Title: Advanced General Relativity - 240131 (afternoon)

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Collection: Advanced General Relativity (PHYS7840)

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$$ds^2 = -f\,dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \;, \quad f = 1 - 2M/r$$

$$u^\alpha = \gamma (1, 0, \Omega, 0) \qquad \Omega = \text{const}$$

$$\gamma = \big(\underbrace{1 - 2M/r - \Omega^2 r^2}_{>0}\big)^{-1/2}$$

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{3}\Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$a^\alpha = u^\beta \nabla_\beta u^\alpha$$

$$a^r = -\gamma^2 \frac{(r - 2M)(M - \Omega^2 r^3)}{r^3}$$

$$(\text{geodesic} - \quad \Omega^2 = M/r^3 \;; \; r > 3M)$$

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad f = 1 - 2M/r$$

$$u^\alpha = \gamma(1, 0, \Omega, 0) \qquad \Omega = \text{const}$$

$$\gamma = \big(\underbrace{1 - 2M/r - \Omega^2 r^2}_{>0}\big)^{-1/2}$$

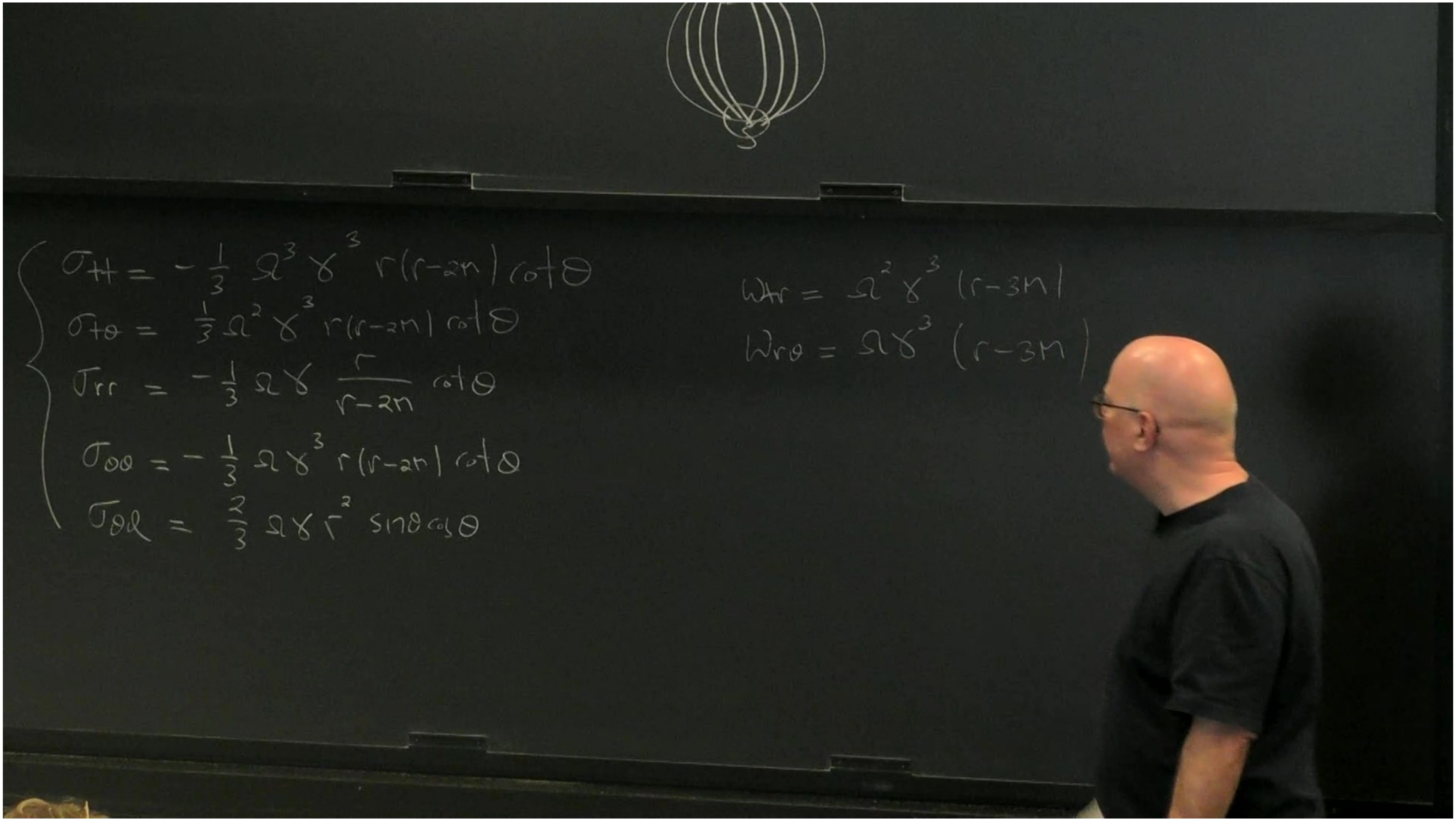
$$a^\alpha = u^\beta \nabla_\beta u^\alpha$$

