

Title: Advanced General Relativity - 240131 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

Date: January 31, 2024 - 1:30 PM

URL: <https://pirsa.org/24010007>

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad f = 1 - 2M/r$$

$$U^\alpha = \gamma(1, 0, \Omega, 0) \quad \Omega = \text{const}$$

$$\gamma = \left(\underbrace{1 - 2M/r - \Omega^2 r^2}_{> 0} \right)^{-1/2}$$

$$\nabla_\alpha U_\beta = -U_\alpha a_\beta + \frac{1}{3} \Theta \delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$a^\alpha = U^\beta \nabla_\beta U^\alpha$$

$$a^r = -\frac{\gamma^2 (r - 2M)(M - \Omega^2 r^3)}{r^3}$$

$$(\text{geodesic} - \Omega^2 = M/r^3; r > 3M)$$

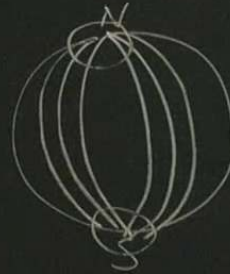
$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad f = 1 - 2M/r$$

$$U^\alpha = \gamma(1, 0, \Omega, 0) \quad \Omega = \text{const}$$

$$\gamma = \underbrace{(1 - 2M/r - \Omega^2 r^2)^{-1/2}}_{> 0}$$

$$\nabla_\alpha U_\beta = -U_\alpha a_\beta + \frac{1}{3} \textcircled{+} P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\textcircled{+} = \nabla_\alpha U^\alpha = \gamma \Omega \cot\theta$$



$$a^\alpha = U^\beta \nabla_\beta U^\alpha$$

$$a^r = -\frac{\gamma^2 (r - 2M) (M - \Omega^2 r^3)}{r^3}$$

(geodesic - $\Omega^2 = M/r^3$; $r > 3M$)



$$\left\{ \begin{aligned} \sigma_{tt} &= -\frac{1}{3} \Omega^3 \gamma^3 r(r-2M) \cos\theta \\ \sigma_{t\theta} &= \frac{1}{3} \Omega^2 \gamma^3 r(r-2M) \sin\theta \\ \tau_{rr} &= -\frac{1}{3} \Omega \gamma \frac{r}{r-2M} \cos\theta \\ \sigma_{\theta\theta} &= -\frac{1}{3} \Omega \gamma^3 r(r-2M) \cos\theta \\ \tau_{\theta\phi} &= \frac{2}{3} \Omega \gamma r^2 \sin\theta \cos\theta \end{aligned} \right.$$

$$\begin{aligned} W_{tr} &= \Omega^2 \gamma^3 (r-3M) \\ W_{r\theta} &= \Omega \gamma^3 (r-3M) \end{aligned}$$



$$\frac{1}{3} \Omega^3 \gamma^3 r(r-2M) \cot \theta$$

$$\frac{1}{3} \Omega^2 \gamma^3 r(r-2M) \cot \theta$$

$$-\frac{1}{3} \Omega \gamma \frac{r}{r-2M} \cot \theta$$

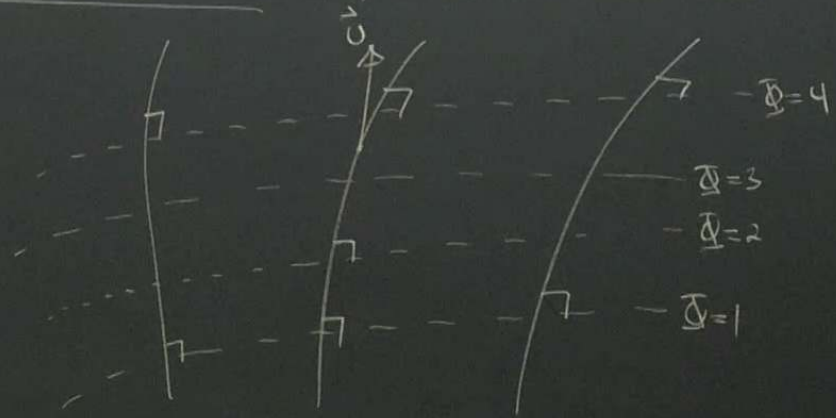
$$\frac{1}{3} \Omega \gamma^3 r(r-2M) \cot \theta$$

$$\frac{2}{3} \Omega \gamma r^2 \sin \theta \cot \theta$$

$$\left\{ \begin{array}{l} \omega_{tr} = \Omega^2 \gamma^3 (r-3M) \\ \omega_{r\theta} = \Omega \gamma^3 (r-3M) \end{array} \right. \quad (\text{check this!})$$

