Title: Advanced General Relativity - 240124 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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$$\boxed{\Theta \equiv \vec{\nabla}\cdot\vec{V} = \frac{1}{V}\frac{DV}{dt} = \text{"rate of expansion"}}$$

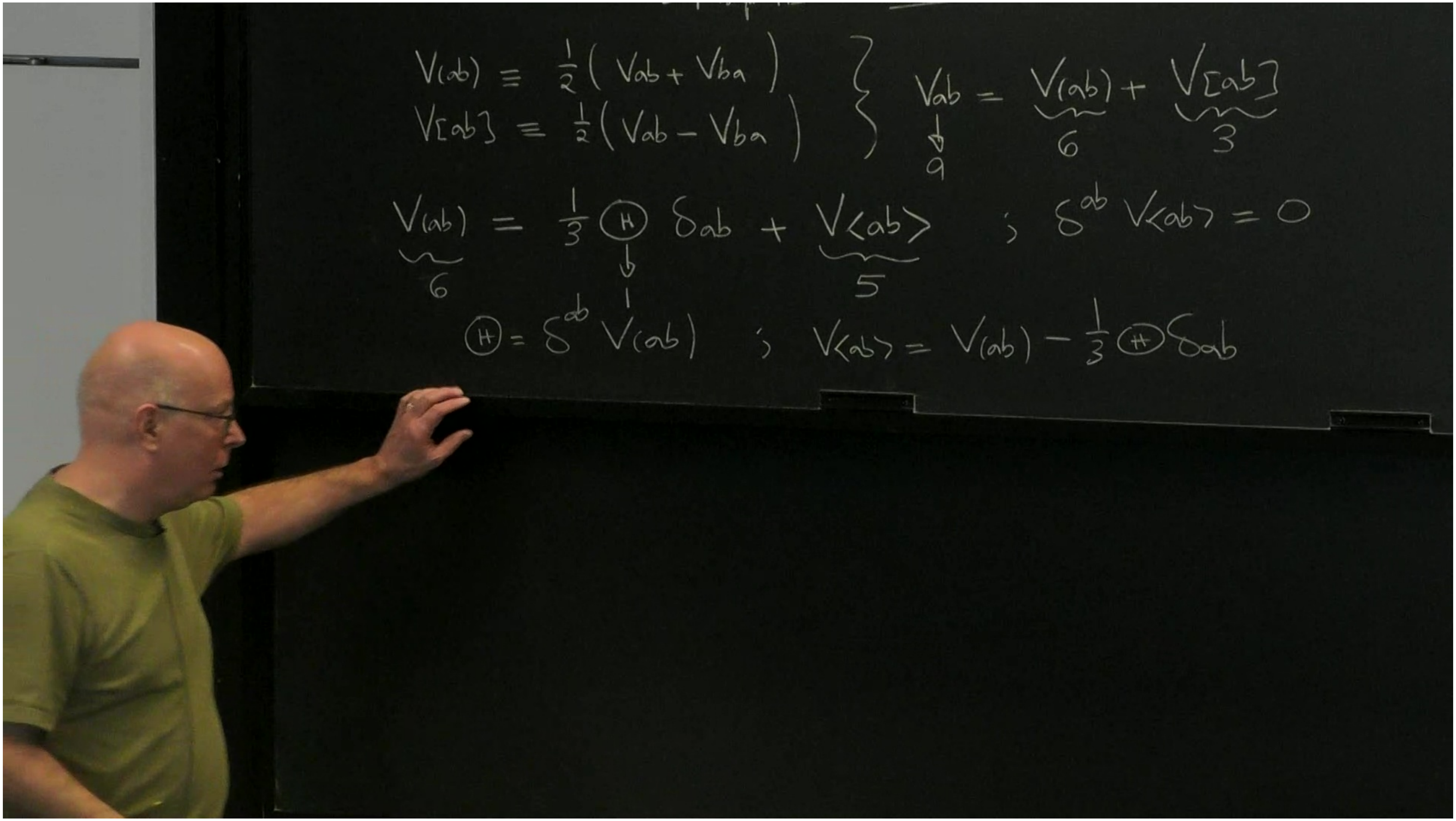
Decomposition of $\partial_a V_b$

$\hookrightarrow$ 9 components

$$\boxed{V_{ab} \equiv \partial_a V_b}$$

$$\left.\begin{aligned} V_{(ab)} &\equiv \frac{1}{2}\left(V_{ab} + V_{ba}\right) \\ V_{[ab]} &\equiv \frac{1}{2}\left(V_{ab} - V_{ba}\right) \end{aligned}\right\} \quad \underset{\downarrow\, 9}{V_{ab}} = \underbrace{V_{(ab)}}_{6} + \underbrace{V_{[ab]}}_{3}$$

$$\underbrace{V_{(ab)}}_{6} = \frac{1}{3}\underset{\downarrow\, 1}{\Theta}\,\delta_{ab} + \underbrace{V_{\langle ab\rangle}}_{5} \quad ; \quad \delta^{ab} V_{\langle ab\rangle} = 0$$



