Title: Advanced General Relativity - 240117 (afternoon)

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

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URL: https://pirsa.org/24010005

Killing's eqn: $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

Geodesic eqn:

$$\begin{cases} t^\alpha = \dfrac{\partial x^\alpha}{\partial t} \rightarrow \dfrac{Dt^\alpha}{\partial t} = k\, t^\alpha \\[2ex] u^\alpha = \dfrac{\partial x^\alpha}{\partial \tau} \rightarrow \dfrac{Du^\alpha}{\partial \tau} = 0 \rightarrow \tau \rightarrow a\tau + b \end{cases}$$

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Geodesic eqn:

$$\begin{cases} t^\alpha = \dfrac{dx^\alpha}{dt} \to \dfrac{Dt^\alpha}{dt} = k\, t^\alpha \\[2ex] u^\alpha = \dfrac{dx^\alpha}{d\tau} \to \boxed{\dfrac{Du^\alpha}{d\tau} = 0} \to \tau \to \underbrace{a\tau + b}_{\text{"affine parameter"}} \end{cases}$$

$$\frac{Du^\alpha}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}$$

$$\to \boxed{\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}}$$

0

$$\frac{Dt^\alpha}{d t} = k\, t^\alpha$$

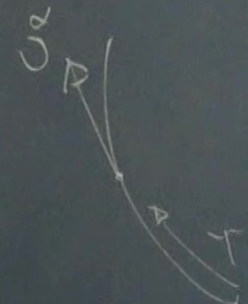
$$\to \boxed{\frac{Du^\alpha}{d\tau} = 0} \longrightarrow \tau \to \underbrace{a\tau + b}_{\text{"affine parameter"}}$$

$$\frac{Du^\alpha}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$

$$\to \boxed{\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0}$$

Conserved quantities

Assume: geodesic: $\frac{DU^\alpha}{d\tau} = 0$



Killing vector ξ^α

Example

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\xi^\alpha_{(t)} = (1,0,0,0)$$

Conserved quantity

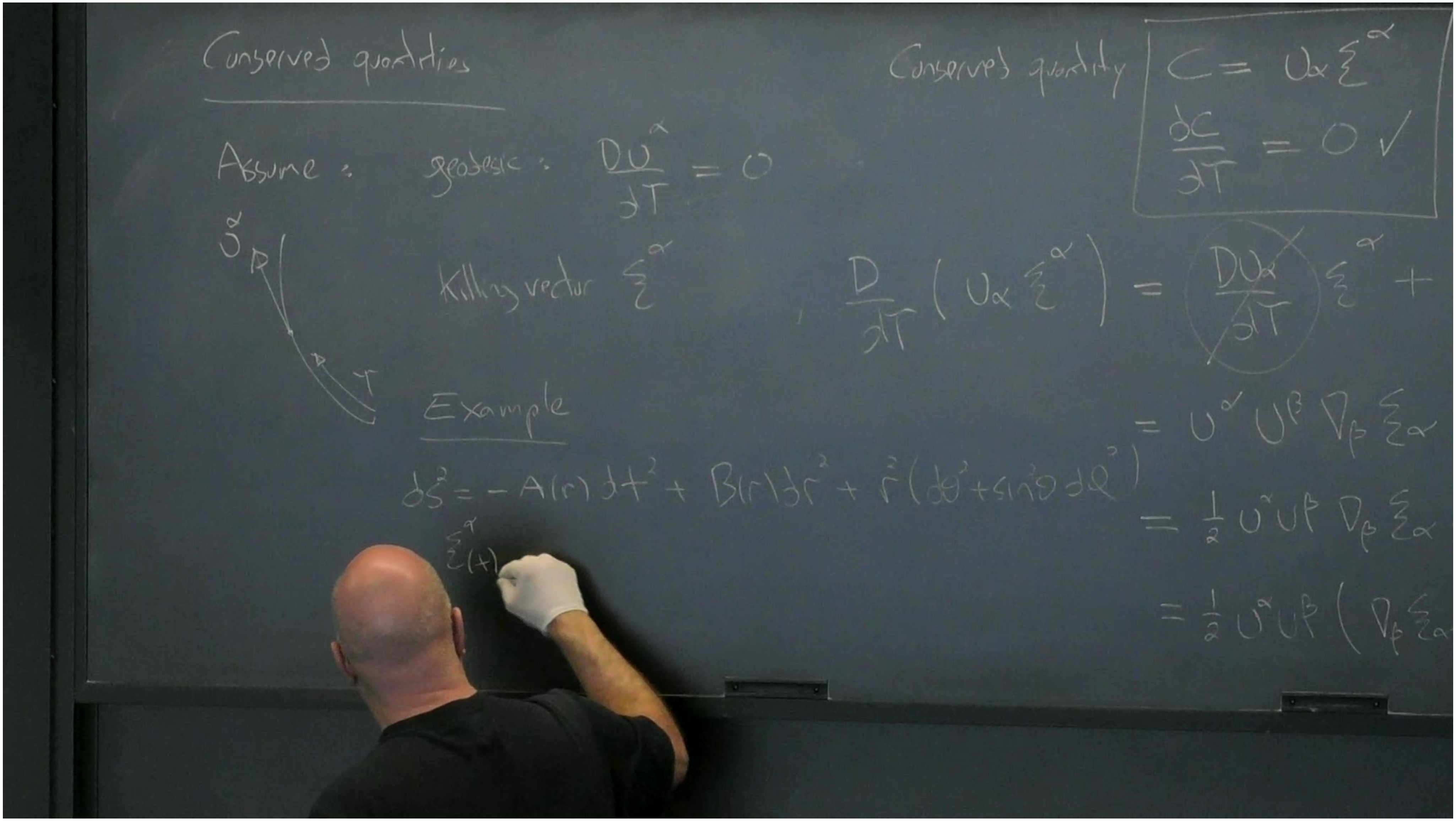
$$\boxed{\begin{aligned} C &= U_\alpha \xi^\alpha \\ \frac{dC}{d\tau} &= 0 \ \checkmark \end{aligned}}$$

