Title: Quantum Field Theory for Cosmology - Lecture 20240125

Speakers: Achim Kempf

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QFT for Cosmology, Achim Kempf, Lecture 6

Recall:

- There are two basic mechanisms to increase the amplitudes of oscillators, i.e., also to excite a field's mode oscillators, i.e. to create particles:

  a) A time-varying driving force $J(t)$

  b) A time-varying spring "constant" $\omega(t)$

- We are presently considering case **a)**:

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

- We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^+(t)a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}J(t)\left(a^+(t) + a(t)\right)$

- By using $a(t)$, the problem reduced to solving:

  $*$ $\quad i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}J(t)$ $\qquad$ **(EOM)**

  $*$ $\quad [a(t), a^+(t)] = 1$ for all t $\qquad$ **(CCR)**

We gave a convenient name to $a(t=0)$:

$$a_{in} := a(t=0)$$

(an operator on Hilbert space that we still have to choose.)

Then, as is easy to verify, the solution is:

$$a(t) = a_{in}e^{-i\omega t} + 1\frac{i}{\sqrt{2\omega}}e^{-i\omega t}\int_0^t J(t')e^{i\omega t'}dt'$$

$$= \left(a_{in} + 1\frac{i}{\sqrt{2\omega}}\int_0^t J(t')e^{i\omega t'}dt'\right)e^{-i\omega t}$$

And so, with the definition: $J_0 := 1\frac{i}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}dt'$

$$\begin{cases} a_{in}e^{-i\omega t} & \text{for } t<0 \end{cases}$$

Before and after the force action, we have an undriven harmonic oscillator, solved as always, by $a(t) = a_0\, e^{-i\omega t}$

Here:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

1/27

1

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a.) A time-varying driving force $J(t)$

b.) A time-varying spring "constant" $\omega(t)$

We are presently considering case **a)**:

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

By using $a(t)$, the problem reduced to solving:

$$* \quad i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}} J(t) \qquad \textbf{(EOM)}$$

$$* \quad [a(t), a^+(t)] = 1 \quad \text{for all } t \qquad \textbf{(CCR)}$$

We gave a convenient name to $a(t=0)$:

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And so, with the definition: $J_0 := 1\frac{i}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}dt'$

$$a(t) = \begin{cases} a_{in}e^{-i\omega t} & \text{for } t<0 \\ \text{see above} & \text{for } 0 \le t \le T \\ \underbrace{(a_{in} + J_0)}_{!!}e^{-i\omega t} & \text{for } T<t \end{cases}$$

Define: a_{out}

Before and after the force action, we have an undriven harmonic oscillator, solved as always, by $a(t) = a_0\,e^{-i\omega t}$

Here:

$$a(t) = \begin{cases} a_{in}\,e^{-i\omega t} & \text{for } t<0 \\ a_{out}\,e^{-i\omega t} & \text{for } t>T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a^+_{in}] = 1$ implies $[a_{out}, a^+_{out}] = 1$.

In fact:

1 / 27

1

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And so, with the definition: $J_0 := \frac{1}{\sqrt{2\omega}} \int_0 j(t)$ ...

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ \text{see above} & \text{for } 0 \le t \le T \\ \underbrace{(a_{in} + J_0)}_{!!}\, e^{-i\omega t} & \text{for } T<t \end{cases}$$

Define: a_{out}

$\left(a_{out}\, e \right.$ ... for $t > T$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a_{in}^+] = 1$ implies $[a_{out}, a_{out}^+] = 1$.

In fact:

Proposition: If we can arrange for $[a_{in}, a_{in}^+] = 1$, then $[a(t), a^+(t)] = 1$ follows for all $t \in \mathbb{R}$!

Proof:

Assume $[a_{in}, a_{in}^+] = 1$. Then:

$$[a(t), a^+(t)] = \Big[a_{in} e^{-i\omega t} + \underbrace{1 \tfrac{i}{\sqrt{2\omega}} \int_0 \dots dt'}_{\text{number}},\ a_{in}^+ e^{+i\omega t} - \underbrace{1 \tfrac{i}{\sqrt{2\omega}} \int_0 \dots dt'}_{\text{number}}\Big]$$

$$= \underbrace{[a_{in}, a_{in}^+]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad \checkmark$$

The initial period, $t < 0$:

- The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$\ast \quad \hat{q}(t) = \frac{1}{\sqrt{2\omega}} \left(a_{in}^+ e^{i\omega t} + a_{in} e^{-i\omega t} \right)$$

$$\ast \quad \hat{p}(t) = i\sqrt{\frac{\omega}{2}} \left(a_{in}^+ e^{i\omega t} - a_{in} e^{-i\omega t} \right)$$