$V_{(ab)} \equiv \frac{1}{2}(V_{ab} + V_{ba})$
$V_{[ab]} \equiv \frac{1}{2}(V_{ab} - V_{ba})$
$\underbrace{V_{ab}}_{9} = \underbrace{V_{(ab)}}_{6} + \underbrace{V_{[ab]}}_{3}$
$\underbrace{V_{(ab)}}_{6} = \frac{1}{3} H \delta_{ab} + \underbrace{V_{\langle ab \rangle}}_{5}$; $\delta^{ab} V_{\langle ab \rangle} = 0$
$H = \delta^{ab} V_{(ab)}$; $V_{\langle ab \rangle} = V_{(ab)} - \frac{1}{3} H \delta_{ab}$

$$\Theta = \delta^{ab} V_{(ab)} \quad ; \quad V_{\langle ab\rangle} = V_{(ab)} - \frac{1}{3}\Theta\delta_{ab}$$

$$\boxed{\underbrace{\partial_a V_b}_{9} = \frac{1}{3}\underbrace{\Theta}_{1}\delta_{ab} + \underbrace{V_{\langle ab\rangle}}_{5} + \underbrace{V_{[ab]}}_{3}}$$

$$\boxed{\begin{aligned} \sigma_{ab} &\equiv V_{\langle ab\rangle} = \text{"rate of sheer"} \\ \omega_{ab} &\equiv V_{[ab]} = \text{"rate of rotation"} \end{aligned}}$$

$$V_{[xy]} = \frac{1}{2}\left(\partial_x V_y - \partial_y V_x\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_z$$

$$V_{[yz]} = \frac{1}{2}\left(\partial_y V_z - \partial_z V_y\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_x$$

$$V_{[zx]} = \frac{1}{2}\left(\partial_z V_x - \partial_x V_z\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_y$$

$$\underbrace{\partial_a V_b}_{9} = \frac{1}{3} \underbrace{\Theta}_{1} \delta_{ab} + \underbrace{V_{\langle ab \rangle}}_{5} + \underbrace{V_{[ab]}}_{3}$$

$\sigma_{ab} \equiv V_{\langle ab \rangle}$ = "rate of shear"

$\omega_{ab} \equiv V_{[ab]}$ = "rate of rotation"

$$V_{[xy]} = \frac{1}{2}\left(\partial_x V_y - \partial_y V_x\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_z$$

$$V_{[yz]} = \frac{1}{2}\left(\partial_y V_z - \partial_z V_y\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_x$$

$$V_{[zx]} = \frac{1}{2}\left(\partial_z V_x - \partial_x V_z\right) = \frac{1}{2}\left(\vec{\nabla}\times\vec{V}\right)_y$$

Focus on 2D flow in x-y plane $V_z = 0$, $\partial_z V_x = \partial_z V_y = 0$

Shear

Situation $\underbrace{\Theta}_{\partial_x v_x + \partial_y v_y = 0} = 0 = \omega_{ab} = 0$

$$\omega_{xy} = \frac{1}{2}\left(\partial_x v_y - \partial_y v_x\right) = 0 \Rightarrow \partial_x$$

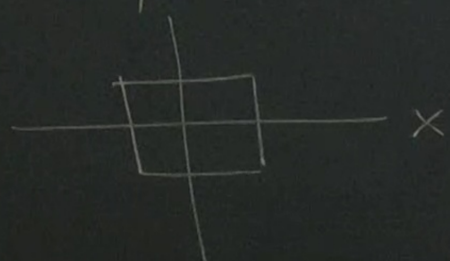
$$\sigma_{xx} = \frac{1}{2}\left(\partial_x v_x + \partial_x v_x\right) = \partial_x v_x \equiv \sigma_+$$

$$\sigma_{xy} = \frac{1}{2}\left(\partial_x v_y + \partial_y v_x\right) = \partial_x v_y = \partial_y v_x = \sigma_\times$$

$$\sigma_{yy} = \partial_y v_y = -\partial_x v_x = -\sigma_+$$

$$\boxed{\sigma_{ab} = \begin{pmatrix} \sigma_+ & \sigma \\ \sigma_\times & - \end{pmatrix}}$$