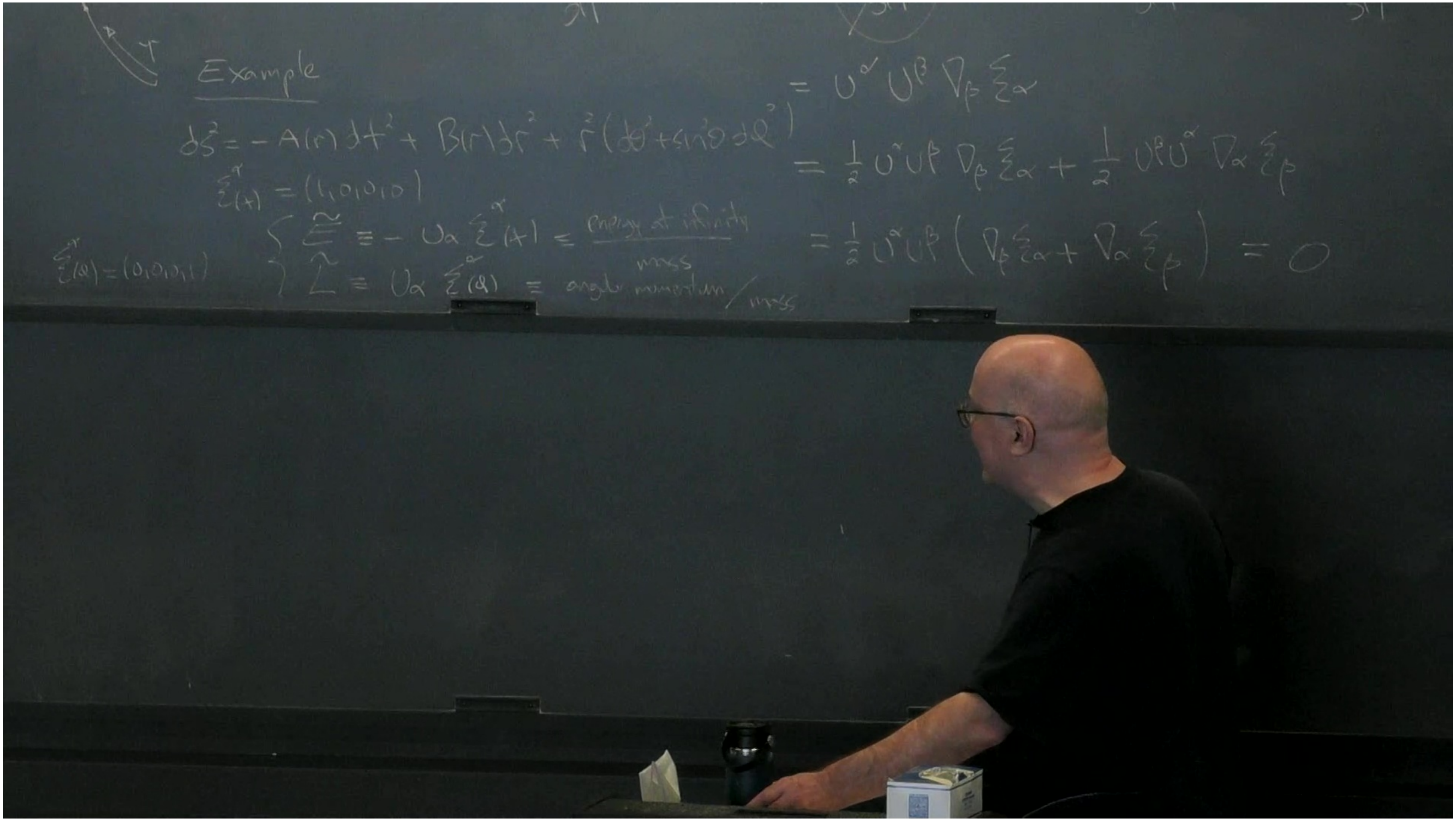
$$\frac{D}{d\tau}\left(U_\alpha \xi^\alpha\right) = \cancel{\frac{DU_\alpha}{d\tau}}\,\xi^\alpha +$$