Exercise: verify

$$\ast \quad \hat{H}(t) = \omega \left(a^+(t)\, a(t) + \tfrac{1}{2} \right)$$

$$= \omega \left(a_{in}^+ e^{i\omega t} a_{in} e^{-i\omega t} + \tfrac{1}{2} \right)$$

$$= \omega \left(a_{in}^+ a_{in} + \tfrac{1}{2} \right) \quad \text{is constant in time!}$$

3/27

3

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = [a_{in} e^{-i\omega t} + \underbrace{1 \tfrac{1}{\sqrt{\omega}} \int \ldots dt'}_{\text{number}},\ a^+_{in} e^{+i\omega t} - \underbrace{1 \tfrac{1}{\sqrt{\omega}} \int \ldots dt'}_{\text{number}}]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad ✓$$

* $\hat{q}(t) = \frac{1}{\sqrt{2\omega}} \left(a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t} \right)$
* $\hat{p}(t) = i\sqrt{\frac{\omega}{2}} \left(a^+_{in} e^{i\omega t} - a_{in} e^{-i\omega t} \right)$

Exercise: verify

* $\hat{H}(t) = \omega \left(\hat{a}^+(t)\, a(t) + \frac{1}{2} \right)$

$$= \omega \left(a^+_{in} e^{i\omega t} a_{in} e^{-i\omega t} + \frac{1}{2} \right)$$

$$= \omega \left(a^+_{in} a_{in} + \frac{1}{2} \right) \quad \text{is constant in time!}$$

The Hilbert space of states:

* As always, we can write arbitrary Hilbert space vectors as linear combinations of an arbitrary set of basis vectors.
* We could use, for example, the eigenbasis of $\hat{q}(t)$ (or the eigenbasis of $\hat{p}(t)$).

  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigen basis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).

  ⇝ Let us construct and use its eigenbasis:

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

$$\hat{H}_{t<0} = \omega \left(a^+_{in} a_{in} + \frac{1}{2} \right)$$

with:

$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in} |0_{in}\rangle = 0$$

← the Hilbert space vector with zero length

* Then it is eigenvector of $\hat{H}_{t<0}$:

$$\hat{H}_{t<0} |0_{in}\rangle = \omega \left(a^+_{in} a_{in} + \frac{1}{2} \right) |0_{in}\rangle = \frac{1}{2}\omega\, |0_{in}\rangle$$

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$, i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

⇒ We recognize $|0\rangle$: it is the lowest energy eigenvector of $\hat{H}$ (and thus it indeed exists)

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  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigenbasis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).
  $\rightsquigarrow$ Let us construct and use its eigenbasis:

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}|0_{in}\rangle = 0 \quad \leftarrow \text{the Hilbert space vector with zero length}$$

* Then it is eigenvector of $H_{t<0}$:

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar=1$).

$$\hat{H}_{t<0}|0_{in}\rangle = \omega(a_{in}^+ a_{in} + \tfrac{1}{2})|0_{in}\rangle = \tfrac{1}{2}\omega|0_{in}\rangle$$

$\Rightarrow$ We recognize $|0\rangle$: it is the lowest energy eigenvector of $\hat{H}$ (and thus it indeed exists)

* Consider now the state $|1_{in}\rangle := a_{in}^+|0_{in}\rangle$:

$$\hat{H}_{t<0}|1_{in}\rangle = \hat{H}_{t<0}a_{in}^+|0_{in}\rangle = \omega(a_{in}^+ a_{in} + \tfrac{1}{2})a_{in}^+|0_{in}\rangle$$

$$= \left(\omega a_{in}^+(\cancel{a_{in}^+ a_{in}} + 1) + \tfrac{\omega}{2}a_{in}^+\right)|0_{in}\rangle$$

$$= \omega\tfrac{3}{2}a^+|0_{in}\rangle$$

$$= \tfrac{3}{2}\omega|1_{in}\rangle$$

$\Rightarrow$ The state $|1_{in}\rangle$ is eigenstate of $\hat{H}$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1^{st} excited state.