Square fluid element:



$$+\partial_x V_x \Big) = \partial_x V_x \equiv \sigma_+$$

$$y + \partial_y V_x \Big) = \partial_x V_y = \partial_y V_x = \sigma_\times$$

$$\sigma_{ab} = \begin{pmatrix} & \\ \sigma_\times & -\sigma_+ \end{pmatrix}$$

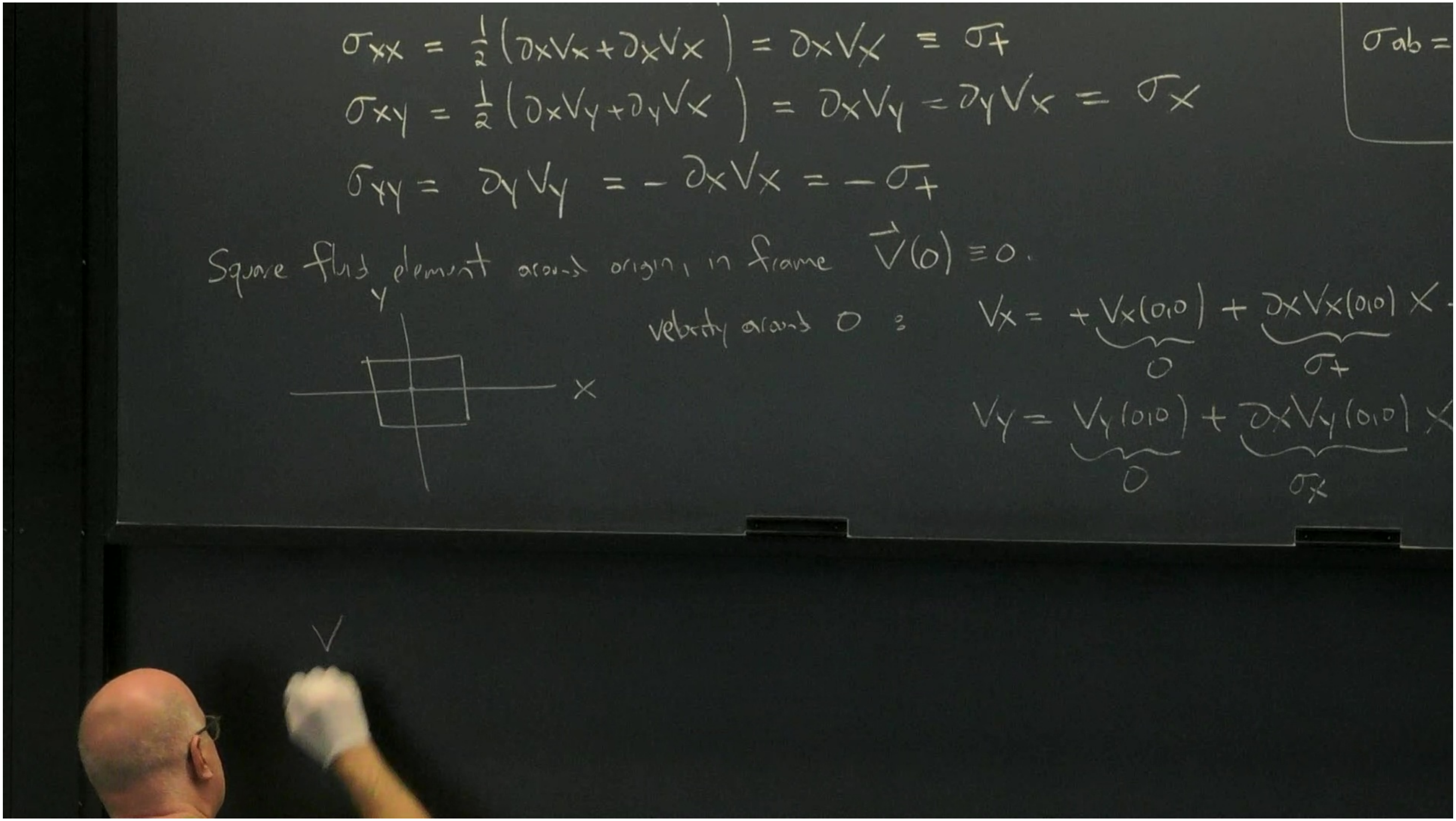
$$= -\partial_x V_x = -\sigma_+$$

origin, in frame $\vec{V}(0) \equiv 0$.