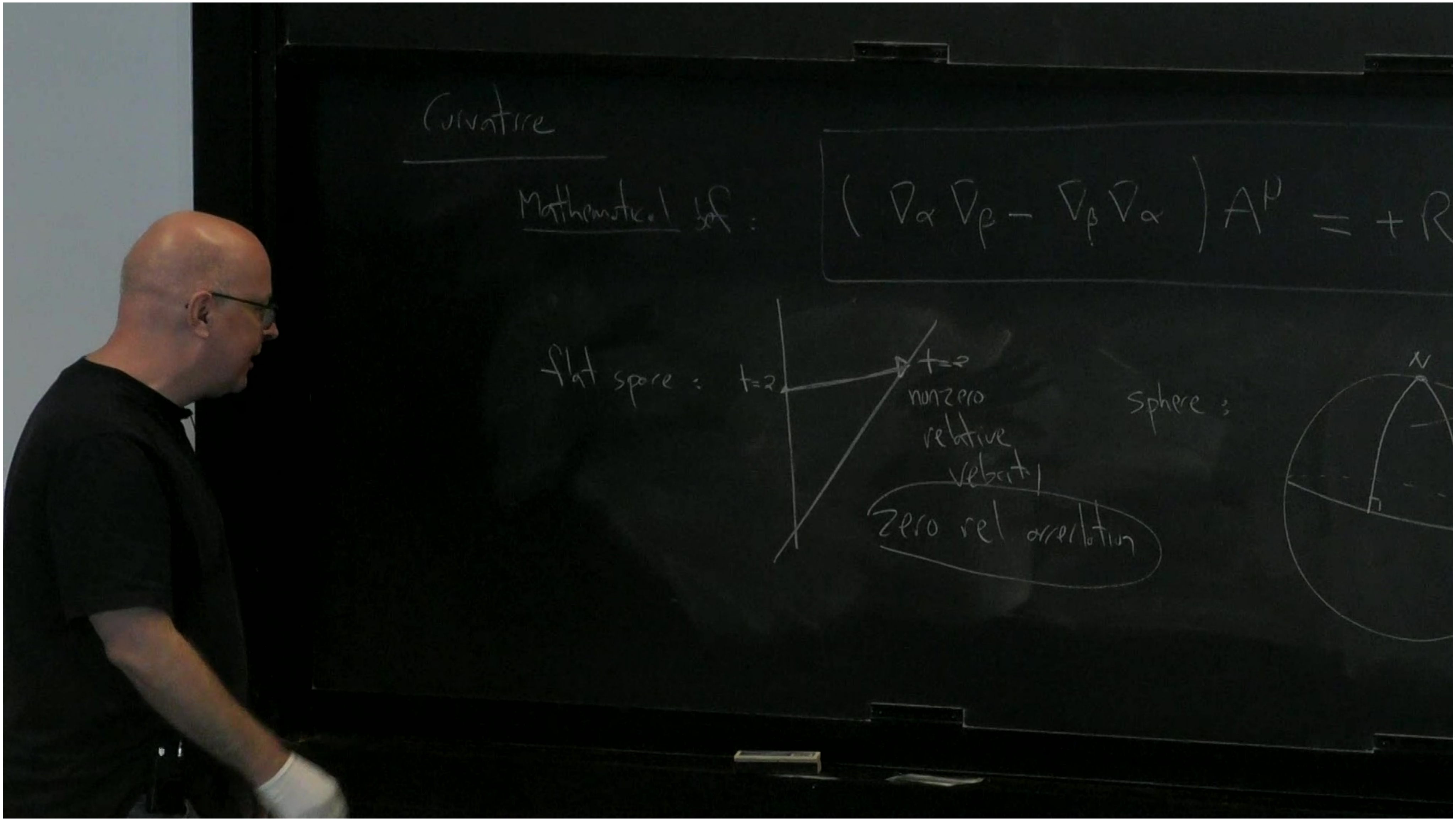
$$= U^\alpha U^\beta \nabla_\beta \xi_\alpha$$

