

Title: Quantum Field Theory for Cosmology - Lecture 20240111

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (PHYS785/AMATH872)

Date: January 11, 2024 - 4:00 PM

URL: <https://pirsa.org/24010009>

QFT for Cosmology, Achim Kempf, Lecture 2

A taste of quantum fields

Intuition:

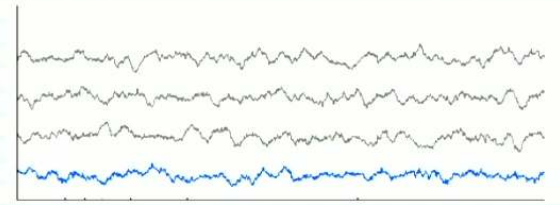
* Consider water waves:



* Probe them locally with cork:



* Multiple cork's oscillations are correlated



→ System of coupled (harmonic) oscillators!

not harmonic for water, not quite in QFT either.



1. Harmonic oscillators

Classical:

Hamiltonian: $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$

Eqs of motion: $\dot{p} = -\omega^2 q, \quad \dot{q} = p$

Lowest energy solution: (later relevant for "vacuum")



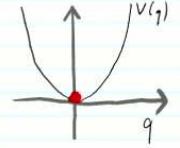
Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization
4. The harmonic oscillators of fields & their vacuum fluctuations

□ Lowest energy solution: (later relevant for "vacuum")

$$q(t) = 0, p(t) = 0$$

i.e., $H(t) = 0$ for all t :



□ "Nothing moves, with certainty"

□ $\Rightarrow \hat{q}(t), \hat{p}(t), \hat{H}$ etc are operator-valued.

□ Lowest energy solution now?

The lowest energy state, $|\psi_0\rangle$, obeys:

$$\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$$

$$\text{with } E_0 = \frac{1}{2}\hbar\omega$$

□ We notice:

Lowest energy is elevated! Why?

(later for quantum fields \Rightarrow nonzero vacuum energy)

Quantum:

As always when quantizing:

- H and Eqs of motion unchanged.
- But, the canonically conjugate pairs of variables (here, q and p) no longer commute:

□ Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{q}^2$

□ Eqs of motion: $\dot{\hat{p}} = -\omega^2\hat{q}, \dot{\hat{q}} = \hat{p}$

□ And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

commute:

$$\square \text{ Hamiltonian: } \hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2} \hat{q}^2$$

$$\square \text{ Eqs of motion: } \dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \hat{p}$$

□ And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

$$\text{with } E_0 = \frac{1}{2} \hbar \omega$$

□ We notice:

Lowest energy is elevated! Why?

(later for quantum fields \Rightarrow nonzero vacuum energy)

□ Lowest energy state $|\psi_0\rangle$?

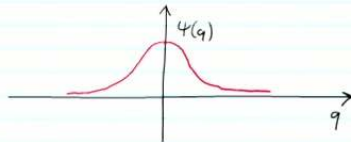
Consider eigenbasis $|q\rangle$ of \hat{q} :

$$\hat{q}|q\rangle = q|q\rangle \quad \text{for } q \in \mathbb{R}$$

$$\langle q|q'\rangle = \delta(q-q')$$

Then, recall:

$$\psi_0(q) = \langle q|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar} q^2}$$



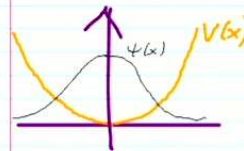
□ Is oscillator at resting position $q=0$?

In lowest energy state, $|\psi_0\rangle$, we have:

$$\bar{q} = \langle \psi_0|\hat{q}|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_0(q) q \psi_0(q) dq = 0$$

i.e. the position expectation vanishes, as in classical mechanics.

□ But, there are quantum fluctuations!



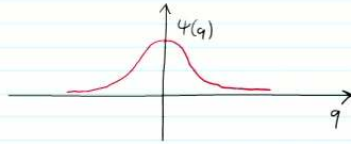
$$\Delta q = \langle \psi_0|(\hat{q} - \bar{q})^2|\psi_0\rangle^{1/2} = \sqrt{\frac{\hbar}{2m\omega}}$$

i.e., actual measurements yield values spread around $q=0$.
 \Rightarrow plausible why energy is elevated

$$\langle q | q' \rangle = \delta(q - q')$$

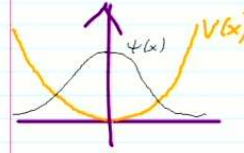
Then, recall:

$$\Psi_0(q) = \langle q | \Psi_0 \rangle = \left(\frac{\omega}{\pi \hbar}\right)^{1/4} e^{-\frac{i\omega}{2\hbar} q^2}$$



i.e. the position expectation vanishes, as in classical mechanics.