$$a^r = -\frac{\gamma^2 (r-2M)(M - \Omega^2 r^3)}{r^3}$$

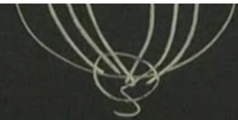
$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{3}\Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

(geodesic — $\Omega^2 = M/r^3$; $r > 3M$)

$$\Theta = \nabla_\alpha u^\alpha = \gamma\,\Omega \cot\theta$$



$\sigma_{tt} = -\frac{1}{3}\Omega^3\gamma^3 r(r-2m)\cot\theta$
$\sigma_{t\theta} = \frac{1}{3}\Omega^2\gamma^3 r(r-2m)\cot\theta$
$\sigma_{rr} = -\frac{1}{3}\Omega\gamma\frac{r}{r-2m}\cot\theta$
$\sigma_{\theta\theta} = -\frac{1}{3}\Omega\gamma^3 r(r-2m)\cot\theta$
$\sigma_{\theta\phi} = \frac{2}{3}\Omega\gamma r^2\sin\theta\cos\theta$
$\omega_{tr} = \Omega^2\gamma^3(r-3m)$
$\omega_{r\theta} = \Omega\gamma^3(r-3m)$



$\frac{1}{3}\Omega^3\gamma^3\, r(r-2M)\cot\theta$

$\frac{1}{3}\Omega^2\gamma^3\, r(r-2M)\cot\theta$

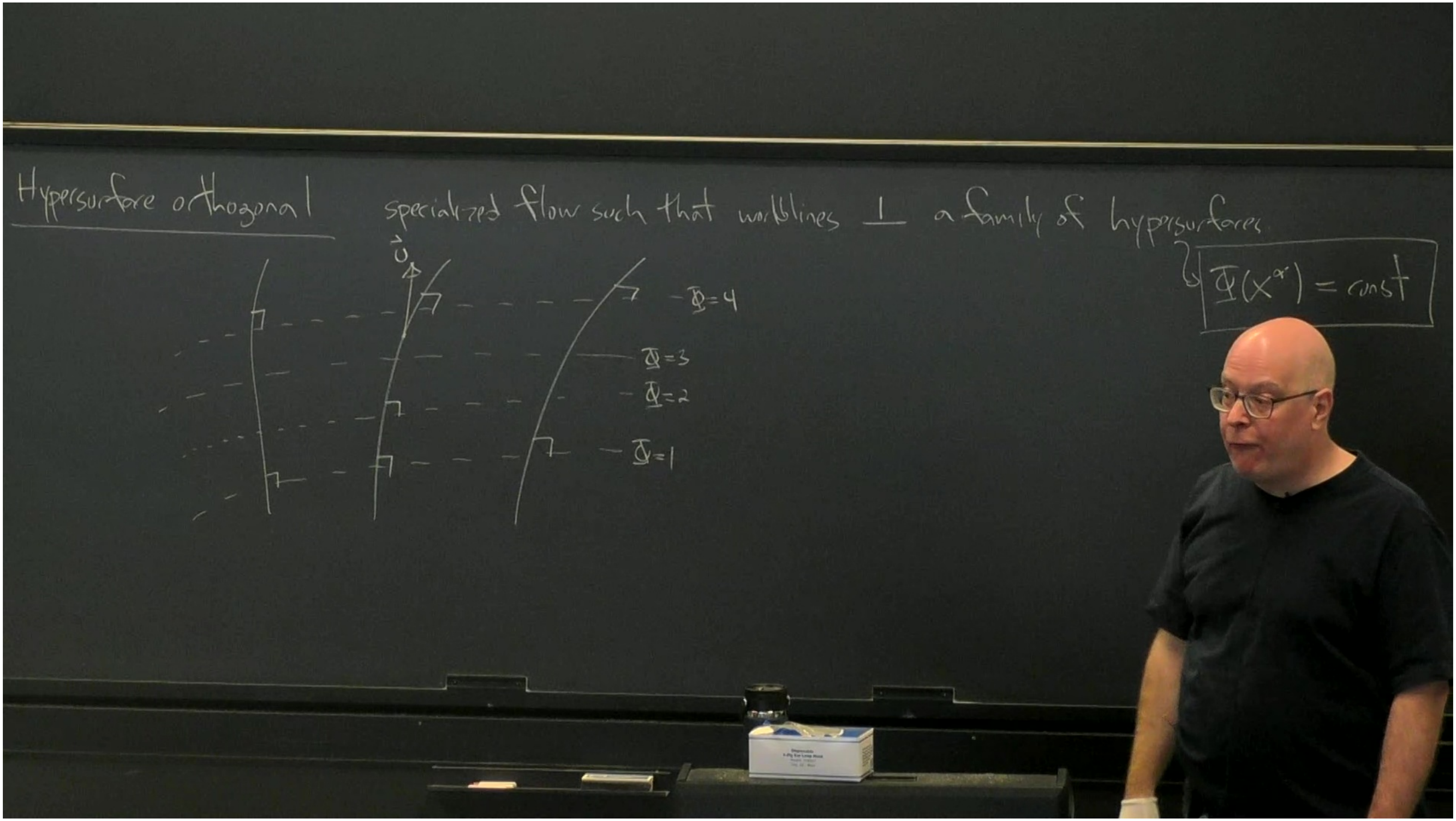
$-\frac{1}{3}\Omega\gamma\,\frac{r}{r-2M}\cot\theta$

