Title: Quantum Field Theory for Cosmology - Lecture 20240123

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

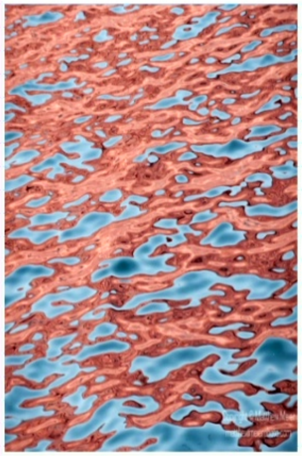
Date: January 23, 2024 - 4:00 PM

URL: https://pirsa.org/24010012

QFT for Cosmology, Achim Kempf, Lecture 5

Particles in QFT

Back in the Heisenberg picture,
to solve the QFT is to solve:



- The hermiticity conditions:

$$\hat{\phi}^+(x,t) = \hat{\phi}(x,t)\ , \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$$

- The canonical commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta(x-x')$$

- The equations of motion:

$$\dot{\hat{\pi}}(x,t) - \Delta\hat{\phi}(x,t) + m^2\hat{\phi}(x,t) = 0$$

$$\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$$

To simplify:

- Infrared regularization:

Box size $L\times L\times L$ with periodic boundary conditions.

↑ Project: uses Dirichlet boundary conditions.

- Then Fourier series expansion:

$$\hat{\phi}(x,t) = L^{-3/2}\sum_k \hat{\phi}_k(t)\,e^{ikx}$$

↑ $k = \frac{2\pi}{L}(n_1,n_2,n_3),\ n_i \in \mathbb{Z}$

Obtain:

$$\ddot{\hat{\phi}}_k(t) = -(k^2+m^2)\hat{\phi}_k(t) \text{ and } [\hat{\phi}_k, \dot{\hat{\phi}}_{k'}^+] = i\,\delta_{k,k'}$$

$$\hat{H} = \sum_k \hat{H}_k \quad \text{with } \hat{H}_k = \frac{1}{2}\hat{\pi}_k^+\hat{\pi}_k + \frac{1}{2}\hat{\phi}_k^+(k^2+m^2)\hat{\phi}_k$$

i.e.:

$$\hat{H} = \sum_k\left(\frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)(k^2+m^2)\hat{\phi}_k(t)\right)$$

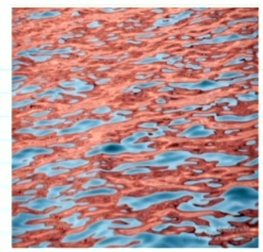
- Crucial observations:

* For each wave vector $k=(k_1,k_2,k_3)$ there is an independent harmonic oscillator with frequency $\omega_k = \sqrt{k^2+m^2}$ and spectrum $\mathrm{spec}(H_k) = \hbar\omega_k\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots\}$.

⇒ The excitation levels of H_k differ by the energy

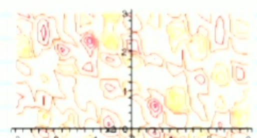
$$E = \omega_k = \sqrt{k^2+m^2} \qquad (\hbar = 1)$$

Water: $\phi(x,t)$



Probe amplitudes, e.g., with a cork:

Quantum field: $\hat{\phi}(x,t)$



Probe amplitudes, e.g., with atoms.

Use as a

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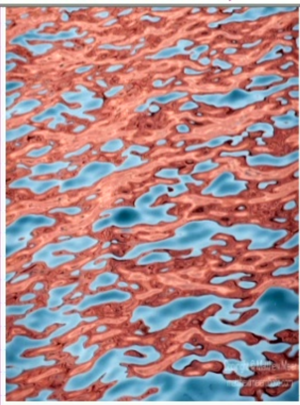

□ The hermiticity conditions:

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i.e.: $\hat{H} = \sum_k \left(\tfrac{1}{2}\hat{\pi}_k^{+}(t)\hat{\pi}_k(t) + \tfrac{1}{2}\hat{\phi}_k^{+}(t)(k^2+m^2)\hat{\phi}_k(t)\right)$

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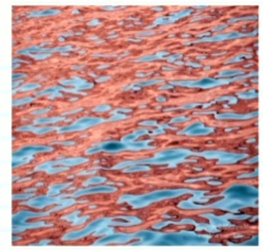
* This is also the energy of a particle of momentum k!

