\dagger_{out}}] + e^{J_0 a^\dagger_{out}} a_{out} - J_0 e^{J_0 a^\dagger_{out}} \right) |0_{out}\rangle$$

(*) $$= e^{-\frac{1}{2}|J_0|^2} \left((J_0 - J_0) e^{J_0 a^\dagger_{out}} + e^{J_0 a^\dagger_{out}} a_{out} \right) |0_{out}\rangle = 0 \quad \checkmark$$

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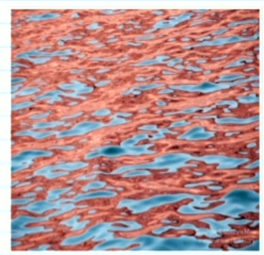

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7 / 22

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Hint: Show that: $[a,a^+]=1,\ [a,a^{+2}] = 2a^+,\ [a,a^3] = 3a^2, \dots, [a,a^{+n}] = n(a^+)^{n-1}$

Exercise: Verify that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$ obeys $\langle 0_{in}|0_{in}\rangle = 1$.

Recall:

$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + J(x,t)\hat{\phi}(x,t)\, d^3x$$

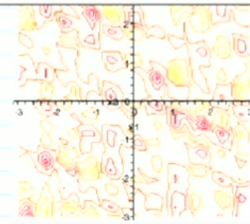
Example interpretation:

* $\hat{\phi}(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.

* $J(x,t)$ may be viewed as a simplified version of a given classical electric charge and current density function.

Example:

A (Klein-Gordon) charge traveling a path $\vec{x}'(t)$:
Then: $J(x,t) = q\,\delta(x - \vec{x}(t))$



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In- and out periods

Let us consider the case where

$$J(x,t) = 0 \quad \text{for all} \quad t \notin [0,T]$$

$\Rightarrow$ It suffices to consider the periods $t<0$ and $t>T$ in both of which $J(x,t)=0$ (and then to relate the bases).

The free (i.e., undriven) QFT: $(t<0 \text{ or } t>T)$

$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t)\, d^3x$$

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classical electric charge and current density function.

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A (Klein-Gordon) charge traveling a path $\vec{x}'(t)$:

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$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t)\, d^3x$$

* We need to solve:

$$\dot{\hat{\pi}}(x,t) - (\Delta - m^2)\hat{\phi}(x,t) = 0$$

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta^3(x-x')$$

* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \hat{\phi}(x,t)\, e^{-ikx}\, d^3x$$

we need to solve:

$$\ddot{\hat{\phi}}_k(t) + \underbrace{(k^2 + m^2)}_{\text{Definition: } =: \omega_k^2}\hat{\phi}_k(t) = 0 \qquad \text{(EoM)}$$

$$k^2 = \sum_{i=1}^{3} k_i^2$$

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\,\delta^3(k+k') \qquad \text{(CCRs)}$$

* Recall: $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$ means $\hat{\phi}_k^\dagger(t) = \hat{\phi}_{-k}(t)$.

Solution strategy due to Fock:

* Proceed analogously to the driven oscillator, e.g., during $t < 0$:

- Introduce new variables:

QM: $a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$

QFT: $a_k(t) := \sqrt{\frac{\omega_k}{2}}\,\hat{\phi}_k(t) + i\frac{1}{\sqrt{2\omega_k}}\,\hat{\pi}_k(t)$

- Equation of motion and CCRs:

QM: $\dot{a}(t) = -i\,\omega\, a(t) \qquad [a(t), a^\dagger(t)] = 1$

Exercise: verify

QFT: $\dot{a}_k(t) = -i\,\omega_k\, a_k(t) \qquad [a_k(t), a_{k'}^\dagger(t)] = \delta^3(k-k')$

- Remark: Valid only while no force and while ω is constant.

11 / 22

11

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$$\ddot{\hat\phi}_k(t) + \big(\underbrace{k^2 + m^2}_{\text{Definition: } =:\, \omega_k^2}\big)\,\hat\phi_k(t) = 0 \qquad \text{(EoM)}$$

($k^2 = \sum_{i=1}^{3} k_i^2$)

$$[\hat\phi_k(t), \hat\pi_{k'}(t)] = i\,\delta^3(k+k') \qquad \text{(CCRs)}$$

* **Recall:** $\hat\phi^\dagger(x,t) = \hat\phi(x,t)$ means $\hat\phi_k^\dagger(t) = \hat\phi_{-k}(t)$.

Equation of motion and CCRs:

QM: $\dot a(t) = -i\omega a(t)$ $\qquad [a(t), a^+(t)] = 1$

QFT: $\dot a_k(t) = -i\,\omega_k a_k(t)$ $\qquad [a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$ (Exercise: verify)

Remark: Valid only while no force <u>and</u> while ω is constant.