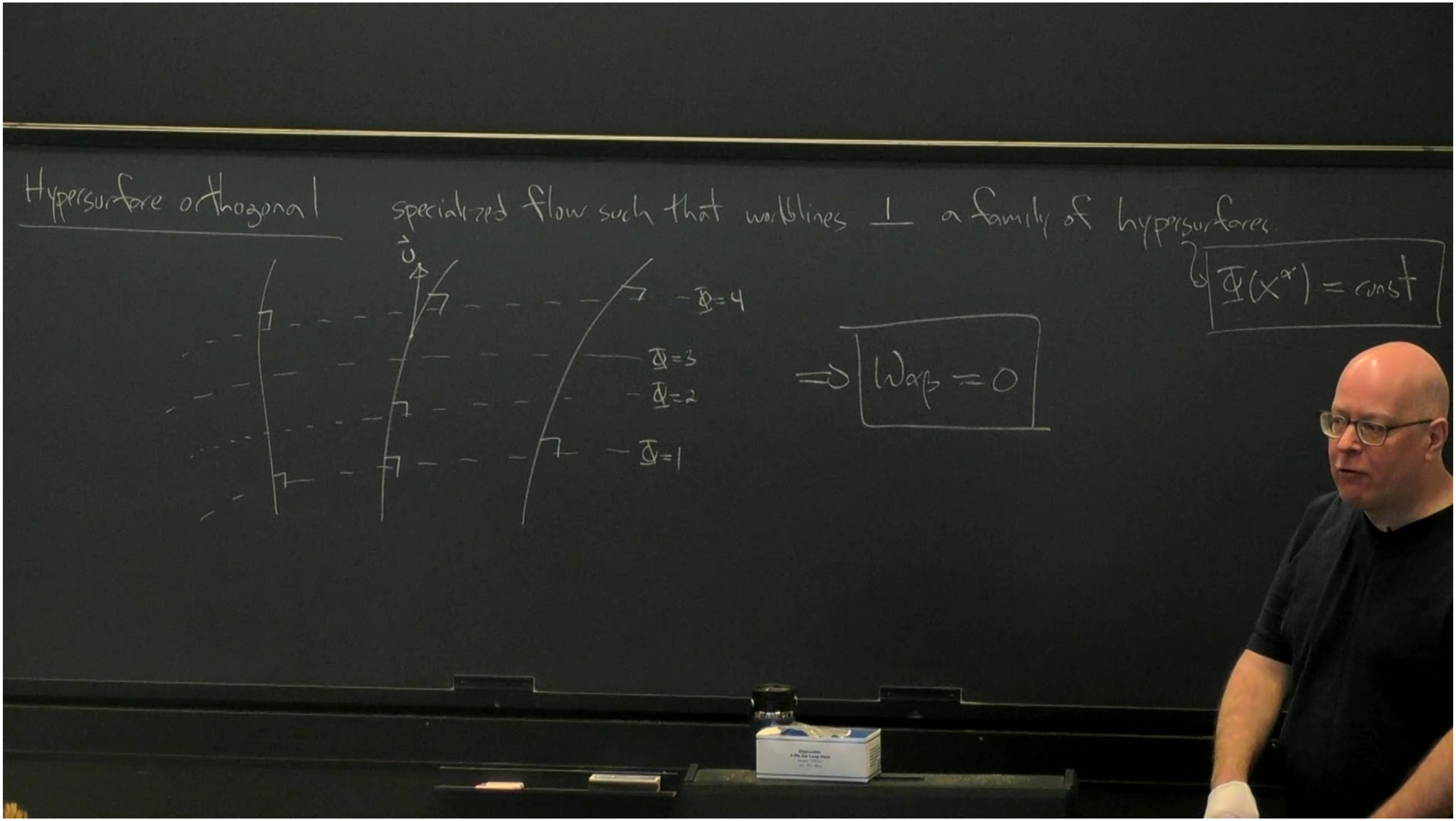
$\frac{1}{3}\Omega\gamma^3\, r(r-2M)\cot\theta$