But, there are quantum fluctuations!



$$\Delta q = \langle \Psi_0 | (q - \bar{q})^2 | \Psi_0 \rangle^{1/2} = \sqrt{\frac{\hbar}{2m\omega}}$$

i.e., actual measurements yield values spread around $q=0$.
 ⇒ plausible why energy is elevated

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields
3. 2nd quantization
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

2. Relativistic fields

How to make the Schrödinger equation, say

choose simple case without a potential

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2m} \Delta \Psi(x,t) \quad (S)$$

relativistically covariant?

Laplacian: $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$

Klein & Gordon:

Recall: $p_i = -i\hbar \frac{\partial}{\partial x_i}$ and $E = i\hbar \frac{\partial}{\partial t}$, i.e., the

↙ because $\hat{H} = \text{energy}$

Schrödinger equation can be written in this form:

$$E\psi = \frac{\vec{p}^2}{2m} \psi, \text{ i.e.: } \boxed{E = \frac{\vec{p}^2}{2m}} \quad \text{i.e. } E = \frac{1}{2} m \dot{x}^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \quad (\text{Namely: } p_\mu p^\mu = m^2 c^4)$$

$$\text{i.e.: } \left(-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \Delta\right) \Psi = m^2 c^2 \Psi$$

2. Relativistic fields

How to make the Schrödinger equation, say

choose simple case without a potential

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (S)$$



relativistically covariant?

Laplacian: $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$

This "Klein Gordon equation" is usually written as:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \psi = 0 \quad (\text{units chosen so that } c=1, \hbar=1)$$

Or, also $(\square + m^2)\psi = 0$ with d'Alembertian $\square = \partial_t^2 - \Delta$

Nonrelativistic limit ok?

Must show that KG eqn reduces to Schrödinger eqn for small momenta:

$$E\psi = \frac{\vec{p}^2}{2m} \psi, \text{ i.e.: } \boxed{E = \frac{\vec{p}^2}{2m}} \quad \text{i.e. } E = \frac{1}{2} m \dot{x}^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \quad (\text{Namely: } p_\mu p^\mu = m^2 c^4)$$

$$\text{i.e.: } \left(-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \Delta \right) \psi = m^2 c^2 \psi$$

Assume K.G. Eqn, i.e.: $E^2 = m^2 c^4 + \vec{p}^2 c^2$

$$\Rightarrow E = \pm \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Choose positive energy solution:

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Taylor expansion for small \vec{p}^2 : (or large c)

$$E = m c^2 + \frac{1}{2} \frac{c^2}{\sqrt{m^2 c^4 + m^2 c^4}} \Big|_{\vec{p}=0} \vec{p}^2 + \mathcal{O}((\vec{p}')^4)$$

$$\Rightarrow E = m c^2 + \frac{\vec{p}^2}{2m} + \mathcal{O}((\vec{p}')^4)$$

Or, also $(\square + m^2)\psi = 0$ with d'Alembertian $\square = \partial_\mu^2 - \Delta$

□ Nonrelativistic limit ok?

Must show that KG eqn reduces
to Schrödinger eqn for small momenta:

⇒ For small momenta the K.G. eqn becomes the Schrödinger eqn:

$$E\psi = \left(\frac{\vec{p}^2}{2m} + mc^2\right)\psi$$

i.e.: $i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m} \Delta + mc^2\right)\psi$

Note: We obtain an extra term:

$$\hat{H} = \frac{\vec{p}^2}{2m} + \underbrace{mc^2}$$

In QM irrelevant: (use Heisenberg picture)

$$i\hbar \frac{d}{dt} \hat{f} = [\hat{f}, \hat{H} + \text{const } 1] = [\hat{f}, \hat{H}]$$

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Taylor expansion for small \vec{p}^2 : (or large c)

$$E = mc^2 + \frac{1}{2} \frac{c^2}{\sqrt{p^2 c^2 + m^2 c^4}} \Big|_{\vec{p}^2=0} \vec{p}^2 + \mathcal{O}((\vec{p}^2)^2)$$

$$\Rightarrow E = mc^2 + \frac{\vec{p}^2}{2m} + \mathcal{O}((\vec{p}^2)^2)$$

□ Remarks:

1a) The negative energy solutions spoil the interpretation of the $\psi(x,t)$ as a probability amplitude density!

1b) This problem is deep and led to quantum field theory, where this is solved in terms of anti-particles.

2a) There are many ways to generalize the Schrödinger equation to obtain a relativistically covariant equation.

Namely:

Require the negative energy solutions to propagate backwards in time: anti-particles!
They look like travelling forward in time with opposite properties.