$$= \frac{1}{2} U^\alpha U^\beta \nabla_\beta \xi_\alpha$$

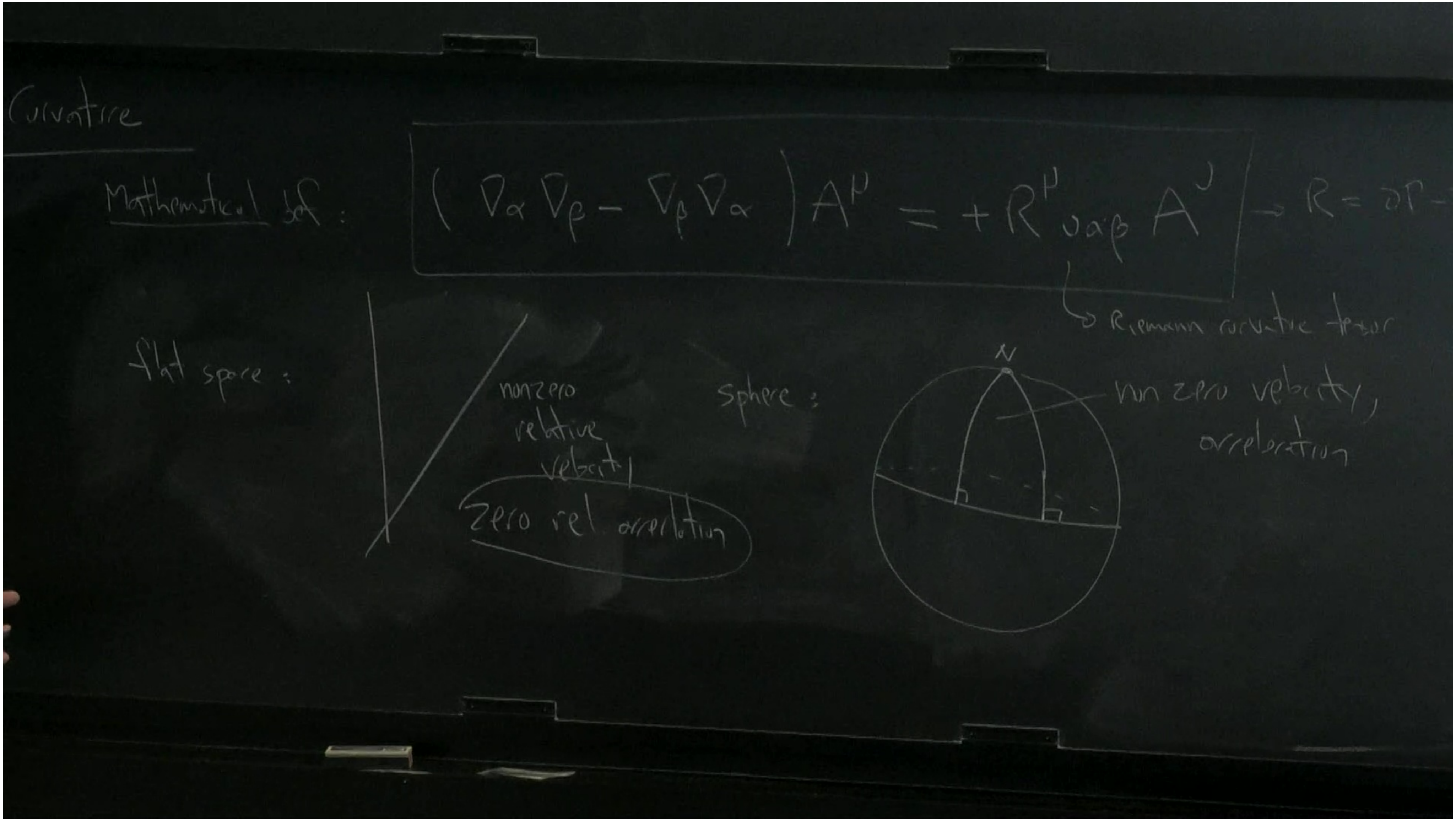
$$= \frac{1}{2} U^\alpha U^\beta \left(\nabla_\beta \xi_\alpha \right.$$