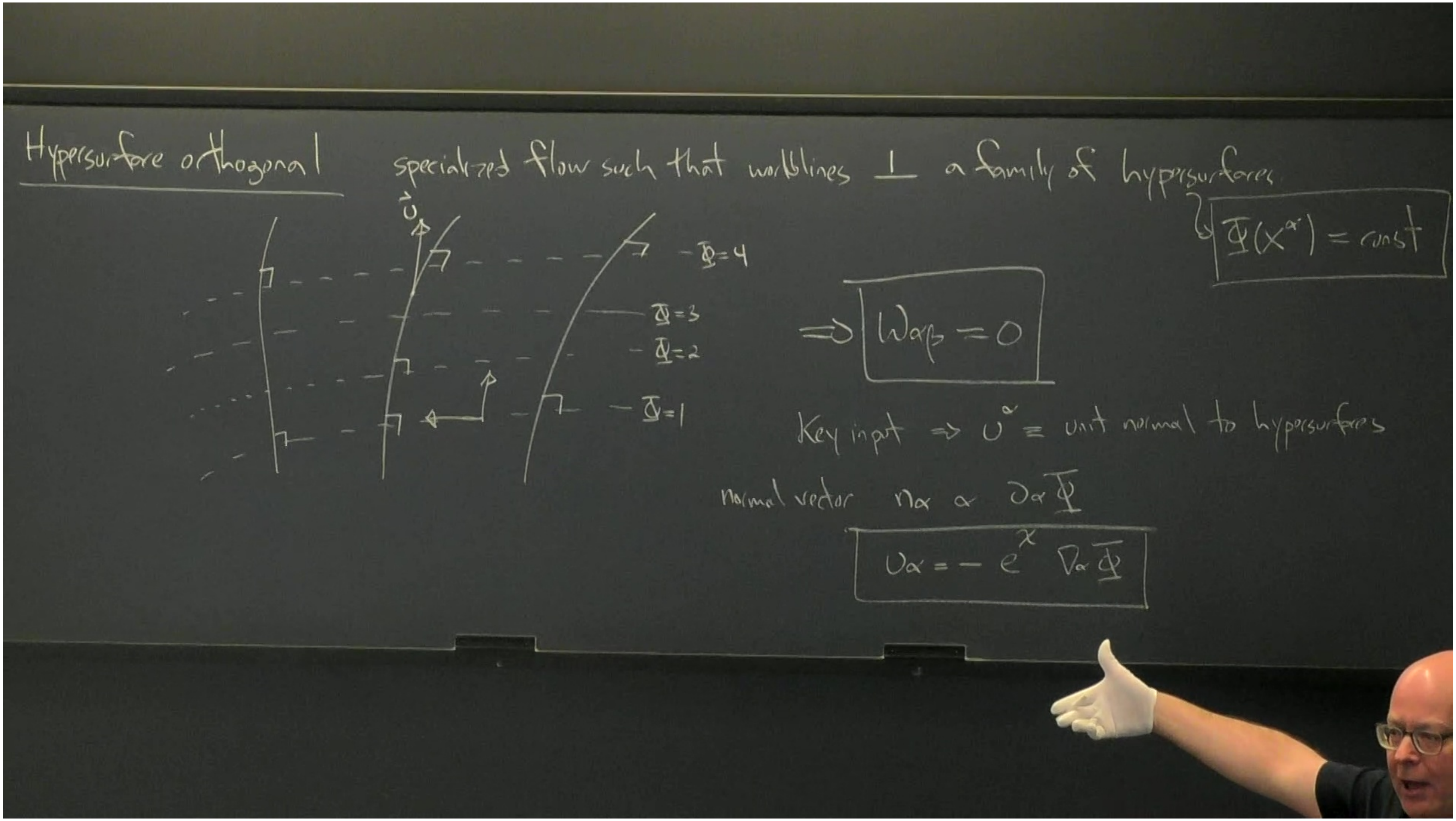
$\frac{2}{3}\Omega\gamma\, r^2\sin\theta\cos\theta$