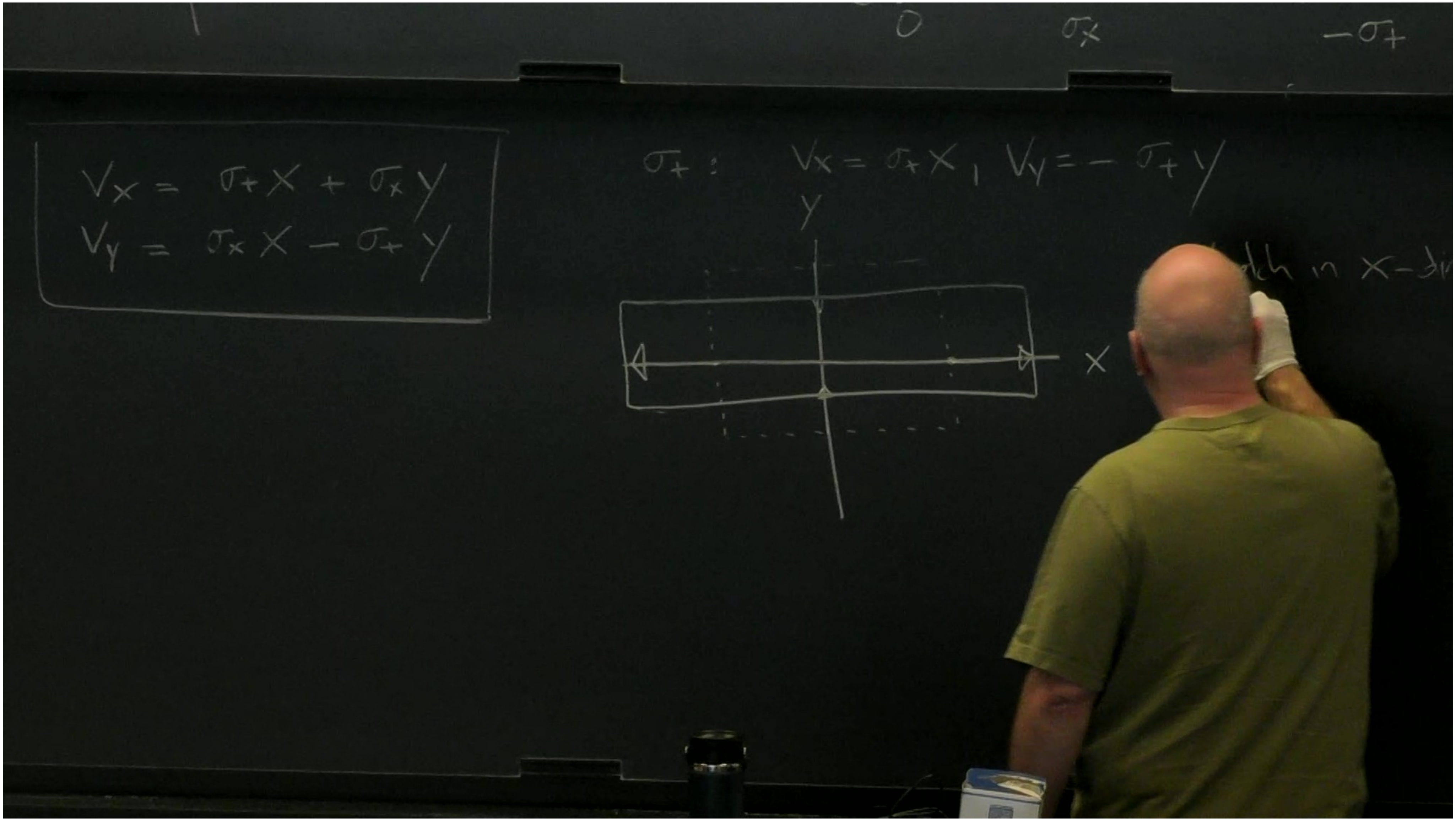
velocity around O :

$$V_x = +\underbrace{V_x(0,0)}_{0} + \underbrace{\partial_x V_x(0,0)}_{\sigma_+} x + \underbrace{\partial_y V_x(0,0)}_{\sigma_\times} y + \cdots$$