* Is the vector $|1_{in}\rangle$ normalized?

$$\langle 1_{in}|1_{in}\rangle = \langle 0|a_{in}a_{in}^+|0\rangle = \langle 0|a_{in}^+a_{in}+1|0\rangle = \langle 0_{in}|0_{in}\rangle = 1 \checkmark$$

* **Proposition:**

  The set of vectors $\{|n_{in}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}}(a_{in}^+)^n|0_{in}\rangle$$

  is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

* **Proposition:**

  The vectors $|n_{in}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\hat{H}_{t<0}|n_{in}\rangle = E_n|n_{in}\rangle$$

  with

$$E_n = \omega(n+\tfrac{1}{2})$$

  Exercise: verify

* **Proposition:** $\{|n_{in}\rangle\}$ is complete eigenbasis of $\hat{H}$.

7 / 27

7

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And so, with the definition: $J_0 := 1\frac{i}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}dt'$

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ \text{see above} & \text{for } 0\le t\le T \\ \underbrace{(a_{in}+J_0)}_{!!}\, e^{-i\omega t} & \text{for } T<t \end{cases}$$

Define: a_{out}

Proposition: If we can arrange for $[a_{in}, a^+_{in}] = 1$,

then $[a(t), a^+(t)] = 1$ follows for all $t\in\mathbb{R}$!

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = [a_{in}e^{-i\omega t} + \underbrace{1\frac{i}{\sqrt{2\omega}}\int_0 \ldots dt'}_{\text{number}},\ a^+_{in}e^{+i\omega t} - \underbrace{1\frac{i}{\sqrt{2\omega}}\int_0 \ldots dt'}_{\text{number}}]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t}e^{i\omega t}$$

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a^+_{in}] = 1$ implies $[a_{out}, a^+_{out}] = 1$.

In fact:

The initial period, $t<0$:

The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

- $\hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(a^+_{in}e^{i\omega t} + a_{in}e^{-i\omega t}\right)$
- $\hat{p}(t) = i\sqrt{\frac{\omega}{2}}\left(a^+_{in}e^{i\omega t} - a_{in}e^{-i\omega t}\right)$

Exercise: verify

- $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right)$

$$= \omega\left(a^+_{in}e^{i\omega t}\, a_{in}e^{-i\omega t} + \frac{1}{2}\right)$$

3/27

3

$$= \omega \tfrac{3}{2} a^\dagger |0_{in}\rangle$$

$$= \tfrac{3}{2}\omega\, |1_{in}\rangle$$

$\Rightarrow$ The state $|1_{in}\rangle$ is eigenstate of $\hat H$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1st excited state.

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$$\langle 1_{in}|1_{in}\rangle = \langle 0|a_{in}\, a^\dagger_{in}|0\rangle = \langle 0|a^\dagger_{in} a_{in} + 1|0\rangle = \langle 0_{in}|0_{in}\rangle = 1 \checkmark$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

* Proposition:

The vectors $|n_{in}\rangle$ are eigenvectors of $\hat H_{t<0}$:

$$\hat H_{t<0}|n_{in}\rangle = E_n\, |n_{in}\rangle$$

with

$$E_n = \omega\left(n + \tfrac{1}{2}\right)$$

Exercise: verify

* Proposition: $\{|n_{in}\rangle\}$ is complete eigenbasis of $\hat H$.

Summary re choice of basis for $t<0$:

- The Hamiltonian $\hat H(t)$ is constant for $t<0$.
- Thus it has one eigenbasis for all $t<0$, namely $\{|n_{in}\rangle\}$.
- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n\, |n_{in}\rangle$$

- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

- The system always stays in state $|\psi\rangle = |5_{in}\rangle$.

Recall:
- But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat H(t)$ for $t>0$!

The period $t>T$: (after the force ceased to act)

- Once the driving force acts, $\hat H(t)$ starts to change.
- **But:** After the force finished, $t>T$, the Hamiltonian simply reads

$$\hat H(t) = \omega\left(a^\dagger(t)\,a(t) + \tfrac{1}{2}\right) - \frac{a^\dagger(t)+a(t)}{\sqrt{2\omega}}\, \underbrace{\cancel{J(t)}}_{=0} \quad \text{for } t>T$$

and from above, therefore:

$$\hat H(t) = \omega\left(a^\dagger_{out}\cancel{e^{i\omega t}}\, a_{out}\cancel{e^{-i\omega t}} + \tfrac{1}{2}\right) \quad \text{with } a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat H_{t>T} = \omega\left(a^\dagger_{out} a_{out} + \tfrac{1}{2}\right) \Rightarrow \hat H \text{ is then constant again!}$$

- Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

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- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

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Recall: o But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat{H}(t)$ for $t>0$!