Conserved quantities
Assume :
geodesic : $\frac{DU^\alpha}{d\tau} = 0$
Killing vector ξ^α
Example
$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
$\xi^\alpha_{(t)}$
Conserved quantity
$C = U_\alpha \xi^\alpha$
$\frac{dC}{d\tau} = 0$ ✓
$\frac{D}{d\tau}(U_\alpha \xi^\alpha) = \frac{DU_\alpha}{d\tau}\xi^\alpha +$
$= U^\alpha U^\beta \nabla_\beta \xi_\alpha$
$= \frac{1}{2} U^\alpha U^\beta \nabla_\beta \xi_\alpha$
$= \frac{1}{2} U^\alpha U^\beta (\nabla_\beta \xi_\alpha$

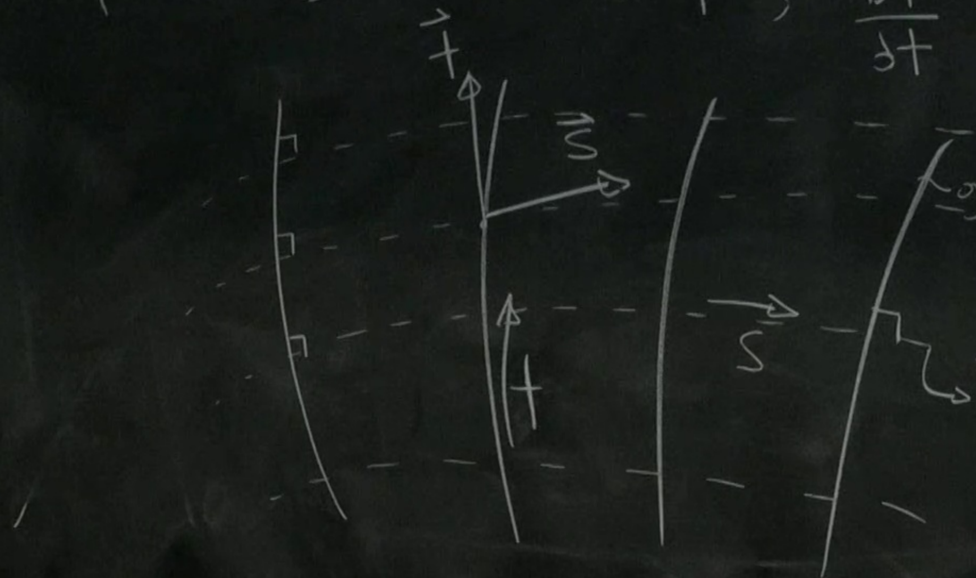

Example
$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
$\xi^\alpha_{(t)} = (1,0,0,0)$
$\xi^\alpha_{(\varphi)} = (0,0,0,1)$
$\tilde{E} \equiv -U_\alpha \xi^\alpha_{(t)} = \frac{\text{energy at infinity}}{\text{mass}}$
$\tilde{L} \equiv U_\alpha \xi^\alpha_{(\varphi)} = \text{angular momentum/mass}$
$= U^\alpha U^\beta \nabla_\beta \xi_\alpha$
$= \frac{1}{2} U^\alpha U^\beta \nabla_\beta \xi_\alpha + \frac{1}{2} U^\beta U^\alpha \nabla_\alpha \xi_\beta$
$= \frac{1}{2} U^\alpha U^\beta (\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta) = 0$



Curvature
Mathematical def:
$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = +R$
flat space:
t=2
nonzero
relative
velocity
zero rel acceleration
sphere:
N


Curvature
Mathematical def:
$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = + R^\mu{}_{\nu\alpha\beta} A^\nu} \to R = \partial\Gamma -$
Riemann curvature tensor
flat space:
nonzero relative velocity
zero rel acceleration
sphere:
N
non zero velocity, acceleration