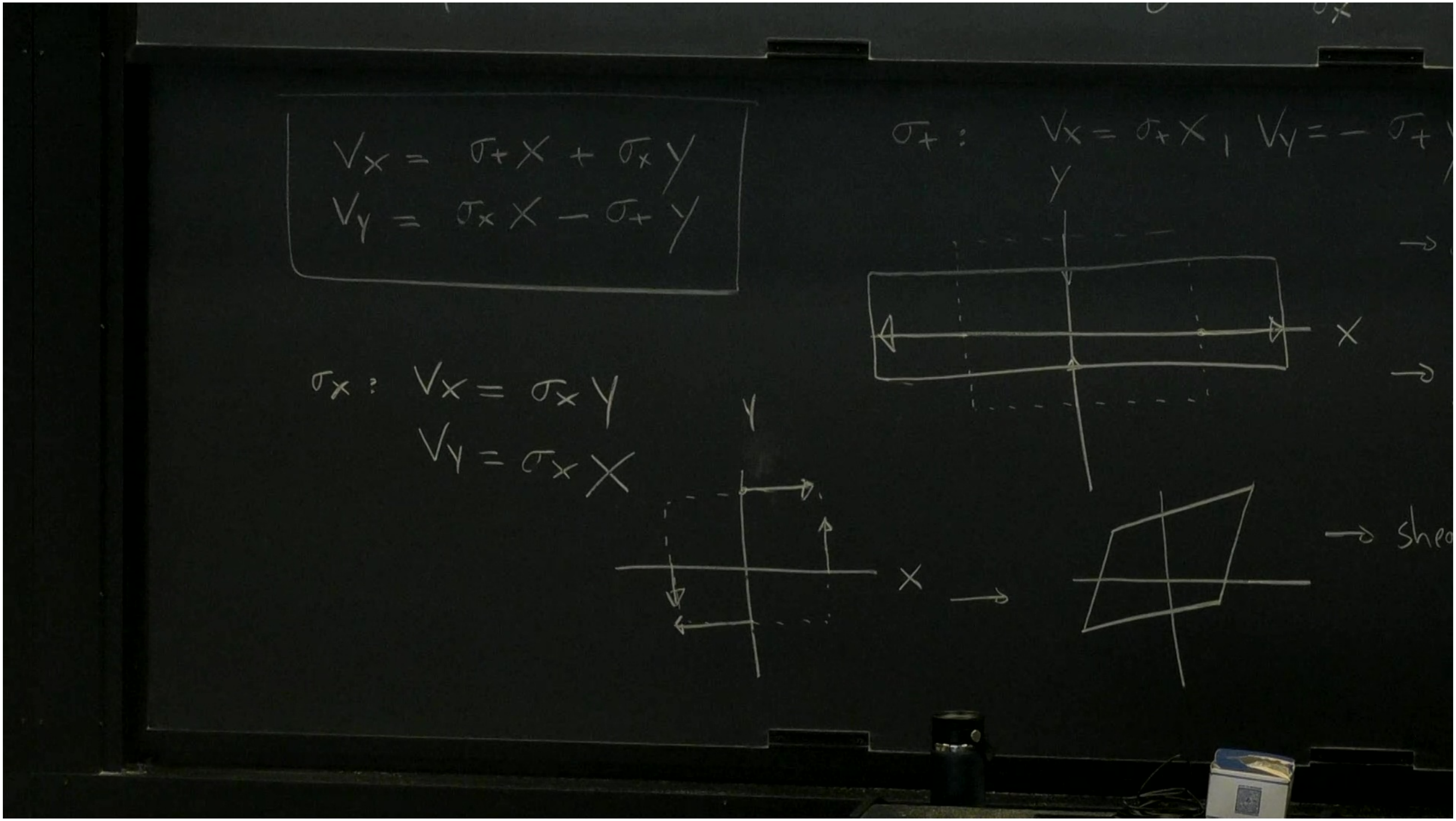
$$V_y = \underbrace{V_y(0,0)}_{0} + \underbrace{\partial_x V_y(0,0)}_{\sigma_\times} x + \underbrace{\partial_y V_y(0,0)}_{-\sigma_+} y + \cdots$$



$\sigma_{xx} = \frac{1}{2}(\partial_x V_x + \partial_x V_x) = \partial_x V_x \equiv \sigma_+$
$\sigma_{xy} = \frac{1}{2}(\partial_x V_y + \partial_y V_x) = \partial_x V_y = \partial_y V_x = \sigma_\times$
$\sigma_{yy} = \partial_y V_y = -\partial_x V_x = -\sigma_+$
$\sigma_{ab} =$
Square fluid element around origin, in frame $\vec{V}(0) \equiv 0$.
velocity around O :
$V_x = +\underbrace{V_x(0,0)}_{0} + \underbrace{\partial_x V_x(0,0)}_{\sigma_+} X$
$V_y = \underbrace{V_y(0,0)}_{0} + \underbrace{\partial_x V_y(0,0)}_{\sigma_\times} X$
y
x
V



