Title: Advanced General Relativity - 240110 (afternoon)

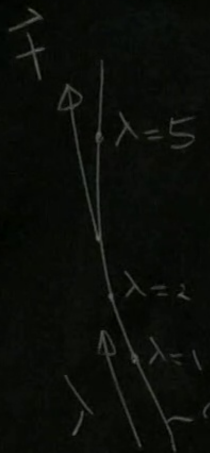
Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

Date: January 10, 2024 - 1:30 PM

URL: https://pirsa.org/24010004

Covariant derivation along a curve



$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma$$

curve γ — $x^\alpha = X^\alpha(\lambda)$

$$t^\alpha = \frac{dX^\alpha}{d\lambda}$$

$$\frac{DV^\beta}{d\lambda} = t^\alpha \nabla_\alpha V^\beta$$ = cov deriv along curve.

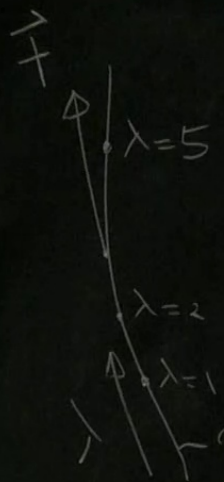
$$\frac{DV^\beta}{\partial\lambda} = t^\alpha \nabla_\alpha V^\beta = \text{cov deriv along curve.}$$

$$= \frac{\partial x^\alpha}{\partial\lambda} \nabla_\alpha V^\beta$$

$$= \frac{\partial x^\alpha}{\partial\lambda} \left(\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\beta \right) = \frac{dV^\beta}{\partial\lambda} + \Gamma^\beta_{\alpha\gamma} t^\alpha V^\beta$$

Covariant derivative along a curve

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma$$



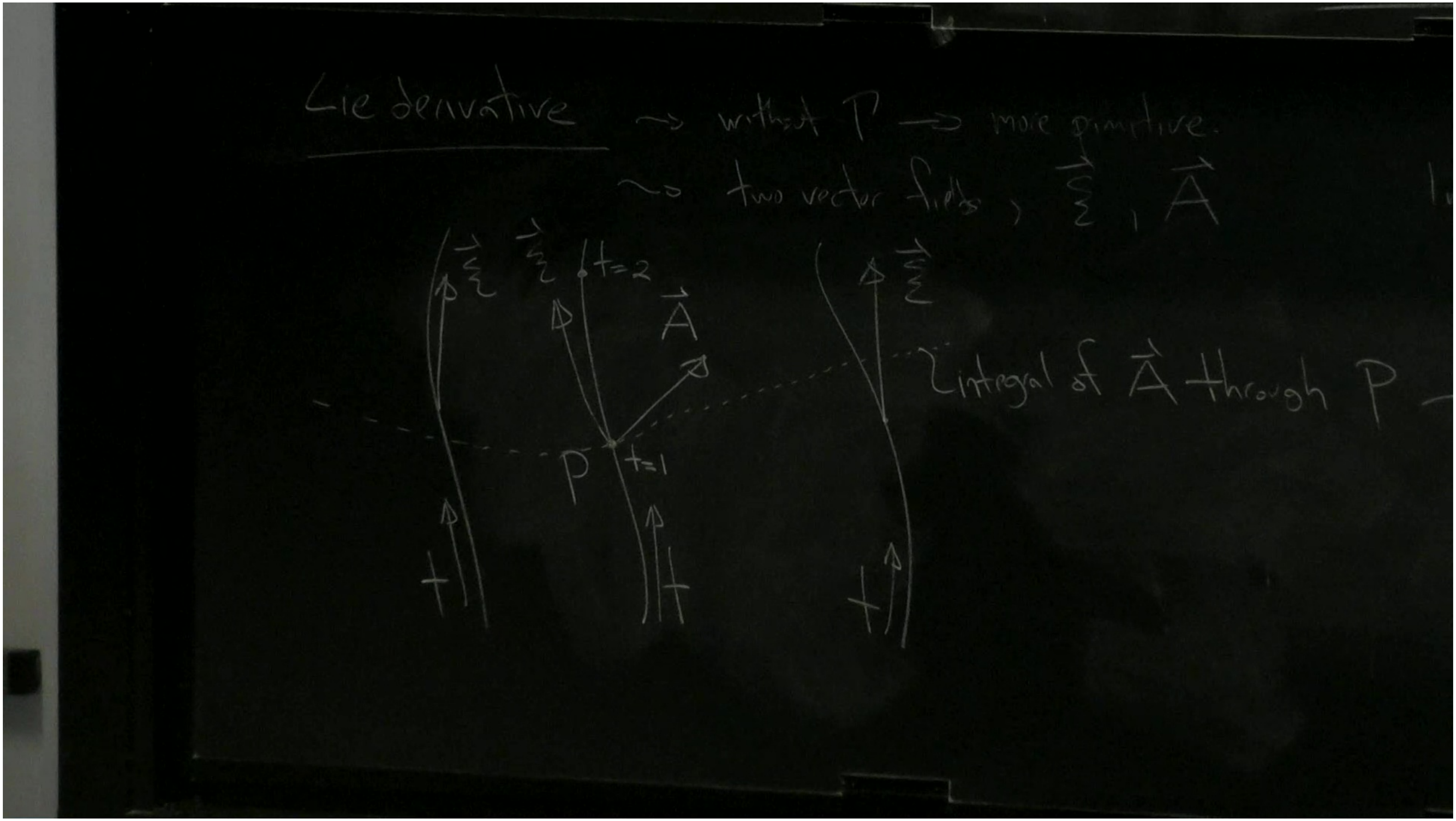
curve γ — $x^\alpha = X^\alpha(\lambda)$