$$\begin{cases} \omega_{tr} = \Omega^2\gamma^3(r-3M) \\ \omega_{r\theta} = \Omega\gamma^3(r-3M) \end{cases}$$

(check this!)


Hypersurface orthogonal
specialized flow such that worldlines ⊥ a family of hypersurfaces
Φ(x^α) = const
u
Φ=4
Φ=3
Φ=2
Φ=1


Hypersurface orthogonal
specialized flow such that worldlines ⊥ a family of hypersurfaces
$\Phi(x^\alpha) = \text{const}$
$\vec{u}$
$\Phi = 4$
$\Phi = 3$
$\Phi = 2$
$\Phi = 1$
$\Rightarrow \boxed{W_{\alpha\beta} = 0}$


Hypersurface orthogonal
specialized flow such that worldlines $\perp$ a family of hypersurfaces
$\boxed{\Phi(x^\alpha) = \text{const}}$
$\vec{u}$
$\Phi = 4$
$\Phi = 3$
$\Phi = 2$
$\Phi = 1$
$\Rightarrow \boxed{W_{\alpha\beta} = 0}$
Key input $\Rightarrow$ u^α = unit normal to hypersurfaces
normal vector $n_\alpha \propto \partial_\alpha \Phi$
$\boxed{u_\alpha = - e^{\chi} \nabla_\alpha \Phi}$

$$\boxed{U_\alpha = -e^{\chi} \nabla_\alpha \Phi}$$

$$-1 = g^{\alpha\beta} U_\alpha U_\beta = e^{2\chi} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \Rightarrow \boxed{e^{-2\chi} = -g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi}$$

$$+e^{-2\chi}(-\cancel{2}\nabla_\mu \chi) = -\nabla_\mu (g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi) = -g^{\alpha\beta} (\nabla_\mu \nabla_\alpha \Phi \nabla_\beta \Phi + \nabla_\alpha \Phi \nabla_\mu \nabla_\beta \Phi)$$

$$= -(\nabla^\alpha \Phi \nabla_{\mu\alpha} \Phi + \nabla^\alpha \Phi \nabla_{\mu\alpha} \Phi) = -2 \nabla^\alpha \Phi \nabla_{\mu\alpha} \Phi = +\cancel{2} e^{-\chi} U^\alpha \nabla_{\mu\alpha} \Phi$$

$$\boxed{\nabla_\mu \chi = -e^{\chi} U^\alpha \nabla_{\mu\alpha} \Phi}$$

$$(\nabla_{\alpha\mu} \Phi = \nabla_{\mu\alpha} \Phi)$$

$$\nabla_\beta U_\alpha = -e^{\chi}\left(\nabla_\beta \chi \, \nabla_\alpha \Phi + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\left(-e^{\chi} U^\mu \nabla_{\mu\beta}\Phi \, \nabla_\alpha \Phi + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\left(U^\mu U_\alpha \nabla_{\mu\beta}\Phi + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\left(U^\mu U_\alpha + g^\mu_\alpha\right)\nabla_{\mu\beta}\Phi$$