Geometrical set up: geodesic vector field t^α ; $\frac{Dt^\alpha}{dt} = 0$; $\boxed{t^\beta \nabla_\beta t^\alpha = 0}$



geodesics (param t)
↳ tangent vector t^α

curves (param s)
↳ tangent vector s^α

Geometrical set up:

two-parameter family of curves:

$$x^\alpha(t,s)$$

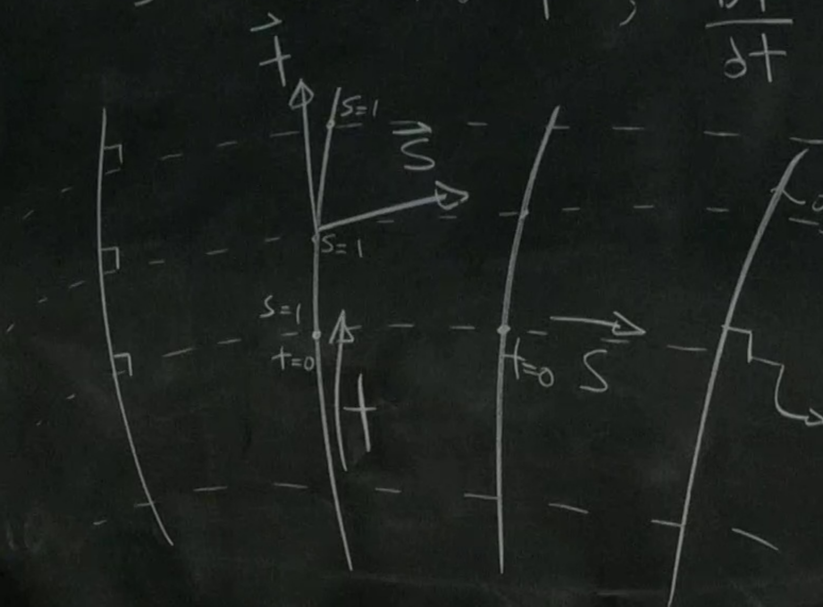
s fixed, t varies → geodesic

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_s$$

t fixed, s varies → cross curves

$$s^\alpha = \left(\frac{\partial x^\alpha}{\partial s}\right)_t$$

geodesic vector field t^α ; $\frac{Dt^\alpha}{dt} = 0$; $t^\beta \nabla_\beta$



geodesics (param t

↳ tangent vector

curves (param s

↳ tangent vector

≡ de

$L \equiv U_\alpha \, \Sigma(\alpha) \equiv$ angular momentum / mass

$$\mathcal{L}_s t^\alpha = 0 = \mathcal{L}_t s^\alpha \rightarrow \boxed{s^\beta \nabla_\beta t^\alpha = t^\beta \nabla_\beta s^\alpha}$$

relative separation : s^α

relative velocity : $\dfrac{Ds^\alpha}{dt} = t^\beta \nabla_\beta s^\alpha$

relative acceleration : $\underbrace{\dfrac{D}{dt}\left(\dfrac{Ds^\alpha}{dt}\right)}_{\frac{D^2 s^\alpha}{dt^2}} = t^\gamma \nabla_\gamma \left(t^\beta \nabla_\beta s^\alpha\right)$

$$\frac{D^2 S^\alpha}{d t^2} = t^\gamma \nabla_\gamma \left(S^\beta \nabla_\beta t^\alpha \right) = \left(t^\gamma \nabla_\gamma S^\beta \right) \nabla_\beta t^\alpha + t^\gamma S^\beta \nabla_\gamma \nabla_\beta t^\alpha$$

$$= \left(t^\gamma \nabla_\gamma S^\beta \right) \nabla_\beta t^\alpha + t^\gamma S^\beta \underbrace{\left(\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma \right) t^\alpha}_{R^\alpha{}_{\mu\gamma\beta} t^\mu} + \underbrace{t^\gamma S^\beta \nabla_\beta \nabla_\gamma t^\alpha}_{S^\beta \nabla_\beta (\underbrace{t^\gamma \nabla_\gamma t^\alpha}_{0}) - (}$$

$$= \left(\underbrace{t^\beta \nabla_\beta S^\gamma - S^\beta \nabla_\beta t^\gamma}_{0} \right) \nabla_\gamma t^\alpha + R^\alpha{}_{\mu\gamma\beta} t^\mu t^\gamma S^\beta$$