$$\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) - \frac{a^+(t)+a(t)}{\sqrt{2\omega}}\,\underbrace{\cancel{J(t)}}_{=0} \quad \text{for } t>T$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^+_{out}\cancel{e^{i\omega t}}\,a_{out}\cancel{e^{-i\omega t}} + \tfrac{1}{2}\right) \quad \text{with } a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat{H}_{t>T} = \omega\left(a^+_{out}a_{out} + \tfrac{1}{2}\right) \Rightarrow \hat{H} \text{ is then constant again!}$$

Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

Compare $t<0$ to $t>T$:

QFT:

A. Motion: $\bar{q}(t)$ (large $\bar{q}$ means large $\bar{\phi}_k$ means large waves)

B. Resonance: best $J(t)$? (consider e.g. antenna)

C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum):

$$|\psi\rangle = |0_{in}\rangle$$

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle\psi|\,\hat{q}(t)\,|\psi\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)|0_{in}\rangle$$

* For $t<0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}}\langle 0_{in}|\,a^+_{in}e^{i\omega t} + a_{in}e^{-i\omega t}\,|0_{in}\rangle$$

$$= 0$$

This was expected since for $t<0$ the system's state $|0_{in}\rangle$ is the ground state of $\hat{H}(t)$.

11

11 / 27

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C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum):

$$|\psi\rangle = |0_{in}\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}(a^\dagger(t) + a(t))|0_{in}\rangle$$

* For $t<0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}}\langle 0_{in}|a^\dagger_{in}e^{i\omega t} + a_{in}e^{-i\omega t}|0_{in}\rangle$$

$$= 0$$

This was expected since for $t<0$ the system's state $|0_{in}\rangle$ is the ground state of $\hat{H}(t)$.

* For $t>T$ we obtain:

$$a_{out} = a_{in} + J_0$$

$$\bar{q}(t) = \langle\psi|\hat{q}(t)|\psi\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}(a^\dagger(t) + a(t))|0_{in}\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}(a^\dagger_{out}e^{i\omega t} + a_{out}e^{-i\omega t})|0_{in}\rangle$$

$$= \langle 0_{in}|\frac{1}{\sqrt{2\omega}}((\cancel{a^\dagger_{in}}+J_0^*)e^{i\omega t} + (\cancel{a_{in}}+J_0)e^{-i\omega t})|0_{in}\rangle$$

$$= \frac{1}{\sqrt{2\omega}}(J_0^* e^{i\omega t} + J_0 e^{-i\omega t}) \qquad (*)$$

Exercise: verify

$$= \int_0^T \frac{\sin((t-t')\omega)}{\omega} J(t')\,dt'$$

(Remark: same as classical $q(t)$ due to Ehrenfest theorem)

$\Rightarrow$ $\bar{q}$ oscillates with frequency ω, as expected.

B. Resonance:

* The amplitude of the excited motion of the oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}}\int_0^T J(t')e^{i\omega t'}\,dt'$$

Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

13/27

13

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$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a}^+_{in} + J_0^*) e^{i\omega t} + (\cancel{a}_{in} + J_0) e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t} \right) \qquad (*)$$

Exercise: verify → $$= \int_0^T \frac{\sin((t-t')\omega)}{\omega} J(t')\,dt'$$ (Remark: same as classical $q(t)$ due to Ehrenfest theorem)

$\Rightarrow$ $\bar{q}$ oscillates with frequency ω, as expected.

... a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'}\,dt'$$

Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

C. Energy expectation

* For $t<0$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^+_{in} \cancel{a}_{in} + \tfrac{1}{2}) | 0_{in} \rangle \qquad (\text{for } t<0)$$

$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^+_{out} a_{out} + \tfrac{1}{2}) | 0_{in} \rangle \quad (\text{for } t>T)$$

$$= \omega \langle 0_{in} | (\cancel{a}^+_{in} + J_0^*)(\cancel{a}_{in} + J_0) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | J_0^* J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2 \right) \qquad \text{which is elevated!}$$

Remark: We notice that the oscillator's energy increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the closer a mode's ω_k is to the frequency of the current, the more this mode gets excited.

15 / 27

15

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$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

$$\bar{H}(t) = \langle \chi | \hat{H}(t) | \chi \rangle \quad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^{+}_{out} a_{out} + \tfrac{1}{2}) | 0_{in} \rangle \quad (\text{for } t>T)$$

increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the closer a mode's ω_k is to the frequency of the current, the more this mode gets excited.

