Title: Quantum Field Theory for Cosmology - Lecture 20240116

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

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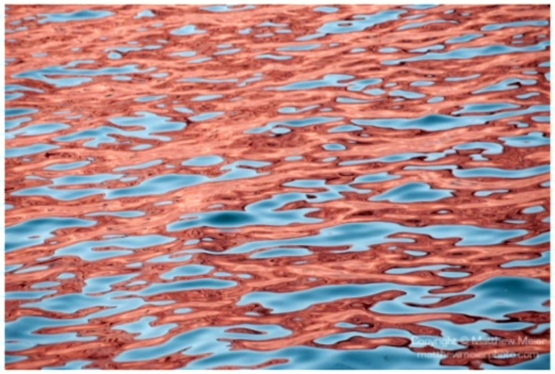
QFT for Cosmology, Achim Kempf, Lecture 3



- Quantization conditions:

analogous to:

$$\left[\hat{\phi}(x,t), \hat{\pi}(x',t)\right] = i\hbar\,\delta^3(x-x') \qquad [\hat{q}_a(t), \hat{p}_{a'}(t)] = i\hbar\,\delta_{aa'}$$

$$\left[\hat{\phi}(x,t), \hat{\phi}(x',t)\right] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$\left[\hat{\pi}(x,t), \hat{\pi}(x',t)\right] = 0 \qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

- We keep the equations of motion:

$$\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \quad \textbf{(E1)} \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

$$\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \quad \textbf{(E2)} \qquad \dot{\hat{p}}_a(t) = -K_a \hat{q}_a(t)$$

- Note: $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$

- Proposition:

$E1, E_2$ follow from the Heisenberg eqns

analogous to:

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

4. Harmonic oscillators in quantum fields

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Proposition:

$E1, E_2$ follow from the Heisenberg eqns

$$i\hbar\, \dot{\hat\phi}(x,t) = [\hat\phi(x,t), \hat H]$$

$$i\hbar\, \dot{\hat\pi}(x,t) = [\hat\pi(x,t), \hat H]$$

analogous to:

$$i\hbar\, \dot{\hat q}_a(t) = [\hat q_a(t), \hat H]$$

$$i\hbar\, \dot{\hat p}_a(t) = [\hat p_a(t), \hat H]$$

with this QFT Hamiltonian:

$$\hat H = \int_{\mathbb{R}^3} \frac{1}{2}\hat\pi^2(x',t) + \frac{1}{2}\hat\phi(x',t)(m^2-\Delta)\hat\phi(x',t)\, d^3x'$$

$$\hat H = \sum_a \frac{\hat p_a^2}{2} + \frac{\omega_a^2}{2}\hat q_a^2$$

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

4. Harmonic oscillators in quantum fields

From the above, we need to solve 2 equations:

a.) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\hat\phi(x,t) = 0$ ($x = (x_1, x_2, x_3)$)

b.) The commutation rels: $[\hat\phi(x,t), \dot{\hat\phi}(x',t)] = i\hbar\,\delta(x-x')$

Q: How to solve these eqns?

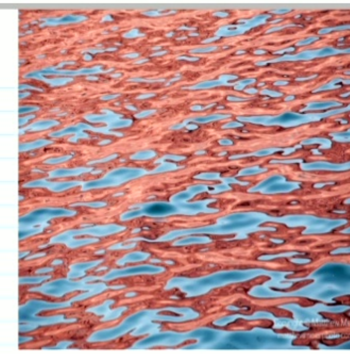
A: Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

1st Difficulty: (in reducing the QFT problem to harmonic oscillators)

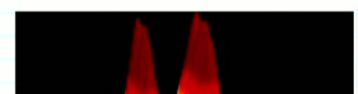
In the K.G. equation,

$$\dot{\hat\pi}(x,t) = -(-\Delta + m^2)\hat\phi(x,t) \quad \overset{\text{Analogy}}{\longleftrightarrow} \quad \dot{\hat p}_a(t) = -K_a\, \hat q_a(t)$$

The local field oscillators are coupled.
⇒ Excitations spread.