0
σx
−σ+
Vx = σ+ X + σx y
Vy = σx X − σ+ y
σ+ : Vx = σ+ X , Vy = − σ+ y
y
X


$V_x = \sigma_+ x + \sigma_x y$
$V_y = \sigma_x x - \sigma_+ y$
$\sigma_+ :\ V_x = \sigma_+ x,\ V_y = -\sigma_+$
$\sigma_x :\ V_x = \sigma_x y$
$V_y = \sigma_x x$
y
x
y
x

Rotation: $\Theta = 0$, $\sigma_{ab} = 0$, $\omega_{ab} \neq 0$

$\downarrow$

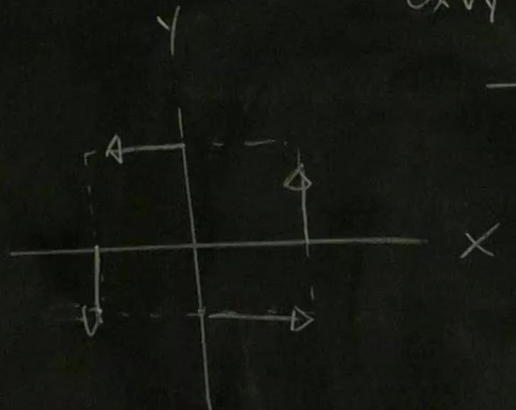
$\partial_x V_x + \partial_y V_y = 0$

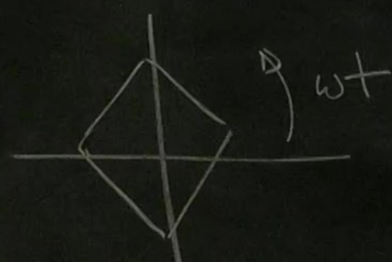
$\partial_x V_x = 0$

$\partial_y V_y = 0$

$\partial_x V_y + \partial_y V_x = 0$

$\rightarrow \partial_x V_y$





$$\omega_{xy} = \frac{1}{2}\left(\partial_x V_y - \partial_y V_x\right) = \omega_z$$

$$= \frac{1}{2}\left(\partial_x V_y + \partial_x V_y\right) \Rightarrow \begin{matrix} \partial_x V_y = \\ \partial_y V_x = \end{matrix}$$

$$\boxed{\begin{aligned} V_x &= -\omega_z y \\ V_y &= \omega_z x \end{aligned}}$$

$\vec{V} = \vec{\omega} \times \vec{r}$

$=$ rotati

$\vec{\omega} =$ angular v

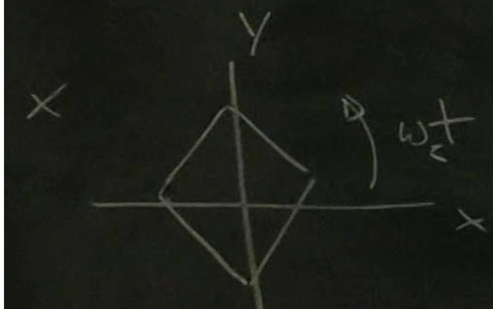
$= 0, \quad \omega_{ab} \neq 0$

$\partial_x V_x = 0$

$\partial_y V_y = 0$

$\partial_x V_y + \partial_y V_x = 0$

$\to \partial_x V_y$



$$\omega_{xy} = \frac{1}{2}(\partial_x V_y - \partial_y V_x) = \omega_z$$

$$= \frac{1}{2}(\partial_x V_y + \partial_x V_y) \Rightarrow \begin{array}{l} \partial_x V_y = \omega_z \\ \partial_y V_x = -\omega_z \end{array}$$

$$\boxed{\begin{array}{l} V_x = -\omega_z y \\ V_y = \omega_z x \end{array}}$$

$\vec{V} = \vec{\omega} \times \vec{r}$

= rotation

$\vec{\omega}$ = angular velocity ≡ rate of rotation

$\Theta = 0\ ,\ \sigma_{ab} = 0\ ,\ \omega_{ab} \neq 0$

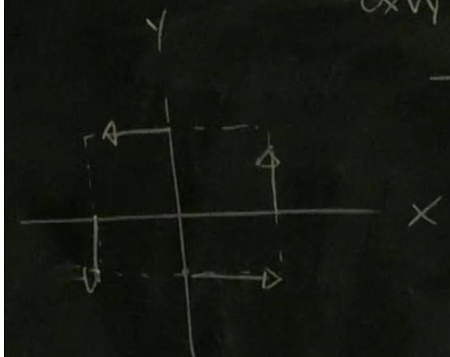
$\partial_x V_x + \partial_y V_y = 0$

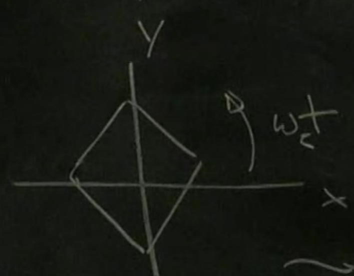
$\partial_x V_x = 0$

$\partial_y V_y = 0$

$\partial_x V_y + \partial_y V_x = 0$

$\rightarrow \partial_x V_y$





$\rightsquigarrow$ no change in volume

$$\omega_{xy} = \frac{1}{2}(\partial_x V_y - \partial_y V_x) = \omega_z$$

$$= \frac{1}{2}(\partial_x V_y + \partial_x V_y) \Rightarrow \begin{cases} \partial_x V_y = \omega_z \\ \partial_y V_x = -\omega_z \end{cases}$$