Hypersurface orthogonal

specialized flow such that worldlines \perp a family of hypersurfaces

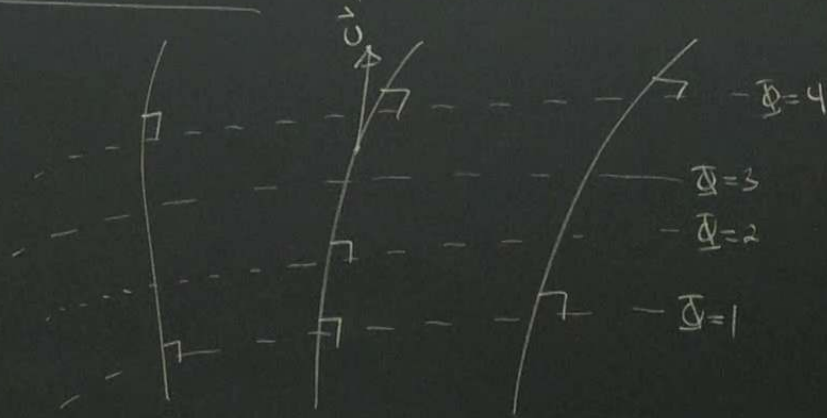


$$\Phi(x^\alpha) = \text{const}$$



Hypersurface orthogonal

specialized flow such that worldlines \perp a family of hypersurfaces.



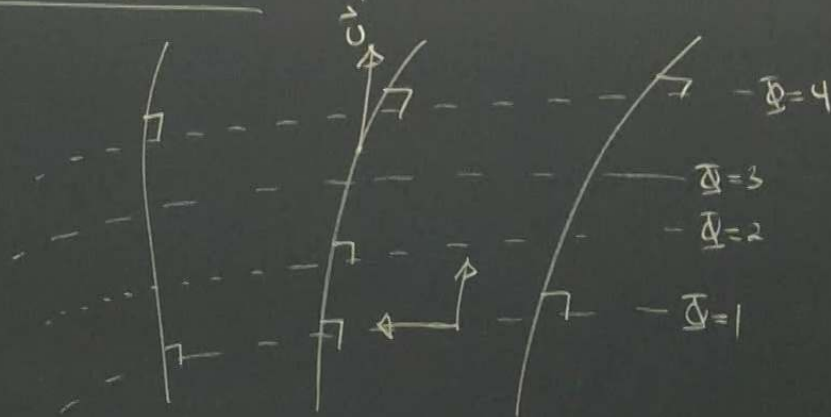
$$\Rightarrow \boxed{W_{ab} = 0}$$

$$\boxed{\Phi(x^\alpha) = \text{const}}$$



Hypersurface orthogonal

specialized flow such that worldlines \perp a family of hypersurfaces



$$\Rightarrow \boxed{W_{\alpha\beta} = 0}$$

Key input $\Rightarrow u^\alpha \equiv$ unit normal to hypersurfaces

normal vector $n_\alpha \propto \partial_\alpha \Phi$

$$\boxed{u_\alpha = -e^\alpha \nabla_\alpha \Phi}$$

$$\Phi(x^\alpha) = \text{const}$$

$$U_\alpha = -e^{\chi} \nabla_\alpha \Phi$$

$$\begin{aligned}
 -1 &= g^{\alpha\beta} U_\alpha U_\beta = e^{2\chi} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \Rightarrow \left[e^{-2\chi} = -g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \right] \\
 + e^{-2\chi} (-2 \nabla_\rho \chi) &= -\nabla_\rho (g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi) = -g^{\alpha\beta} (\nabla_\rho \nabla_\alpha \Phi \nabla_\beta \Phi + \nabla_\alpha \Phi \nabla_\rho \nabla_\beta \Phi) \\
 &= -(\nabla^\alpha \Phi \nabla_{\rho\alpha} \Phi + \nabla^\alpha \Phi \nabla_{\mu\alpha} \Phi) = -2 \nabla^\alpha \Phi \nabla_{\rho\alpha} \Phi = +2e^{-\chi} \nabla^\alpha \nabla_{\rho\alpha} \Phi \\
 \nabla_\rho \chi &= -e^{\chi} \nabla^\alpha \nabla_{\rho\alpha} \Phi \quad \left(\nabla_{\rho\alpha} \Phi = \nabla_{\alpha\rho} \Phi \right)
 \end{aligned}$$

$$\begin{aligned}
\nabla_\beta U_\alpha &= -e^\chi \left(\nabla_\beta \chi \nabla_\alpha \Phi + \nabla_{\beta\alpha} \Phi \right) \\
&= -e^\chi \left(-e^\chi U^\mu \nabla_{\mu\beta} \Phi \nabla_\alpha \Phi + \nabla_{\beta\alpha} \Phi \right) \\
&= -e^\chi \left(U^\mu U_\alpha \nabla_{\mu\beta} \Phi + \nabla_{\beta\alpha} \Phi \right) \\
&= -e^\chi \left(U^\mu U_\alpha + \delta_\alpha^\mu \right) \nabla_{\mu\beta} \Phi
\end{aligned}$$