The oscillators that are local in



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with this QFT Hamiltonian:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)\left(m^2-\Delta\right)\hat{\phi}(x',t)\, d^3x'$$

$$\hat{H} = \sum_a \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

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a.) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\hat{\phi}(x,t) = 0$, with $x = (x_1, x_2, x_3)$

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  A: Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

1st Difficulty: (in reducing the QFT problem to harmonic oscillators)

- In the K.G. equation,

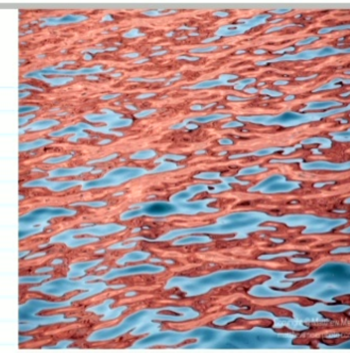
$$\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t) \quad \xleftrightarrow{\text{Analogy}} \quad \dot{\hat{p}}_a(t) = -K_a\,\hat{q}_a(t)$$

we notice that $(-\Delta + m^2)$, unlike K_a, is not a number!

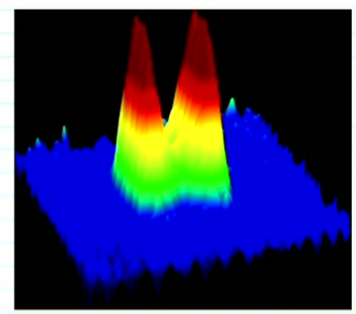
- **Q:** Can we "transform" $(-\Delta + m^2)$ into a number?

  A: Yes: Fourier transform turns derivatives into numbers!

The local field oscillators are coupled.
$\Rightarrow$ Excitations spread.



The oscillators that are local in momentum space are uncoupled.
$\Rightarrow$ Excitations don't spread in momentum space.



Fourier transform of the spatial variables x:

- Proposition: (Exercise: show this)

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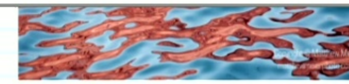

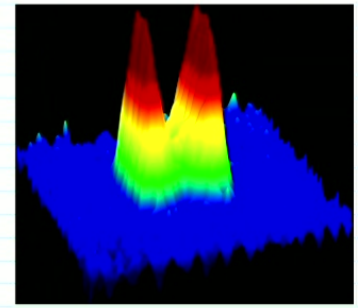

The oscillators that are local in momentum space are uncoupled.
⇒ Excitations don't spread in momentum space.

Fourier transform of the spatial variables x:

- Definition:

$xk = \sum_{i=1}^{3} x_i k_i$; $k = (k_1, k_2, k_3)$

$$\hat{\phi}(k,t) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ixk}\,\hat{\phi}(x,t)\,d^3x$$

Traditional notation: $\hat{\phi}_k(t) := \hat{\phi}(k,t)$

Traditional terminology: $\hat{\phi}_k(t)$ is called the field's k-mode.

- Inverse Fourier transform:

$$\hat{\phi}(x,t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}_k(t)\,d^3k$$

- Proposition: (Exercise: show this)

Analogous to:

a) $k^2 = \sum_{i=1}^{3} k_i^2$

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)\left(k^2+m^2\right)\hat{\phi}_k(t)\,d^3k \qquad \hat{H} = \sum_a \frac{1}{2}\hat{p}_a\hat{p}_a + \frac{1}{2}\omega_a\hat{q}_a\hat{q}_a$$

b)
$$\left[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)\right] = i\hbar\,\delta^3(k+k') \qquad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$$

$$\left[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)\right] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$\left[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)\right] = 0 \qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

c)
$$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t) \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

$$\dot{\hat{\pi}}_k(t) = -\left(k^2+m^2\right)\hat{\phi}_k(t) \qquad \dot{\hat{p}}_a(t) = -\omega_a^2\,\hat{q}_a(t)$$

⇒ For each mode $\vec{k}$ we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2+m^2}$.

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Traditional notation: $\hat{\phi}_k(t) := \hat{\phi}(k,t)$

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- Inverse Fourier transform:

$$\hat{\phi}(x,t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ixk}\, \hat{\phi}_k(t)\, d^3k$$

- Exercise:

Show that a) + b) + Heisenberg eqn $\dot{\hat{f}}(t) = \frac{1}{i\hbar}[\hat{f}(t), \hat{H}]$ yields c)

($\hat{f}$ is arbitrary. E.g. $\hat{f} = \hat{\phi}_x$ or $\hat{f} = \hat{\pi}_x$)

2nd Difficulty: (in reducing the QFT problem to harmonic oscillators)

- We notice that the commutation relations

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta^3(k+k') \quad \text{and} \quad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\, \delta_{a,a'}$$

do not match, because the Kronecker δ is only either 0 or 1, unlike the Dirac δ!