$$\boxed{\begin{aligned} V_x &= -\omega_z y \\ V_y &= \omega_z x \end{aligned}}$$

$$\vec{V} = \vec{\omega} \times \vec{r} = \text{rotation}$$

$\vec{\omega}$ = angular velocity ≡ rate of rotation

$$\Rightarrow \Theta = \vec{\nabla} \cdot \vec{V} = \partial_x V_x + \partial_y V_y$$

→ no change in volume.

$$\nabla_a V_b = \frac{1}{3}\Theta\,\delta_{ab} + \sigma_{ab} + \omega_{ab}$$

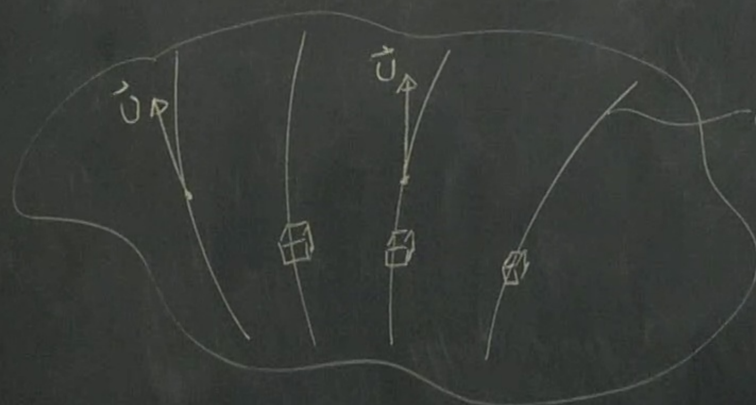
rate of expansion

rate of shear

rate of rotation

Spacetime velocity field

u^α : normalized velocity field , $g_{\alpha\beta}u^\alpha u^\beta = -1$, not necessarily geodesic.



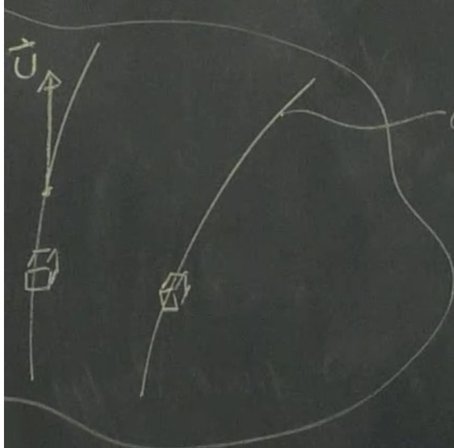
congruence of timelike worldlines, to which u^α is tangent
& of a fluid element.

Acceleration vector $\boxed{a^\alpha \equiv u^\beta \nabla_\beta u^\alpha}$ $\boxed{u_\alpha a^\alpha = 0}$

Proof: $u_\alpha a^\alpha = u_\alpha u^\beta \nabla_\beta u^\alpha = \frac{1}{2} u^\beta \nabla_\beta (u_\alpha u^\alpha)$

$= \frac{1}{2}(u^\beta \nabla_\beta u_\alpha) u^\alpha + \frac{1}{2} u$

eld, $g_{\alpha\beta}U^\alpha U^\beta = -1$, not necessarily geodesic.



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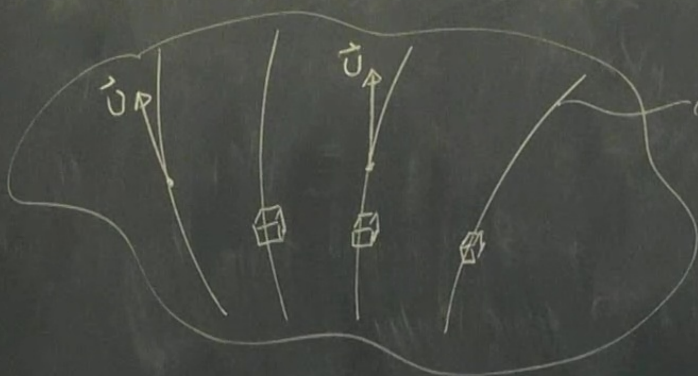
Proof: $U_\alpha a^\alpha = U_\alpha U^\beta \nabla_\beta U^\alpha = \frac{1}{2} U^\beta \nabla_\beta (U_\alpha U^\alpha)$

$$= \underbrace{\frac{1}{2}(U^\beta \nabla_\beta U_\alpha) U^\alpha}_{\frac{1}{2}(U^\beta \nabla_\beta U^\alpha) U_\alpha} + \frac{1}{2} U_\alpha (U^\beta \nabla_\beta U^\alpha)$$