$\rightarrow -\nabla_\alpha(e^{2\chi})$

$$-1 = g^{\sigma\rho} U_\alpha U_\rho = e^{2\chi} g^{\sigma\rho} \nabla_\alpha \Phi \nabla_\rho \Phi \Rightarrow \boxed{e^{-2\chi} = -g^{\sigma\rho} \nabla_\alpha \Phi \nabla_\rho \Phi}$$
$$+ e^{-2\chi} (-2 \nabla_\rho \chi) = -\nabla_\rho (g^{\sigma\rho} \nabla_\alpha \Phi \nabla_\rho \Phi) = -g^{\sigma\rho} (\nabla_\rho \nabla_\alpha \Phi \nabla_\rho \Phi + \nabla_\alpha \Phi \nabla_\rho \nabla_\rho \Phi)$$
$$= -(\nabla^\sigma \Phi \nabla_{\rho\sigma} \Phi + \nabla^\sigma \Phi \nabla_{\rho\sigma} \Phi) = -2 \nabla^\sigma \Phi \nabla_{\rho\sigma} \Phi = +2 e^{-\chi} \overset{\circ}{\nabla}_\rho \chi$$

$$\boxed{\nabla_\rho \chi = -e^\chi \overset{\circ}{\nabla}_{\rho\alpha} \Phi}$$

$$(\nabla_{\sigma\rho} \Phi = \nabla_{\rho\sigma} \Phi)$$

$$\boxed{\nabla_\rho \omega_\alpha = -e^\chi P^\mu_{\alpha} \nabla_{\mu\beta} \Phi}$$

$$a_\alpha = U^\beta \nabla_\beta U_\alpha = -e^{\chi} U^\beta P_\alpha^\mu \nabla_{\mu\beta} \Phi$$

$$\nabla_\beta U_\alpha + U_\beta a_\alpha = \frac{1}{3} (\Theta) P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi} \left(P_\alpha^\mu \nabla_{\mu\beta} \Phi + U_\beta U^\nu P_\alpha^\mu \nabla_{\mu\nu} \Phi \right) = -e^{\chi} \left(g_{\beta\nu} + U_\beta U^\nu \right) P_\alpha^\mu \nabla_{\mu\nu} \Phi$$

$$\nabla_\beta U_\alpha + U_\beta a_\alpha = -e^{\chi} P_\alpha^\mu P_\beta^\nu \nabla_{\mu\nu} \Phi$$

$$= -e^{\chi} \left(U^{\mu} U_{\alpha} + \gamma_{\alpha}^{\mu} \right) \nabla_{\mu} \Phi$$

$$a_{\alpha} = U^{\beta} \nabla_{\beta} U_{\alpha} = -e^{\chi} U^{\beta} P_{\alpha}^{\mu} \nabla_{\mu} \Phi$$

$$\underbrace{\nabla_{\beta} U_{\alpha} + U_{\beta} a_{\alpha}} = \left(\frac{1}{3} \oplus \right) P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi} \left(P_{\alpha}^{\mu} \nabla_{\mu} \Phi + U_{\beta} U^{\nu} P_{\alpha}^{\mu} \nabla_{\mu} \Phi \right) = -e^{\chi} \left(g_{\beta}^{\nu} + U_{\beta} U^{\nu} \right) P_{\alpha}^{\mu} \nabla_{\mu} \Phi$$

$$\boxed{\nabla_{\beta} U_{\alpha} + U_{\beta} a_{\alpha} - e^{\chi} P_{\alpha}^{\mu} P_{\beta}^{\nu} \nabla_{\mu} \Phi}$$

symmetric in $\alpha\beta$ ✓

$$= -e^{\chi} \left(U^{\mu} U_{\alpha} + \mathfrak{J}_{\alpha}^{\mu} \right) \nabla_{\mu} \Phi$$

$$a_{\alpha} = U^{\beta} \nabla_{\beta} U_{\alpha} = -e^{\chi} U^{\beta} P_{\alpha}^{\mu} \nabla_{\mu} \Phi$$

$$\nabla_{\beta} U_{\alpha} + U_{\beta} a_{\alpha} = \left(\frac{1}{3} \oplus \right) P_{\beta\alpha} + \mathcal{T}_{\beta\alpha} + W_{\beta\alpha}$$