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0 \qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

c) $$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t) \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

$$\dot{\hat{\pi}}_k(t) = -(k^2+m^2)\,\hat{\phi}_k(t) \qquad \dot{\hat{p}}_a(t) = -\omega_a^2\, \hat{q}_a(t)$$

$\Rightarrow$ For each mode $\vec{k}$ we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2+m^2}$.

- Idea: If we Fourier series instead, should have discrete values of k, thus Kronecker δ for CCR!

- Strategy:

1. Put system into a large box $[-L/2, L/2]^{\times 3}$
2. Assume (for example) periodic boundary conditions.
   (If box large enough it should not matter here what happens at the boundary of the box)
3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called "Infrared regularization".

[because "long" wavelengths are removed.]

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2nd Difficulty: (in reducing the QFT problem to harmonic oscillators)

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3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called "Infrared regularization".
↑ because "long" wavelengths are removed.

- Infrared regularization:

  * $(k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ with $n_1, n_2, n_3 \in \mathbb{Z}$
  * $V = L^3$ (Volume of box)
  * Fourier series expansion coefficients:

$$\hat{\phi}_k(t) = V^{-1/2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \hat{\phi}(x,t)\, e^{-ixk}\, d^3x$$

  * The inverse is the Fourier series:

$$\hat{\phi}(x,t) = V^{-1/2} \sum_k \hat{\phi}_k(t)\, e^{ixk}$$

k ← discrete set of vectors!

- The QFT problem in the box:

analogous to

a) $\hat{H} = \sum_k \frac{1}{2} \hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2}\omega_k^2 \hat{\phi}_k^+ \hat{\phi}_k$ $\qquad \hat{H} = \sum_a \frac{1}{2}\hat{p}_a \hat{p}_a + \frac{1}{2}\omega_a^2 \hat{q}_a \hat{q}_a$

↑ $\omega_k^2 = k^2 + m^2$

↓ Kronecker δ

b) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'}$ $\qquad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$ $\qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$

$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0$ $\qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$

c) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$ $\qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -(k^2 + m^2)\,\hat{\phi}_k(t)$ $\qquad \dot{\hat{p}}_a(t) = -\omega_a^2 \hat{q}_a(t)$

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* Fourier series expansion coefficients:

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* The inverse is the Fourier series:

$$\hat{\phi}(x,t) = V^{-1/2} \sum_{k} \hat{\phi}_k(t)\, e^{ixk}$$

$\leftarrow$ discrete set of vectors!

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'} \qquad [\hat{q}_n, \hat{p}_{n'}] = i\hbar\, \delta_{n,n'}$$

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0 \qquad [\hat{q}_n(t), \hat{q}_{n'}(t)] = 0$$

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3rd Difficulty: (in reducing the QFT problem to harmonic oscillators)

Hermiticity:

We notice that $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^+(x,t) = \hat{\pi}(x,t)$ implies

$$\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t), \quad \hat{\pi}_k^+(t) = \hat{\pi}_{-k}(t) \qquad \text{(H)}$$

(Indeed:

$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}^+(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t)$$
)

But eqns (H) do not match:

$$\hat{q}_n^+(t) = \hat{q}_n(t) \qquad \hat{p}_n^+(t) = \hat{p}_n(t)$$

Namely: Our $\hat{\phi}_k, \hat{\pi}_k$ are not hermitean!

Correspondingly:

The analogy between

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'} \quad \text{and} \qquad [\hat{q}_n, \hat{p}_{n'}] = i\hbar\, \delta_{n,n'}$$

suffers from $\delta_{k,-k'}$ instead of $\delta_{k,k'}$. (we do have $[\hat{\phi}_k(t), \hat{\pi}_{k'}^+(t)] = i\hbar\, \delta_{k,k'}$)

Mukhanov:

Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators but with complex, i.e., non hermitean amplitudes.