2. and quantization

4. Harmonic oscillators in fields \Rightarrow vacuum fluctuations

2. Relativistic fields

How to make the Schrödinger equation, say

choose simple case without a potential

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (S)$$

relativistically covariant?

Laplacian: $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$

Schrödinger equation can be written in this form:

$$E\psi = \frac{\vec{p}^2}{2m} \psi, \text{ i.e.:}$$

$$E = \frac{\vec{p}^2}{2m}$$

$$\text{i.e. } E = \frac{1}{2} m v^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \quad (\text{Namely: } p_\mu p^\mu = m^2 c^4)$$

$$\text{i.e.: } \left(-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \Delta \right) \psi = m^2 c^2 \psi$$

This "Klein Gordon equation" is usually written as:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \psi = 0 \quad (\text{units chosen so that } c=1, \hbar=1)$$

Or, also $(\square + m^2)\psi = 0$ with d'Alembertian $\square = \partial_\mu \partial^\mu - \Delta$

Nonrelativistic limit ok?

Must show that KG eqn reduces to Schrödinger eqn for small momenta:

Assume K.G. Eqn., i.e.: $\frac{E^2}{c^2} = m^2 c^2 + \vec{p}^2$

$$\Rightarrow E = \pm \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Choose positive energy solution:

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Taylor expansion for small \vec{p}^2 : (or large c)

$$E = m c^2 + \frac{1}{2} \frac{c^2}{\sqrt{p^2 c^2 + m^2 c^4}} \Big|_{\vec{p}^2=0} \vec{p}^2 + \mathcal{O}((\vec{p}^2)^2)$$

$$\rightarrow E = m c^2 + \frac{\vec{p}^2}{2m} + \mathcal{O}((\vec{p}^2)^2)$$

2. Relativistic fields

How to make the Schrödinger equation, say

choose simple case without a potential

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (S)$$



relativistically covariant?

Laplacian: $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$

This "Klein Gordon equation" is usually written as:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \psi = 0 \quad (\text{units chosen so that } c=1, \hbar=1)$$

Or, also $(\square + m^2)\psi = 0$ with d'Alembertian $\square = \partial_t^2 - \Delta$

Nonrelativistic limit ok?

Must show that KG eqn reduces to Schrödinger eqn for small momenta:

$$E\psi = \frac{\vec{p}^2}{2m} \psi, \text{ i.e.: } \boxed{E = \frac{\vec{p}^2}{2m}} \quad \text{i.e. } E = \frac{1}{2} m \dot{x}^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \quad (\text{Namely: } p_\mu p^\mu = m^2 c^4)$$

$$\text{i.e.: } \left(-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \Delta \right) \psi = m^2 c^2 \psi$$

Assume K.G. Eqn, i.e.: $E^2 = m^2 c^4 + \vec{p}^2 c^2$

$$\Rightarrow E = \pm \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Choose positive energy solution:

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Taylor expansion for small \vec{p}^2 : (or large c)

$$E = m c^2 + \frac{1}{2} \frac{c^2}{\sqrt{p^2 c^2 + m^2 c^4}} \Big|_{\vec{p}^2=0} \vec{p}^2 + \mathcal{O}((\vec{p}^2)^2)$$

$$\Rightarrow E = m c^2 + \frac{\vec{p}^2}{2m} + \mathcal{O}((\vec{p}^2)^2)$$

Note: We obtain an extra term:

$$\hat{H} = \frac{\hat{p}^2}{2m} + mc^2$$

In QM irrelevant: (use Heisenberg picture)

$$i\hbar \frac{d}{dt} \hat{j} = [\hat{j}, \hat{H} + \text{const}] = [\hat{j}, \hat{H}]$$

Variety:
Require the negative energy solutions to propagate backwards in time: anti-particles! They don't like travelling forward in time with opposite properties.

1b) This problem is deep and led to quantum field theory, where this is solved in terms of anti-particles.

2a) There are many ways to generalize the Schrödinger equation to obtain a relativistically covariant equation.

2b) E. Wigner (1940s): Complete classification of relativistically covariant wave equations:

Note: The complete classification allows arbitrarily high spins and distinguishes massive from massless cases. All covariant wave eqns for same spin and mass lead to equivalent QFTs. See, e.g., textbook on QFT by S. Weinberg.

<u>Spin</u>	<u>Standard wave eqn</u>	<u>Examples</u>
0	Klein Gordon eqn.	Higgs, Inflaton, π^0, π^\pm
$\frac{1}{2}$	Dirac eqn.	e^- , quarks, p^+ , n
1	Maxwell YM eqns.	Photons, gluons

Higher spins?