⇝ Hyothesis: $\boxed{\text{Mode excitation = particle creation}}$

Does this interpretation work?

Water: $\phi(x,t)$



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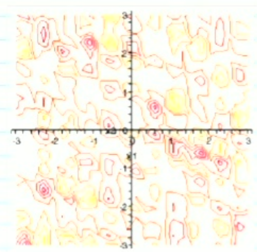

Probe amplitudes, e.g., with atoms.

Use as a detector for the field's particles (e.g. photons for EM field)

One finds:

□ Interpretation works but is acceleration and curvature dependent.

□ Interpretation simple only in Minkowski space for inertial detector.

1/22

1

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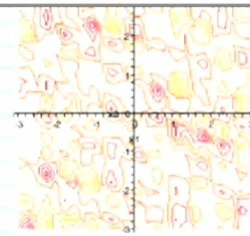

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One finds:

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Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state $|\psi\rangle$ in which we have 3 particles of momentum k_a and 7 particles of momentum k_b?

$$|\psi\rangle = |n_{k_a}=3, n_{k_b}=7, \text{all other } n_k=0\rangle$$
$$= |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left(\bigotimes_{\text{all other } k_c} |n_{k_c}=0\rangle\right)$$

Energy: $$\hat{H}_k|\psi\rangle = \begin{pmatrix} \hbar\omega_k\left(\frac{1}{2}+3\right) & \text{if } k=k_a \\ \hbar\omega_k\left(\frac{1}{2}+7\right) & \text{if } k=k_b \\ \hbar\omega_k\frac{1}{2} & \text{if } k\neq k_a,k_b \end{pmatrix}|\psi\rangle$$

$$\Rightarrow \quad \hat{H}|\psi\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k}\frac{1}{2}\omega_k\right)|\psi\rangle$$

And one can have linear combinations:

Which is, e.g., the state $|\chi\rangle$ in which we have 3 particles of momentum k_a or 7 particles of momentum k_b, with probability amplitudes α, $\beta=\sqrt{1-|\alpha|^2}$?

$$|\chi\rangle = \alpha|n_{k_a}=3, \text{other } n_k=0\rangle + \beta|n_{k_b}=7, \text{other } n_k=0\rangle$$

Notice: This is not a state of fixed particle number!

(**Remark:** Some particle species have a number conservation law, e.g., leptons, i.e. $e^-, \mu^-, \tau^-, \nu_{e^-}, \nu_{\mu^-}, \nu_{\tau^-}$, (where the antiparticles count negatively). "Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.)

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$$\Rightarrow \quad \hat{H}|\psi\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2}\omega_k \right) |\psi\rangle$$

$$|x\rangle = \alpha |n_{k_a}=3, \text{other } n_k=0\rangle + \beta |n_{k_b}=7, \text{other } n_k=0\rangle$$

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Mechanisms for mode excitation / particle creation?

I.e.: What are mechanisms for exciting harmonic oscillators?

- **2 types of mechanism:** (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

we'll begin with this effect → **a.)** A "driving force" shakes the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{J}(t)$$

e.g.: 

b.) A time dependence of ω affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t)\, \hat{q}(t)$$

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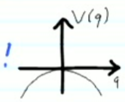
And, $\omega^2(t)$ could even go negative! 

e.g.: 

All occur in QFT:

A) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with J terms, e.g.:

$$H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda\, \hat{\phi}(x,t)\, \hat{\psi}(x,t)\, d^3x$$

- **Wave interpretation:** Nontrivial interaction of waves of different types of fields
- **Particle interpretation:** The collision of particles happens when their mode oscillators drive one other.
  → Collissions can create and annihilate particles.
- **Strongest effects?**

When oscillator "resonates" with driving force.

E.g.: LHC takes high energy particles to make high energy particles

5 / 22

5

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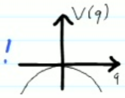

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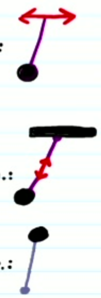

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B) The presence of gravity can effectively influence the $\omega_k(t)$.