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$$\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t)\,, \quad \hat{\pi}_k^+(t) = \hat{\pi}_{-k}(t) \qquad \textbf{(H)}$$

(Indeed:
$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\hat{\phi}^+(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t)$$
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But eqns **(H)** do not match:

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Namely: Our $\hat{\phi}_k, \hat{\pi}_k$ are not hermitean!

$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta_{k,-k'}$ and $[q_k, p_{k'}] = i\hbar\,\delta_{k,k'}$

suffers from $\delta_{k,-k'}$ instead of $\delta_{k,k'}$. (we do have $[\hat{\phi}_k(t), \hat{\pi}_{k'}^+(t)] = i\hbar\,\delta_{k,k'}$)

- **Mukhanov:**

Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators but with complex, i.e., non hermitean amplitudes.

Proper treatment:

- Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

Eqns of motion: $\dot{\hat{p}}_k = \dot{\hat{q}}_k\,, \quad \dot{\hat{p}}_k = -\omega_k^2 q_k$

Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,k'}$

Hermiticity: $\hat{q}_k^+ = \hat{q}_k \quad \hat{p}_k^+ = \hat{p}_k$

- Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{2\omega_k}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad \textbf{(A)}$$

Remark: In practice, it'll be more convenient to work with a_k, a_k^+:

With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\hat{p}_k$ the ansatz reads: $\hat{\phi}_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a_{-k}^+)$

Exercise!

- Now, show that ansatz **(A)** succeeds, i.e., that indeed:

Hamiltonian $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{1}{2}\omega_k^2\hat{q}_k^2 \qquad \textbf{(H)}$

Eqns of motion: $\dot{\hat{\pi}}_k = \dot{\hat{\phi}}_k\,, \quad \dot{\hat{\pi}}_k = -\omega_k^2\hat{\phi}_k$

Canon. com. rels: $[\hat{\phi}_k, \hat{\pi}_{k'}] = i\,\delta_{k,-k'}$

Hermiticity cond.: $\hat{\phi}_k^+ = \hat{\phi}_{-k}\,, \quad \hat{\pi}_k^+ = \hat{\pi}_{-k}$

- Finally, via inverse Fourier series, show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{V}}\sum_k \left\{\cos(xk)\,\hat{q}_k - \frac{1}{\omega_k}\sin(xk)\,\hat{p}_k\right\} \qquad \textbf{(B)}$$

Remark: Ansatz **(A)** was not unique!
The x's and p's could be more mixed! **(H)** could be different!

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Hermiticity: $\hat{q}_k^+ = \hat{q}_k \quad \hat{p}_k^+ = \hat{p}_k$

□ Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{2\omega_k}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad (A)$$

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Remark: Ansatz (A) was not unique!
The x's and p's could be more mixed! (H) could be different!

Significance of non-uniqueness?

"No particles state"

* Ground state of q_k, p_k oscillators ⇝ Vacuum
* This need not be lowest energy state of the QFT Hamiltonian
  → Problem of vacuum identification on curved space.
  → See later.

For now: We solved, using (A), the QFT eqns of the K.G. field.

Namely, we have now solved:

$$\text{Eqns of motion:}\begin{cases}\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)\\ \dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t)\end{cases}$$

Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)\,, \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$

Can. com. rels: $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$

Example: How to calculate quant. fluct. of K.G. field?

1. Solve the system of ∞ many quantum harmonic oscillator degrees of freedom

$$\hat{q}_k\,, \hat{p}_k$$

with $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{\omega_k^2}{2}\hat{q}_k^2, \qquad \omega_k = \sqrt{k^2 + m^2}$

for all $k = (k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ where $n_1, n_2, n_3 \in \mathbb{Z}$

2. Choose a state $|\psi\rangle$ of that quantum system.

Example: □ The oscillators could all be in their lowest energy state.

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Hermiticity: $\hat\phi^\dagger(x,t) = \hat\phi(x,t)$, $\hat\pi^\dagger(x,t) = \hat\pi(x,t)$

Can. com. rels: $[\hat\phi(x,t), \hat\pi(x',t)] = i\hbar\,\delta^3(x-x')$

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1. Solve the system of ∞ many quantum harmonic oscillator degrees of freedom

$$\hat q_k, \hat p_k$$

with $\hat H = \sum_k \frac{1}{2}\hat p_k^2 + \frac{\omega_k^2}{2}\hat q_k^2$, $\omega_k = \sqrt{k^2+m^2}$

for all $k = (k_1,k_2,k_3) = \frac{2\pi}{L}(n_1,n_2,n_3)$ where $n_1, n_2, n_3 \in \mathbb{Z}$

2. Choose a state $|\psi\rangle$ of that quantum system.

Example: The oscillators could all be in their lowest energy state.