- not observed in truly elementary particles.
- appear to lead to incurable "divergences" in QFT.

Note:

- "Graviton" should be a spin 2 particle.

Plan:

- Recall harmonic oscillators ✓
- Relativistic fields ✓
- 2nd quantization
- Harmonic oscillators in fields \Rightarrow vacuum fluctuations

3. 2nd quantization

- We will 2nd quantize only the Klein Gordon equation because:
 - is easiest
 - is only case of cosmological significance that we know of (so far).



Higher spins?

- not observed in truly elementary particles.
- appear to lead to incurable "divergencies" in QFT.

Note:

- "Graviton" should be a spin 2 particle.

- We will 2nd quantize only the Klein Gordon equation because:
 - is easiest
 - is only case of cosmological significance that we know of (so far).

Terminology: We switch from ψ to ϕ and call it a "Field".

Definition:

we will do the general definition later

The canonically conjugate field $\pi(x,t)$ to $\phi(x,t)$

is defined as: $\pi(x,t) = \dot{\phi}(x,t)$ (analogous to $p_i = \dot{q}_i$)

Klein Gordon equation can now be written in the form:

$$\ddot{\pi}(x,t) - \Delta \phi(x,t) + m^2 \phi(x,t) = 0$$

Notice:

The K.G. equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \phi = 0 \quad (\hbar = 1 = c)$$

does not couple $\text{Re}(\phi)$ to $\text{Im}(\phi)$:
each separately fulfills the K.G. eqn.

⇒ It suffices to study real-valued ϕ .

Making ϕ complex is then straightforward.

□ Klein Gordon equation can now be written in the form:

$$\ddot{\pi}(x,t) - \Delta \phi(x,t) + m^2 \phi(x,t) = 0$$

each separately fulfills the K.G. eqn.

⇒ It suffices to study real-valued ϕ .

Making ϕ complex is then straightforward.

□ Quantization conditions:

$$[\hat{\phi}(x,t), \dot{\hat{\pi}}(x',t)] = i\hbar \delta^3(x-x')$$

analogous to:
 $[\hat{q}_a(t), \hat{p}_a(t)] = i\hbar \delta_{aa}$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

$$[\hat{q}_a(t), \hat{q}_a(t)] = 0$$

$$[\dot{\hat{\pi}}(x,t), \dot{\hat{\pi}}(x',t)] = 0$$

$$[\hat{p}_a(t), \hat{p}_a(t)] = 0$$

□ We keep the equations of motion:

(E1) $\dot{\hat{\phi}}(x,t) = \dot{\hat{\pi}}(x,t)$ $\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

(E2) $\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\hat{\phi}(x,t)$ $\dot{\hat{p}}_a(t) = -K_a \hat{q}_a(t)$

□ Note: $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$

□ Is there a Hamiltonian for 2nd quantization? **Yes!**

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2} \dot{\hat{\pi}}^2(x,t) + \frac{1}{2} \hat{\phi}(x,t) (m^2 - \Delta) \hat{\phi}(x,t) d^3x$$

analogous to:
 $\hat{H} = \sum_a \frac{p_a^2}{2} + \frac{K_a}{2} q_a^2$

□ Proposition:

With this definition of \hat{H} , the Heisenberg equations $i\hbar \dot{\hat{f}} = [\hat{f}, \hat{H}]$

$$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] \quad i\hbar \dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar \dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \quad (*) \quad i\hbar \dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

yield the proper eqns of motion: E1, E2.

□ We keep the equations of motion:

(E1) $\hat{\phi}(x,t) = \hat{\pi}(x,t)$ $\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

(E2) $\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\hat{\phi}(x,t)$ $\dot{\hat{p}}_a(t) = -K_a \hat{q}_a(t)$

□ Note: $\hat{\phi}^*(x,t) = \hat{\phi}(x,t)$ now implies hermiticity: $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$

$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$

$i\hbar \dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$

$i\hbar \dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}]$ (*)

$i\hbar \dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$

yield the proper eqns of motion: E1, E2.

Indeed, e.g.:

$$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] = \left[\hat{\phi}(x,t), \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(x',t) + \dots d^3x' \right]$$

$$= \frac{1}{2} \int [\hat{\phi}(x,t), \hat{\pi}^2(x',t)] \hat{\pi}^2(x',t) + \hat{\pi}^2(x',t) [\hat{\phi}(x,t), \hat{\pi}^2(x',t)] d^3x'$$

$$= \frac{1}{2} \int \delta^3(x-x') \hat{\pi}^2(x',t) + \hat{\pi}^2(x',t) \delta^3(x-x') d^3x' = \hat{\pi}^2(x,t) i\hbar \checkmark$$

Exercise: Prove (*)