Spacetime velocity field

u^α : normalized velocity field, $g_{\alpha\beta}u^\alpha u^\beta = -1$, not necessarily geodesic.



congruence of timelike worldlines, to which u^α is tangent

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Acceleration vector $\boxed{a^\alpha \equiv u^\beta \nabla_\beta u^\alpha}$ $\boxed{u_\alpha a^\alpha = 0}$

Proof: $u_\alpha a^\alpha = u_\alpha u^\beta \nabla_\beta u^\alpha = \frac{1}{2} u^\beta \nabla_\beta (\overbrace{u_\alpha u^\alpha}^{-1}) = 0$

$$= \underbrace{\frac{1}{2}(u^\beta \nabla_\beta u_\alpha) u^\alpha}_{\frac{1}{2}(u^\beta \nabla_\beta u^\alpha) u_\alpha} + \frac{1}{2} u_\alpha (u^\beta \nabla_\beta u^\alpha)$$

Projection oper

$$= \underbrace{\tfrac{1}{2}(U^{\beta}\nabla_{\beta}U_{\alpha})U}_{\tfrac{1}{2}(U^{\beta}\nabla_{\beta}U^{\alpha})U_{\alpha}} + \tfrac{1}{2}U_{\alpha}(U$$

Projection operator: $\boxed{P^{\alpha}{}_{\beta} \equiv \underset{\substack{\parallel\\ \delta^{\alpha}_{\beta}}}{g^{\alpha}{}_{\beta}} + U^{\alpha}U_{\beta}}$

$$P^{\alpha}{}_{\beta}U^{\beta} = (g^{\alpha}{}_{\beta} + U^{\alpha}U_{\beta})U^{\beta} = U^{\alpha} - U^{\alpha} = 0$$

$$\boxed{\begin{aligned} &P^{\alpha}{}_{\beta}P^{\beta}{}_{\gamma} = P^{\alpha}{}_{\gamma} \\ &P^{\alpha}{}_{\alpha} = 3 \end{aligned}}$$

$\to$ prove this!

$$\boxed{P^{\alpha}{}_{\beta}U^{\beta} = 0}$$

Decomposition of vector

U^α and its integral curves $\longrightarrow$ preferred time $\longrightarrow$ "time"

orthogonal directions $\longrightarrow$ preferred spacial directions $\longrightarrow$ "space"

$$\boxed{\underset{4}{A^\alpha} = \overset{1}{A} U^\alpha + \underset{3}{A^\alpha_\perp} \qquad U_\alpha A^\alpha_\perp = 0}$$

$$U_\alpha A^\alpha = A \underbrace{(U_\alpha U^\alpha)}_{-1} + \cancel{U_\alpha A^\alpha_\perp}$$

$$\Rightarrow A = - U_\alpha A^\alpha$$

Decomposition of vector

U^α and its integral curves $\longrightarrow$ preferred time $\longrightarrow$ "time"

orthogonal directions $\longrightarrow$ preferred spacial directions $\longrightarrow$ "space"

$$A^\alpha = A\, U^\alpha + A^\alpha_\perp$$

4 $\qquad$ 1 $\qquad$ 3

$$U_\alpha A^\alpha_\perp = 0$$

$$U_\alpha A^\alpha = A\, \underbrace{(U_\alpha U^\alpha)}_{-1} + \cancel{U_\alpha A^\alpha_\perp}$$

$$\Rightarrow A = -U_\alpha A^\alpha$$

$$\boxed{\begin{aligned} A &= -U_\alpha A^\alpha \\ A^\alpha_\perp &= P^\alpha{}_\beta A^\beta \end{aligned}}$$

$$P^\beta{}_\alpha A^\alpha = (\delta^\beta{}_\alpha + U^\beta U_\alpha)(A U^\alpha + A^\alpha_\perp)$$

$$= \cancel{A U^\beta} + A^\beta_\perp - \cancel{A U^\beta} + 0$$

$$= A^\beta_\perp$$