Solution, using an initial condition:

QM: $a(t) = e^{-i\omega t} a_{in}$, $\qquad [a_{in}, a_{in}^+] = 1$

QFT: $a_k(t) = e^{-i\omega_k t} a_{in_k}$, $\qquad [a_{in_k}, a_{in_{k'}}^+] = \delta^3(k-k')$

Explicitly $\Rightarrow$

QM: $$\hat q(t) = \frac{1}{\sqrt2}\left(\frac{e^{-i\omega t}}{\sqrt\omega} a_{in} + \frac{e^{i\omega t}}{\sqrt\omega} a_{in}^+\right)$$

QFT: $$\hat\phi_k(t) = \frac{1}{\sqrt2}\left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_{-k}}^+\right) \qquad (S)$$

Exercise: verify

$$\left(\text{i.e.: } \hat\phi(x,t) = \int \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^+ e^{i\omega_k t - ikx}\right) d^3k\right)$$

The Hilbert space of states:

* Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys: $a_{in_k}|0_{in}\rangle = 0$, now for all vectors k.

$$\left(|0_{in}\rangle = \bigotimes_k |0_{in_k}\rangle\right)$$

* The Hamiltonian reads (for $t<0$):

Exercise: verify

$$\hat H = \frac12 \int_{\mathbb{R}^3} \hat\pi_k^+ \hat\pi_k + \omega^2 \hat\phi_k^+ \hat\phi_k \; d^3k$$

$$= \int_{\mathbb{R}^3} \omega_k \left(a_{in_k}^+ a_{in_k} + \tfrac12 \delta^3(0)\right) d^3k$$

This ∞ is called a "Infrared divergence"

In a box: $\hat H = L^{-3/2}\sum_k \omega_k\left(a_{in_k}^+ a_{in_k} + \tfrac12\right)$ $\qquad$ (because $\delta(k,k')$ is now $\delta_{k,k'}$)

Notice: The divergence $\sum_k L^{-3/2}\omega_k \frac12 = \infty$ is an "ultraviolet divergence".

13 / 22

After the driving ends, $t > T$:

* One obtains: $a_k(t) = e^{-i\omega_k t} a_{out_k}$ with $a_{out_k} = a_{in_k} + J_{0_k}$

$$J_{0_k} := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J_k(t') e^{i\omega t'} dt'$$

Here: $J_k(t)$ is the Fourier transform of $J(x,t)$.

* Construct the out-basis $\{|n_{out_k}\rangle\}$ from:

$$a_{out_k} |0_{out}\rangle = 0$$

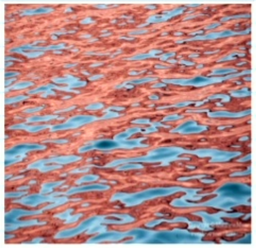
$\rightsquigarrow$ Can calculate, e.g., $|\langle n_{out_k} | 0_{in}\rangle|^2$ i.e., the probability for $J(x,t)$ to have created n particles of momentum k.

Upgrade:

Give the charge $J(x,t)$ its own dynamics:

Notice: The divergence $\sum_k L^{-3/2} \omega_k \frac{1}{2} = \infty$ is an "ultraviolet divergence".

Recall:



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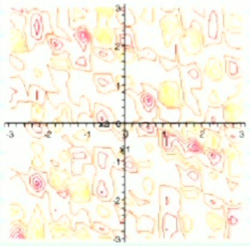

Making EM waves...

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Described by $J(x,t)$.

number-valued, classical

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + J(x,t)\hat{\phi}(x,t)\, d^3x$$

and upgrade it to:

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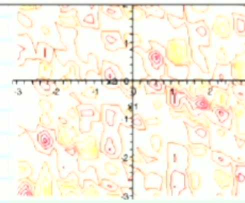
$$a_{out_k}|0_{out}\rangle = 0$$

⤳ Can calculate, e.g., $|\langle n_{out_k}|0_{in}\rangle|^2$ i.e., the probability for $J(x,t)$ to have created n particles of momentum k.



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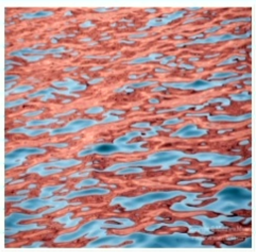
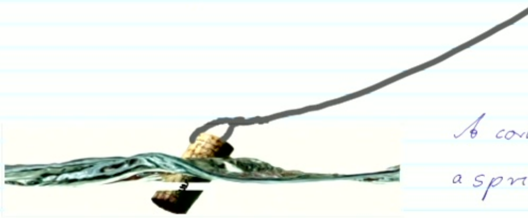
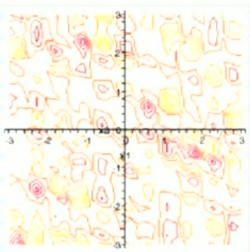
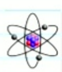
Upgrade: Give the charge $J(x,t)$ its own dynamics:





A cork with a spring.





Often, only one atomic transition is of interest. Then, we can model the atom as a 2-level system.