<u>Implication</u>: $|0_{out}\rangle \neq |0_{in}\rangle = |\chi\rangle$

- Ground state $|0_{out}\rangle$ of

$$H_{t>T} = \omega(a^{+}(t)a(t) + \tfrac{1}{2}) = \omega(a^{+}_{out} a_{out} + \tfrac{1}{2})$$

has eigenvalue $\omega/2$, i.e.:

$$a_{out}|0_{out}\rangle = 0.$$

- Therefore: $a_{out}|\chi\rangle = a_{out}|0_{in}\rangle$

$$= (a_{in} + J_0)|0_{in}\rangle$$

$$= J_0|\chi\rangle \neq 0$$

$\Rightarrow$ At late times: $|\chi\rangle \neq |0_{out}\rangle$

Q: So what kind of excited state is $|\chi\rangle$ at late times?

A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

$$a_{out}|\chi\rangle = J_0|\chi\rangle$$

$|\chi\rangle$ is what is called a Coherent State.

<u>Recall:</u>

Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

$$\Delta q_{|\psi\rangle} \Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

$\rightsquigarrow$ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

$$\hat{q}|\psi\rangle = \langle\hat{q}\rangle|\psi\rangle \text{ and } \hat{p}|\psi\rangle = \langle\hat{p}\rangle|\psi\rangle$$

17 / 27

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$\Rightarrow$ At late times: $|\chi\rangle \neq |0_{out}\rangle$

Q: So what kind of excited state is $|\chi\rangle$ at late times?

Exercise: Show that if $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha \in \mathbb{C}$

and $\hat{q} = \frac{1}{\sqrt{2\omega}}(a^+ + a)$, $\hat{p} = i\sqrt{\frac{\omega}{2}}(a^+ - a)$ $\quad (*)$

Then, $\langle\alpha|\hat{q}|\alpha\rangle = \frac{1}{\sqrt{2\omega}}(\alpha^* + \alpha)$

$\langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\frac{\omega}{2}}(\alpha^* - \alpha)$

and: $\Delta q(t)\,\Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in $(*)$ is chosen to be not the frequency of the harmonic oscillator of the Hamiltonian, then $|\alpha\rangle$

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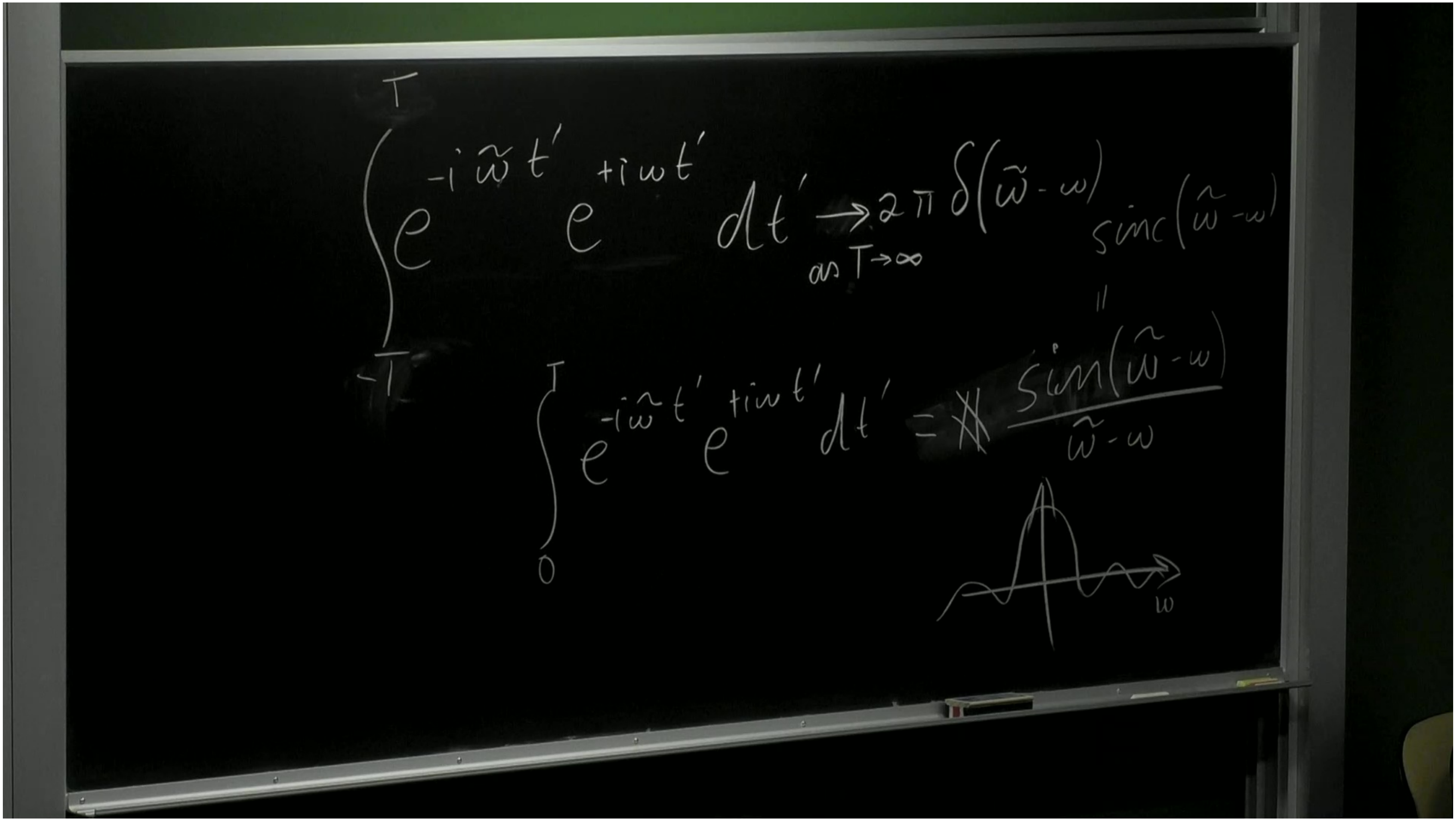
$\rightsquigarrow$ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