$$\longrightarrow -\nabla_\alpha(\ldots\Phi)$$

$$-1 = g^{\alpha\beta} U_\alpha U_\beta = e^{2\chi} g^{\alpha\beta} \nabla_\alpha\Phi \nabla_\beta\Phi \Rightarrow \boxed{e^{-2\chi} = - g^{\alpha\beta} \nabla_\alpha\Phi \nabla_\beta\Phi}$$

$$+ e^{-2\chi}(-\cancel{2}\nabla_\mu\chi) = -\nabla_\mu\left(g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\right) = -g^{\alpha\beta}\left(\nabla_\mu\nabla_\alpha\Phi\;\nabla_\beta\Phi + \nabla_\alpha\Phi\;\nabla_\mu\nabla_\beta\Phi\right)$$

$$= -\left(\nabla^\alpha\Phi\;\nabla_{\mu\alpha}\Phi + \nabla^\alpha\Phi\;\nabla_{\mu\alpha}\Phi\right) = -2\nabla^\alpha\Phi\;\nabla_{\mu\alpha}\Phi = +\cancel{2}e^{-\chi}U^\alpha\nabla_{\mu\alpha}$$

$$\left(\nabla_{\alpha\mu}\Phi = \nabla_{\mu\alpha}\Phi\right)$$

$$\boxed{\nabla_\mu\chi = -e^{\chi}U^\alpha\nabla_{\mu\alpha}\Phi}$$

$$\boxed{\nabla_\beta U_\alpha = -e^{\chi}P^\mu{}_\alpha\nabla_{\mu\beta}\Phi}$$

$$a_\alpha = U^\beta \nabla_\beta U_\alpha = -e^{\chi} U^\beta P_\alpha{}^\mu \nabla_{\mu\beta} \Phi$$

$$\underbrace{\nabla_\beta U_\alpha + U_\beta a_\alpha} = \frac{1}{3} \Theta P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi} \left(P_\alpha{}^\mu \nabla_{\mu\beta} \Phi + U_\beta U^\nu P_\alpha{}^\mu \nabla_{\mu\nu} \Phi \right) = -e^{+\chi} \left(g_\beta{}^\nu + U_\beta U^\nu \right) P_\alpha{}^\mu \nabla_{\mu\nu} \Phi$$

$$\boxed{\nabla_\beta U_\alpha + U_\beta a_\alpha = -e^{\chi} P_\alpha{}^\mu P_\beta{}^\nu \nabla_{\mu\nu} \Phi}$$

$$= -e^{\chi}\left(U^{\mu}U_{\alpha} + g^{\mu}{}_{\alpha}\right)\nabla_{\mu\beta}\Phi$$

$$a_{\alpha} = U^{\beta}\nabla_{\beta}U_{\alpha} = -e^{\chi}\,U^{\beta}P_{\alpha}{}^{\mu}\,\nabla_{\mu\beta}\Phi$$

$$\underbrace{\nabla_{\beta}U_{\alpha} + U_{\beta}a_{\alpha}} = \frac{1}{3}\Theta P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi}\left(P_{\alpha}{}^{\mu}\nabla_{\mu\beta}\Phi + U_{\beta}U^{\nu}P_{\alpha}{}^{\mu}\nabla_{\mu\nu}\Phi\right) = -e^{\chi}\left(g_{\beta}{}^{\nu} + U_{\beta}U^{\nu}\right)P_{\alpha}{}^{\mu}\,\nabla_{\mu\nu}\Phi$$

$$\boxed{\nabla_{\beta}U_{\alpha} + \ldots = -e^{\chi}\underbrace{P_{\alpha}{}^{\mu}P_{\beta}{}^{\nu}}_{\text{symmetric in }\alpha\beta\ \checkmark}\nabla_{\mu\nu}\Phi}$$

$$= -e^{\chi}\left(U^\mu U_\alpha + g^\mu{}_\alpha\right)\nabla_{\mu\beta}\Phi$$

$$a_\alpha = U^\beta \nabla_\beta U_\alpha = -e^{\chi}\, U^\beta P_\alpha{}^\mu\, \nabla_{\mu\beta}\Phi$$