↳ Described by QM, e.g. atom or qubit

number-valued, classical

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + J(x,t)\hat{\phi}(x,t)\, d^3x$$

and upgrade it to:

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} 1 \otimes \left(\hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t)\right) + \hat{J}(x,t) \otimes \hat{\phi}(x,t)\, d^3x$$

an operator acting on the Hilbert space of the atom.

with $\hat{J}(x,t) = \hat{Q}(t)\,\delta(x - \hat{\vec{x}}(t))$

The Hilbert space: $\mathcal{H}_{total} = \mathcal{H}_{atom} \otimes \mathcal{H}_{field}$

Simplified notation:

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \hat{J}(x,t)\hat{\phi}(x,t)\, d^3x$$

17 / 22

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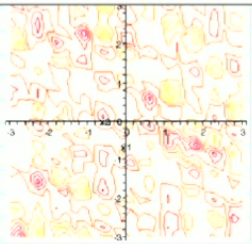

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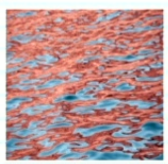
The Hilbert space: $\mathcal{H}_{total} = \mathcal{H}_{atom} \otimes \mathcal{H}_{field}$

Simplified notation:

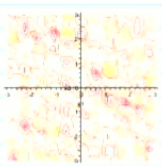
$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \hat{j}(x,t)\hat{\phi}(x,t)\, d^3x$$

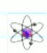
$\Longrightarrow$

The charged systems can act as emitters and as receivers of waves:



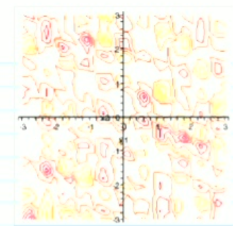
And, quantumly, they can act as emitters and receivers of particles!





Definition: (Unruh, deWitt): A "particle", such as a photon is what a "particle detector", such as an atom, can detect, by getting excited.

With acceleration:



An accelerated atom's charges can excite the field.

This, in turn can excite the atom: the Unruh effect

$\rightsquigarrow$ An accelerated atom may detect particles even if inertial observers only see the vacuum

$\rightsquigarrow$ Related to gravity via the equivalence principle

$\rightsquigarrow$ Related to Hawking radiation.

19 / 23

$$\hat{Q}(t_0)\,|q_n\rangle = q_n|q_n\rangle \qquad \hat{q}(t),\ \hat{p}(t)$$

$$\hat{Q}(t) = \hat{U}^\dagger(t)\hat{Q}(t_0)\hat{U}(t) \qquad a(t),\ a^\dagger(t)$$

$$\hat{H}(t)$$

$$\underbrace{\hat{U}^\dagger(t)\hat{Q}(t_0)\cancel{\hat{U}(t)}}_{\hat{Q}(t)}\ \underbrace{\cancel{\hat{U}^\dagger(t)}|q_n\rangle}_{\|} = q_n\,\underbrace{\hat{U}^\dagger(t)|q_n\rangle}$$

$$|\tilde{q}_n\rangle = q_n\,|\tilde{q}_n\rangle$$

$\hat q, \; \hat p$

$\underline{\hat f(t)} := \cos(\tilde\omega t)\,\hat q(t) + e^{\lambda t}\,\hat p(t)$

$\hat Q(t_0)\,|q_n\rangle = q_n\,|q_n\rangle$

$\hat q(t), \; \hat p(t)$

$\hat Q(t) = \hat U^\dagger(t)\,\hat Q(t_0)\,\hat U(t)$

$a(t), \; a^\dagger(t)$

$\hat H(t)$

$$\underbrace{\hat U^\dagger(t)\,\hat Q(t_0)\,\cancel{\hat U(t)}}_{\hat Q(t)}\;\underbrace{\cancel{\hat U^\dagger(t)}\,|q_n\rangle}_{=} = q_n\,\underbrace{\hat U^\dagger(t)\,|q_n\rangle}$$

$|\tilde q_n\rangle = q_n\,|\tilde q_n\rangle$

$$[\partial_x, e^{wx}] f(x) = w e^{wx} f(x)$$

$$a_{out} |n_{out}\rangle = \cancel{X} |n-1_{out}\rangle$$

$$\vec{k} \; , \; \boxed{w = \sqrt{\vec{k}^2 + m^2}}$$

$$w^2 = \vec{k}^2 + m^2$$

$$\boxed{w^2 - \vec{k}^2 = m^2}$$

$$a a^\dagger - a^\dagger a = 1$$

$$\hbar w$$

$$\partial_x x^n = n x^{n-1}$$

$$\partial_x x - x \partial_x = 1$$

$$\vec{k}$$

$$[\partial_x, x^n] f = n x^{n-1} f$$

$$\partial_x (x f(x)) - x \partial_x f(x) = f(x)$$

$$f(x) + \cancel{x \partial_x f(x)} - \cancel{x \partial_x f(x)} = f(x)$$