We preliminarily call this $|\psi\rangle$ the vacuum state $|\psi_0\rangle$.

3. Given a state $|\psi\rangle$, we can calculate the probability (amplitude density) for finding arbitrary values $q_k(t)$, $p_k(t)$.

Example:

4. Given $|\psi\rangle$, calculate the probability distribution of the Fourier coefficients:

$$\hat\phi_k, \hat\pi_k$$

Can do because they are simply linear combinations of the harmonic oscillator variables $\hat q_k, \hat p_k$. (Exercise: calculate)

Example:

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15

for now: We solved, using (A), the QFT eqns of the K.G. field.

Namely, we have now solved:

Eqns of motion:
$$\begin{cases} \dot{\hat\phi}(x,t) = \hat\pi(x,t) \\ \dot{\hat\pi}(x,t) = -(-\Delta + m^2)\,\hat\phi(x,t) \end{cases}$$

Hermiticity: $\hat\phi^\dagger(x,t) = \hat\phi(x,t)$, $\hat\pi^\dagger(x,t) = \hat\pi(x,t)$

Can. com. rels: $[\hat\phi(x,t), \hat\pi(x',t)] = i\hbar\,\delta^3(x-x')$

with $\hat H = \sum_k \frac{1}{2}\hat p_k^2 + \frac{\omega_k^2}{2}\hat q_k^2$, $\quad \omega_k = \sqrt{k^2+m^2}$

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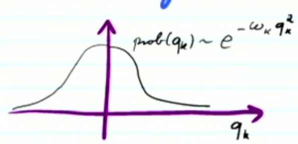
Example:

▫ In vacuum state, we know that the probability distribution of the $\hat q_k$ (and $\hat p_k$ as well) is gaussian:

▫



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Example:

▫ For $|\Psi_0\rangle$, since q_k, p_k are gaussian distributed, also the ϕ_k, π_k are gaussian distributed:

$$\text{prob}(\phi_k) \sim e^{-\omega_k \phi_k^* \phi_k}$$ (straightforward but tedious to show)

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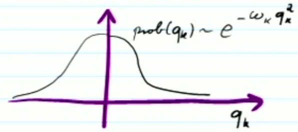

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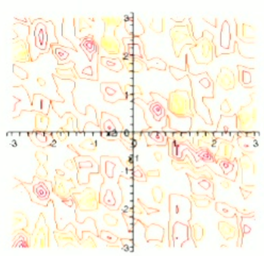
5. Given the prob. distribution of the ϕ_k, use Fourier to obtain prob. distribution of $\phi(x)$!

Example:

- Consider $|\Psi_0\rangle$.
- Draw a field $\phi(x)$ from the above calculated probability distribution for fields $\phi(x)$.

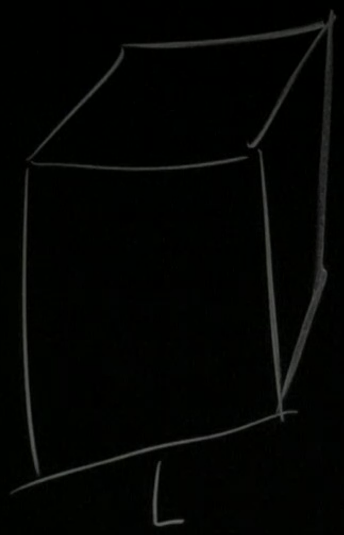
Contour lines of a typical $\phi(x,t)$ drawn from the vacuum's probability distribution for ϕ's.



Actual draw from that distribution.

The fluctuations trace back to the Fourier coefficients and to the $\hat q_k, \hat p_k$ which fluctuate even in lowest energy state.

$$i\hbar\,\dot{\hat{f}}(t) = \left[\hat{f}(t), \underline{\hat{H}(t)}\right]$$



"Infrared cutoff"

X-rays...

UV

blue

green

red

"infrared"

⋮

radio