- Wave interpretation: * E.g., cosmic expansion stretches the wavelength
  ⇒ expect $\omega = \omega(t)$ decreases. True, and also:
  * if wavelength > horizon then $\omega^2(t) < 0$!
  ⇒ runaway harmonic mode "oscillators"

  (then: field amplification but no particle interpretation)
- Particle interpretation:
  Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.
- Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case A: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

- We model the electromagnetic field as a Klein Gordon field.
  (The fact that EM fields have polarization and have $m=0$ is not important here)
- Consider an arbitrary mode of the electromagnetic field:
  $\hat{\phi}_k(t)$
- should really be quantized too
  We model the electric current as a given classical field $j(x,t)$ whose modes are $j_k(t)$.
  should really be vector-valued

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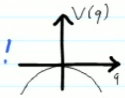
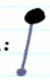
And, $\omega^2(t)$ could even go negative! 

e.g.: 

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7/22

7

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- In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\omega_k^2\hat{\phi}_k^+(t)\hat{\phi}_k(t) + \hat{\phi}_k(t)j_k(t)$$

$\Rightarrow$ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

$\Rightarrow$ **Need to study the quantized driven harmonic oscillator!**

$$\boxed{\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)}$$

for $\hat{H}_k(t)$; for $\hat{\pi}_k(t)$; stands for a field mode $\hat{\phi}_k(t)$

stands for a mode $j_k(t)$ of another classical (or better quantum) field.

Recall: $\hat{H}_S(t) = \hat{H}(t)$ only if $\frac{d}{dt}\hat{H}(t) = 0$

The dynamics is $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_s(t)|\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_s(t) = \hat{U}(t)\hat{H}(t)\hat{U}^+(t)$

Exercise: check

- We will use, equivalently, the Heisenberg picture:

$$\bar{f}(t) = \langle\psi_0|\left(\hat{U}^+(t)\,\hat{f}_0\,\hat{U}(t)\right)|\psi_0\rangle$$

I Preparation:

- Recall that for all observables $\hat{f}$:

$$\bar{f}(t) = \langle\psi_0|\hat{U}^+(t)\,\hat{f}_0\,\hat{U}(t)|\psi_0\rangle$$

state at initial time

operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \quad i\frac{d}{dt}\hat{U}(t) = \hat{U}(t)\hat{H}(t)$$

the original Hamiltonian

"Heisenberg Hamiltonian"

- Schrödinger picture? We write, equivalently:

Exercise: check!

$$\bar{f}(t) = \left(\langle\psi_0|\hat{U}^+(t)\right)\hat{f}_0\left(\overbrace{\hat{U}(t)|\psi_0\rangle}^{=|\psi(t)\rangle}\right)$$

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II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).

- The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc, are time-dependent operators in Hilbert space.

- Important implication:

9 / 22

9

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can excite the mode oscillators, creating photons.

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$$\boxed{\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)}$$

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11

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- Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t)\,|f_n(t)\rangle = f_n(t)\,|f_n(t)\rangle$$

$$\hat{H}(t)\,|E_n(t)\rangle = E_n(t)\,|E_n(t)\rangle$$

11 / 22

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$$\hat{f}(t) | f_n(t) \rangle = f_n(t) \, | f_n(t) \rangle$$

$$\hat{H}(t) | E_n(t) \rangle = E_n(t) \, | E_n(t) \rangle$$

Example: * Assume the driven harmonic oscillator starts out at time t_1 in n'th energy state, say $|\psi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1) | E_n(t_1) \rangle = E_n(t_1) | E_n(t_1) \rangle$$

* State vector of the system stays $|\psi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$ the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t) | E_n(t) \rangle = E_n(t) | E_n(t) \rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1) \;, \; |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

$\Rightarrow$ At time t_2 system is still in state $|\psi\rangle$ and still

$$|\psi\rangle = |E_n(t_1)\rangle$$

but $|\psi\rangle$ is generally no longer n'th (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e., in vacuum): $|\psi\rangle = |E_0(t_1)\rangle$

* Then if $|\psi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

$\Rightarrow$ At t_2 the system's state $|\psi\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2.