$$\underbrace{\nabla_\beta U_\alpha + U_\beta a_\alpha} = \frac{1}{3}\Theta P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi}\left(P_\alpha{}^\mu \nabla_{\mu\beta}\Phi + U_\beta U^\nu P_\alpha{}^\mu \nabla_{\mu\nu}\Phi\right) = -e^{\chi}\left(g_\beta{}^\nu + U_\beta U^\nu\right) P_\alpha{}^\mu \nabla_{\mu\nu}\Phi$$

$$\boxed{\nabla_\beta U_\alpha + U_\beta a_\alpha = -e^{\chi}\,\underbrace{P_\alpha{}^\mu P_\beta{}^\nu}_{\text{symmetric in } \alpha\beta\ \checkmark}\,\nabla_{\mu\nu}\Phi} \longrightarrow \omega_{\alpha\beta} = 0$$

$$e^{2\chi}\, g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \Rightarrow \boxed{e^{-2\chi} = - g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi}$$

$$\chi \,) = - \nabla_\mu \left(g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \right) = - g^{\alpha\beta} \left(\nabla_\mu \nabla_\alpha \Phi \, \nabla_\beta \Phi + \nabla_\alpha \Phi \, \nabla_\mu \nabla_\beta \Phi \right)$$

$$= - \left(\nabla^\alpha \Phi \, \nabla_{\mu\alpha} \Phi + \nabla^\alpha \Phi \, \nabla_{\mu\alpha} \Phi \right) = -2 \nabla^\alpha \Phi \, \nabla_{\mu\alpha} \Phi = + \cancel{2} e^{-\chi} u^\alpha \nabla_{\mu\alpha} \Phi$$

$$\boxed{{}^\alpha \nabla_{\mu\alpha} \Phi}$$

$$\left(\nabla_{\alpha\beta} \Phi = \nabla_{\beta\alpha} \Phi \right)$$

$$\boxed{P^\mu{}_\alpha \nabla_{\mu\beta} \Phi}$$

Raychaudhuri's eqn (evolution equation for Θ)

$\Theta = \nabla_\alpha U^\alpha$

$$\frac{D\Theta}{d\tau} = \cdots$$

$$\frac{D}{d\tau}\left(\nabla_\alpha U_\beta\right) = U^\mu \nabla_\mu \left(\nabla_\alpha U_\beta\right)$$

$$= U^\mu \nabla_{\mu\alpha} U_\beta = \underbrace{U^\mu \left(\nabla_{\mu\alpha} - \nabla_{\alpha\mu}\right) U_\beta}_{-R^\nu{}_{\beta\mu\alpha} U_\nu} + \underbrace{U^\mu \nabla_{\alpha\mu} U_\beta}_{\nabla_\alpha\left(\underbrace{U^\mu \nabla_\mu U_\beta}_{a_\beta}\right) - \nabla_\alpha U^\mu \nabla_\mu U_\beta}$$

$$\frac{D}{d\tau}(\nabla_\alpha U_\beta) = -\underbrace{R^\nu{}_{\beta\mu\alpha} U_\nu U^\mu} + \nabla_\alpha a_\beta - (\nabla_\alpha U^\mu)(\nabla_\mu U_\beta)$$

$$R_{\nu\beta\mu\alpha} U^\mu U^\nu = R_{\mu\alpha\nu\beta} U^\mu U^\nu = R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

$$\frac{D}{d\tau}(\Theta) = -R_{\mu\nu} U^\mu U^\nu - (\nabla_\alpha U_\mu)(\nabla^\mu U^\alpha) + \nabla_\alpha a$$

$$\boxed{\frac{D\Theta}{d\tau} = -R_{\alpha\beta} U^\alpha U^\beta - (\nabla_\alpha U_\beta)(\nabla^\beta U^\alpha) + \nabla_\alpha a^\alpha}$$

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{3}\Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\nabla^\beta u^\alpha = -u^\beta a^\alpha + \frac{1}{3}\Theta P^{\alpha\beta} + \sigma^{\alpha\beta} - \omega^{\alpha\beta}$$

$$(\nabla_\alpha u_\beta)(\nabla^\beta u^\alpha) = \frac{1}{3}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta}$$

$$\boxed{\frac{D\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta + \nabla_\alpha a^\alpha} \quad \text{(R's equation)}$$