$$\hat{q}|\psi\rangle = \langle\hat{q}\rangle|\psi\rangle \text{ and } \hat{p}|\psi\rangle = \langle\hat{p}\rangle|\psi\rangle$$

Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges driving the mode oscillators of the electromagnetic QFT:

- The charges and currents drive the EM oscillators into a coherent state. $\leftarrow$ (In Heisenberg picture: State stays constant but its meaning relative to the then time-dependent operators changes)
- $\Rightarrow$ The $\hat{q}_k$, $\hat{p}_k$ of the EM field (essentially the $\hat{E}_i$ and $\hat{B}_i$ fields) will be as sharp as possible, i.e., the EM QFT's state is close to being eigenstate to $\hat{E}_i$ and $\hat{B}_i$.


$\int_{-T}^{T} e^{-i\tilde{\omega}t'} e^{+i\omega t'} dt' \xrightarrow[\text{as } T\to\infty]{} 2\pi\,\delta(\tilde{\omega}-\omega)$
$\mathrm{sinc}(\hat{\omega}-\omega)$
$= \frac{\sin(\hat{\omega}-\omega)}{\hat{\omega}-\omega}$
$\int_{0}^{T} e^{-i\hat{\omega}t'} e^{+i\omega t'} dt' =$
ω

$$\hat{H}_{tot} = \underline{\hat{H}_A \otimes 1} + \underline{1 \otimes \hat{H}_B}$$

$$i\partial_t \psi(t,x) = \left(-\frac{\Delta}{2m} + \underline{\hat{V}(\vec{x})} + \hat{B}\vec{p}\right)\psi(x,t)$$

$$e^{it\hat{H}_{tot}}(|\psi\rangle \otimes |\phi\rangle) = e^{itH_A \otimes 1}\, \underbrace{e^{it1\otimes H_B}(|\psi\rangle \otimes |\phi\rangle)}_{=\, |\psi\rangle \otimes (e^{itH_B}|\phi\rangle)}$$

$$= \underline{(e^{itH_A}|\psi\rangle)} \otimes \underline{(e^{itH_B}|\phi\rangle)}$$

$$\hat{H}_{tot} = H_A \otimes 1 + 1 \otimes H_B + \hat{A} \otimes \overset{b1}{\overset{\shortparallel}{\hat{B}}}$$

$$= (H_A + \hat{A}) \otimes 1 + 1 \otimes H_B$$

$$\left(e^{itH_A}|\psi\rangle\right) \otimes \left(e^{itH_B}|\phi\rangle\right)$$

$$e^{it(\hat{A} \otimes \hat{B})}|\psi\rangle \otimes |\phi\rangle = |\psi\rangle \otimes |\phi\rangle + it(\hat{A}|\psi\rangle) \otimes (\hat{B}|\phi\rangle) + \ldots$$

Now what if e.g. $\hat{B}|\phi\rangle = b1|\phi\rangle$