$$-e^{\chi} \left(P_{\alpha}^{\mu} \nabla_{\mu} \Phi + U_{\beta} U^{\nu} P_{\alpha}^{\mu} \nabla_{\mu} \Phi \right) = -e^{\chi} \left(g_{\alpha}^{\nu} + U_{\beta} U^{\nu} \right) P_{\alpha}^{\mu} \nabla_{\mu} \Phi$$

$$\nabla_{\beta} U_{\alpha} + U_{\beta} a_{\alpha} = -e^{\chi} \underbrace{P_{\alpha}^{\mu} P_{\beta}^{\nu}}_{\text{symmetric in } \alpha\beta} \nabla_{\mu} \Phi$$

$$\rightarrow W_{\alpha\beta} = 0$$

$$e^{2\chi} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \Rightarrow \boxed{e^{-2\chi} = -g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi}$$

$$\begin{aligned} \chi) &= -\nabla_\rho (g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi) = -g^{\alpha\beta} (\nabla_\rho \nabla_\alpha \Phi \nabla_\beta \Phi + \nabla_\alpha \Phi \nabla_\rho \nabla_\beta \Phi) \\ &= -(\nabla^\alpha \Phi \nabla_{\rho\alpha} \Phi + \nabla^\alpha \Phi \nabla_{\rho\alpha} \Phi) = -2 \nabla^\alpha \Phi \nabla_{\rho\alpha} \Phi = +2e^{-\chi} \nabla^\alpha \nabla_{\rho\alpha} \Phi \end{aligned}$$

$$\boxed{\nabla_{\rho\alpha} \Phi}$$

$$(\nabla_{\alpha\rho} \Phi = \nabla_{\rho\alpha} \Phi)$$

$$\boxed{P^\mu_\alpha \nabla_{\mu\beta} \Phi}$$

Raychaudhuri's eqn

(evolution equation for Θ)

$$\frac{D\Theta}{dT} = \dots$$

$$\frac{D}{dT} (\nabla_\alpha U_\beta) = U^\mu \nabla_\mu (\nabla_\alpha U_\beta)$$

$$= U^\mu \nabla_{\mu\alpha} U_\beta = U^\mu \underbrace{(\nabla_{\mu\alpha} - \nabla_{\alpha\mu}) U_\beta}_{-R^{\nu}{}_{\beta\mu\alpha} U_\nu} + \underbrace{U^\mu \nabla_{\alpha\mu} U_\beta}_{\nabla_\alpha (U^\mu \nabla_\mu U_\beta)} - \nabla_\alpha U^\mu \nabla_\mu U_\beta$$

$\rightarrow -\nabla_\alpha (\frac{x^\alpha}{t})$

$$\frac{D}{dt} (\nabla_\alpha U_\beta) = - \underbrace{R^\nu_{\beta\mu\alpha} U_\nu U^\mu}_{R_{\beta\mu\nu\alpha} U^\mu U^\nu} + \nabla_\alpha a_\beta - (\nabla_\alpha U^\mu) (\nabla_\mu U_\beta)$$

$$R_{\beta\mu\nu\alpha} U^\mu U^\nu = R_{\mu\alpha\nu\beta} U^\mu U^\nu = R_{\alpha\mu\nu\beta} U^\mu U^\nu$$

$$\frac{D}{dt} (\Theta) = -R_{\mu\nu\alpha\beta} U^\mu U^\nu - (\nabla_\alpha U_\beta) (\nabla^\mu U^\alpha) + \nabla_\alpha a^\alpha$$

$$\frac{D\Theta}{dt} = -R_{\alpha\mu\nu\beta} U^\mu U^\nu - (\nabla_\alpha U_\beta) (\nabla^\mu U^\alpha) + \nabla_\alpha a^\alpha$$

$$\nabla_\alpha U_\beta = -U_\alpha a_\beta + \frac{1}{3} \Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\nabla^\alpha U^\alpha = -U^\alpha a_\alpha + \frac{1}{3} \Theta P^{\alpha\beta} + \sigma^{\alpha\beta} - \omega^{\alpha\beta}$$

$$(\nabla_\alpha U_\beta)(\nabla^\alpha U^\alpha) = \frac{1}{3} \Theta^2 + \sigma_{\alpha\beta} \sigma^{\alpha\beta} - \omega_{\alpha\beta} \omega^{\alpha\beta}$$

$$\boxed{\frac{D\Theta}{dT} = -\frac{1}{3} \Theta^2 - \sigma_{\alpha\beta} \sigma^{\alpha\beta} + \omega_{\alpha\beta} \omega^{\alpha\beta} - R_{\alpha\beta} U^\alpha U^\beta + \nabla_\alpha a^\alpha} \quad (\text{R's equation})$$