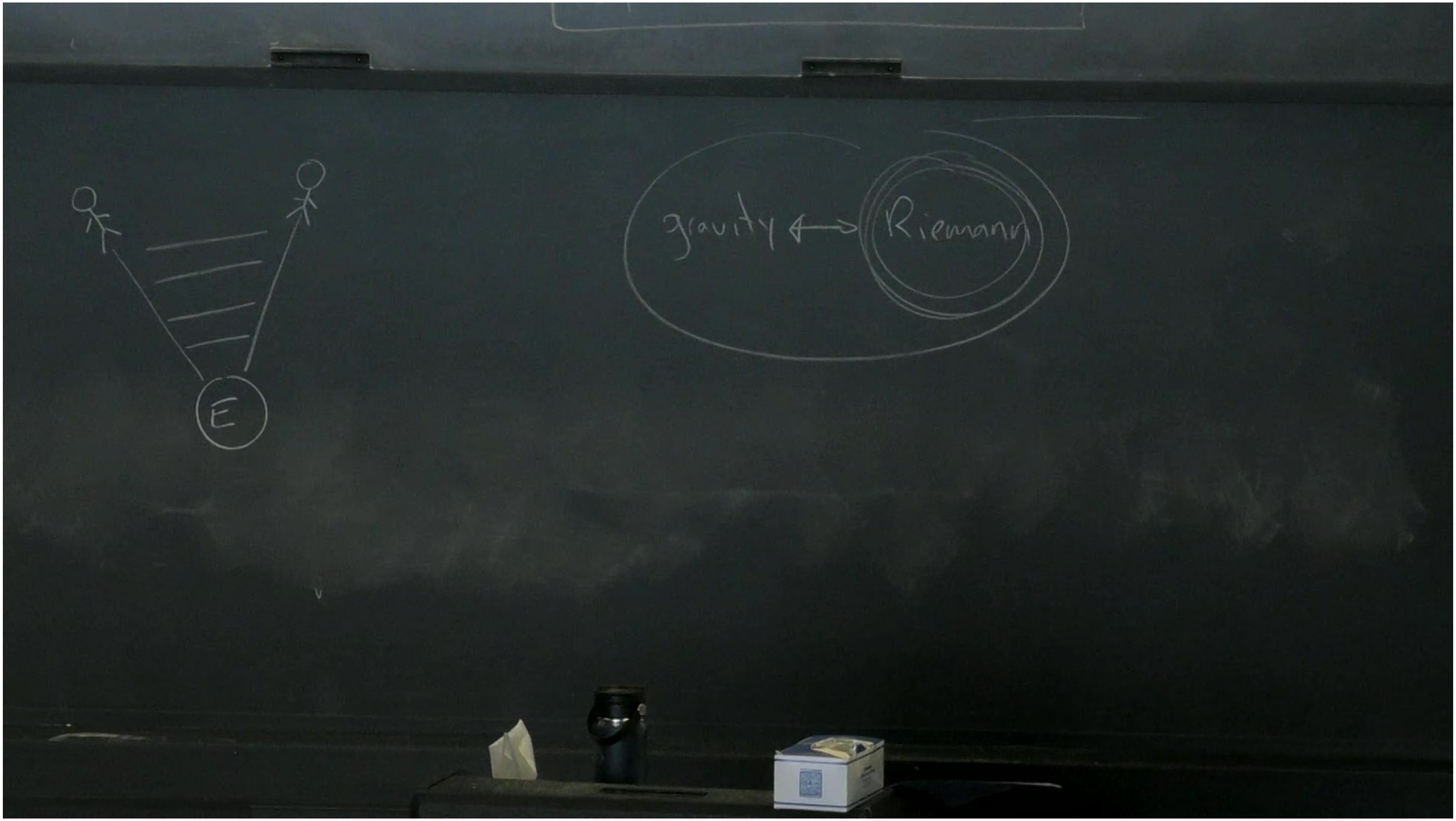
$$\boxed{\frac{D^2 S^\alpha}{d t^2} = - R^\alpha{}_{\mu\beta\gamma} t^\mu S^\beta t^\gamma}$$