13 / 22

13

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III Strategy for solving quantized driven harmonic oscillator

Problem: * CCR: $[\hat q(t), \hat p(t)] = i1$

* Hermiticity: $\hat q^\dagger(t) = \hat q(t)$, $\hat p^\dagger(t) = \hat p(t)$

* Hamiltonian: $\hat H(t) = \frac{1}{2}\hat p(t)^2 + \frac{\omega^2}{2}\hat q(t)^2 - J(t)\hat q(t)$

* Heisenberg eqns $i\dot{\hat f}(t) = [\hat f(t), \hat H(t)]$ yield:

$$\dot{\hat q}(t) = \hat p(t)$$
$$\dot{\hat p}(t) = -\omega^2 \hat q(t) + J(t)$$

This is a good strategy with and without a driving force

Strategy: * Combine

is operator even though no "hat".

$$a(t) := \alpha\,\hat q(t) + i\beta\,\hat p(t)$$

Mukhanov calls it $a^-(t)$

(analogous to "real" & "imaginary" parts)

* Choose α, β so that $\hat H(t)$ and eqn of motion simplify.

IV Determine α and β:

Notice first that once we have $a(t)$ we immediately obtain $\hat q(t), \hat p(t)$: Use of $a^\dagger(t) = \alpha\,\hat q(t) - i\beta\,\hat p(t)$ yields:

$$\hat q(t) = \frac{1}{2\alpha}\left(a^\dagger(t) + a(t)\right)$$

$$\hat p(t) = \frac{i}{2\beta}\left(a^\dagger(t) - a(t)\right)$$

Use this to express $[\hat q, \hat p] = i$ in terms of new variable $a(t)$:

$$\Rightarrow \qquad [a(t), a^\dagger(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

$$\boxed{[a(t), a^\dagger(t)] = 1}$$

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* Hamiltonian: $\hat H(t) = \frac{1}{2}\hat p(t)^2 + \frac{\omega^2}{2}\hat q(t)^2 - J(t)\hat q(t)$

* Heisenberg eqns $i\dot{\hat f}(t) = [\hat f(t), \hat H(t)]$ yield:

This is a good strategy with and without a driving force

$$\dot{\hat q}(t) = \hat p(t)$$
$$\dot{\hat p}(t) = -\omega^2 \hat q(t) + J(t)$$

Strategy: * Combine $a(t) := \alpha\,\hat q(t) + i\beta\,\hat p(t)$ (analogous to "real" & "imaginary" parts)

is operator even though no "hat".

Mukhanov calls it $a^-(t)$

* Choose α, β so that $\hat H(t)$ and eqn of motion simplify.

$$\hat p(t) = \frac{i}{2\beta}\left(a^+(t) - a(t)\right)$$

Use this to express $[\hat q, \hat p] = i$ in terms of new variable $a(t)$:

$$\Rightarrow \quad [a(t), a^+(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

$$\boxed{[a(t), a^+(t)] = 1}$$

Now express $\hat H(t)$ in terms of new variable $a(t)$:

$$\hat H(t) = -\frac{1}{2}\alpha^2\left(a^+(t) - a(t)\right)^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2}\left(a^+(t) + a(t)\right)^2$$
$$- J(t)\frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}\alpha^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2} = 0$$

Thus, we choose: $\alpha = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

Thus, by definition:

$$\boxed{a(t) := \sqrt{\frac{\omega}{2}}\,\hat q(t) + i\frac{1}{\sqrt{2\omega}}\,\hat p(t)}$$

The Hamiltonian simplifies to become:

$$\hat H(t) = \omega\left(a^+(t)a(t) + \frac{1}{2}\right) - J(t)\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)$$

V Solve for $a(t)$:

The Heisenberg equation $i\dot{\hat f}(t) = [\hat f(t), \hat H(t)]$ reads for $a(t)$:

$$i\,\dot a(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}} J(t)$$

Let us give $a(t{=}0)$ a name: $a_{in} = a(0)$. Then:

Exercise: verify.

$$\boxed{a(t) = a_{in}\, e^{-i\omega t} + \frac{i}{\sqrt{2\omega}}\int_0^t J(t')\, e^{i\omega(t'-t)}\, dt'}$$

17 / 22

17

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- Thus, by definition:

$$\boxed{a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)}$$

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VI Case of force of finite duration

- Assume $J(t) = 0$ for all $t \notin [0,T]$

- Define

$$J_0 := \frac{i}{\sqrt{2\omega}}\int_0^T J(t')\, e^{i\omega t'}\, dt'$$

- Then:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0,T] \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } t > T \end{cases}$$

Next:

Implications in terms of particle (e.g. photon) production?

19 / 23

19