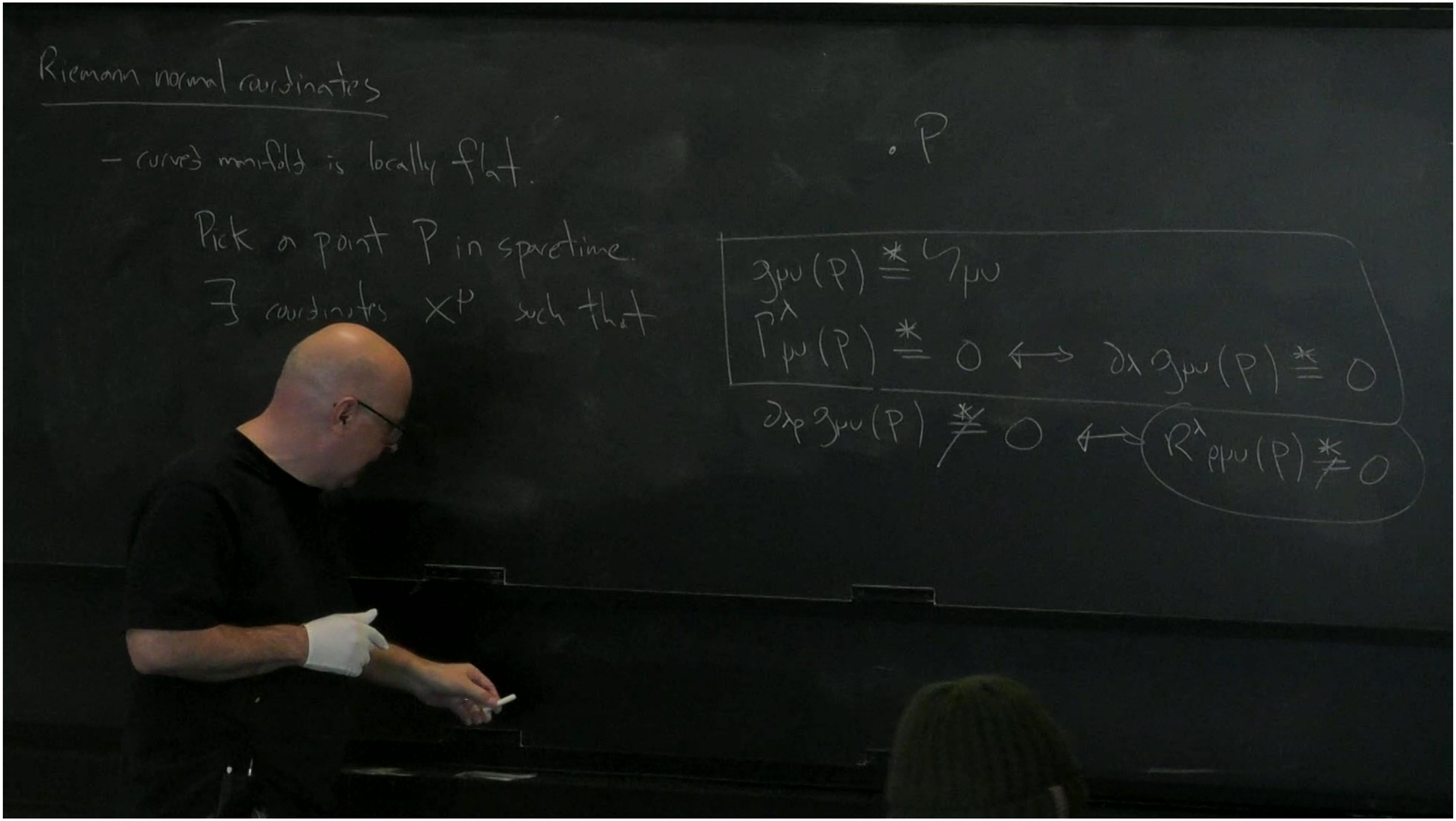
$$\ldots) = (t^\gamma \nabla_\gamma s^\beta)\nabla_\beta t^\alpha + t^\gamma s^\beta \nabla_\gamma \nabla_\beta t^\alpha$$

$$\ldots + t^\gamma s^\beta \underbrace{(\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma) t^\alpha}_{R^\alpha{}_{\mu\gamma\beta}\, t^\mu} + \underbrace{t^\gamma s^\beta \nabla_\beta \nabla_\gamma t^\alpha}_{s^\beta \nabla_\beta (\underbrace{t^\gamma \nabla_\gamma t^\alpha}_{0}) - (s^\beta \nabla_\beta t^\gamma)\nabla_\gamma t^\alpha}$$

$$\ldots s^\beta \nabla_\beta t^\gamma)\nabla_\gamma t^\alpha + R^\alpha{}_{\mu\gamma\beta}\, t^\mu t^\gamma s^\beta$$

$$\boxed{\frac{D^2 s^\alpha}{Dt^2} = -R^\alpha{}_{\mu\beta\gamma}\, t^\mu s^\beta t^\gamma}$$


gravity ⟷ Riemann
E


Riemann normal coordinates
- curved manifold is locally flat.
Pick a point P in spacetime.
∃ coordinates x^ρ such that
• P
$g_{\mu\nu}(P) \overset{*}{=} \eta_{\mu\nu}$
$\Gamma^\lambda_{\mu\nu}(P) \overset{*}{=} 0 \longleftrightarrow \partial_\lambda g_{\mu\nu}(P) \overset{*}{=} 0$
$\partial_{\lambda\rho} g_{\mu\nu}(P) \overset{*}{\neq} 0 \longleftrightarrow R^\lambda{}_{\rho\mu\nu}(P) \overset{*}{\neq} 0$