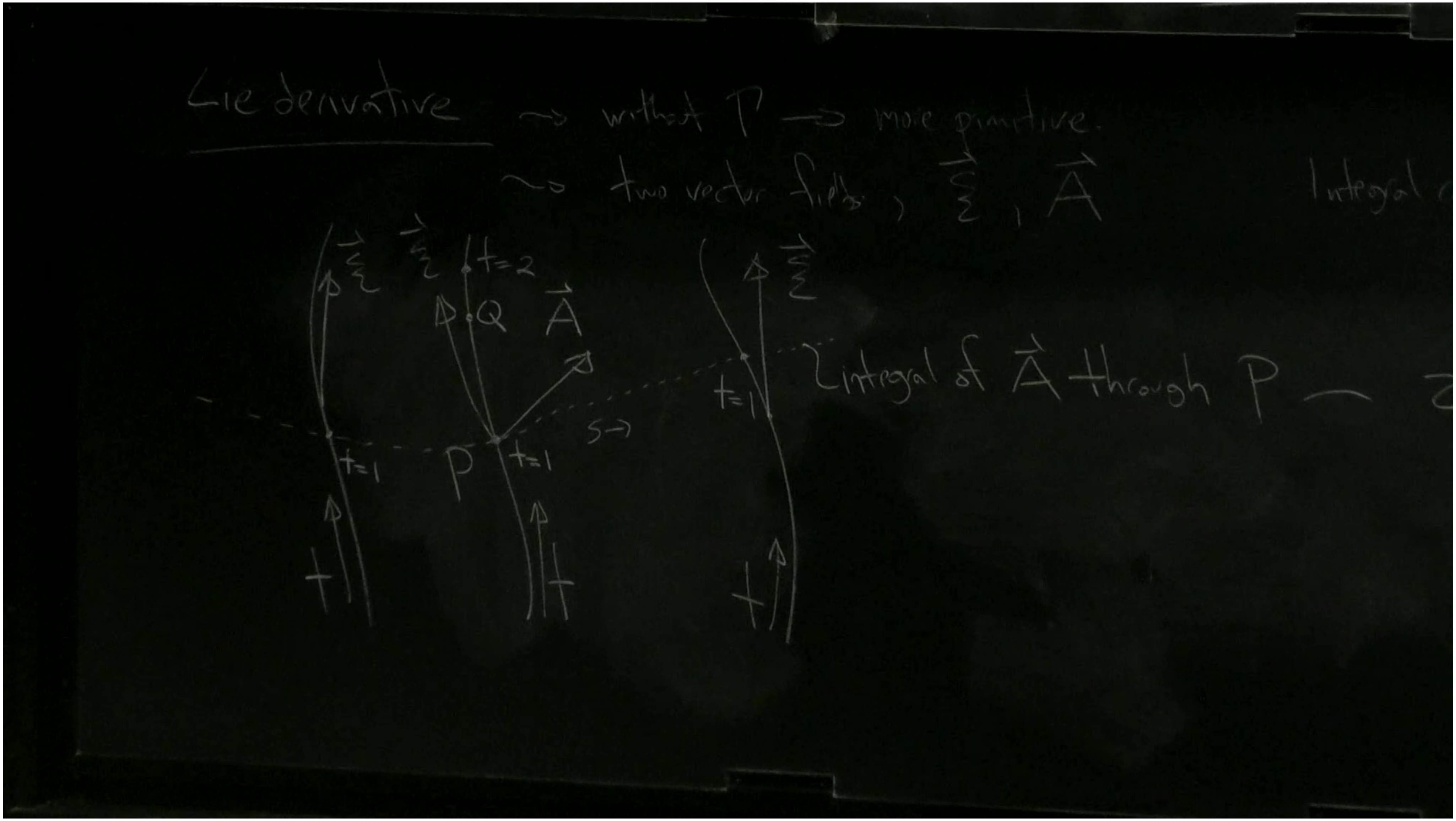
$$t^\alpha = \frac{dX^\alpha}{d\lambda}$$

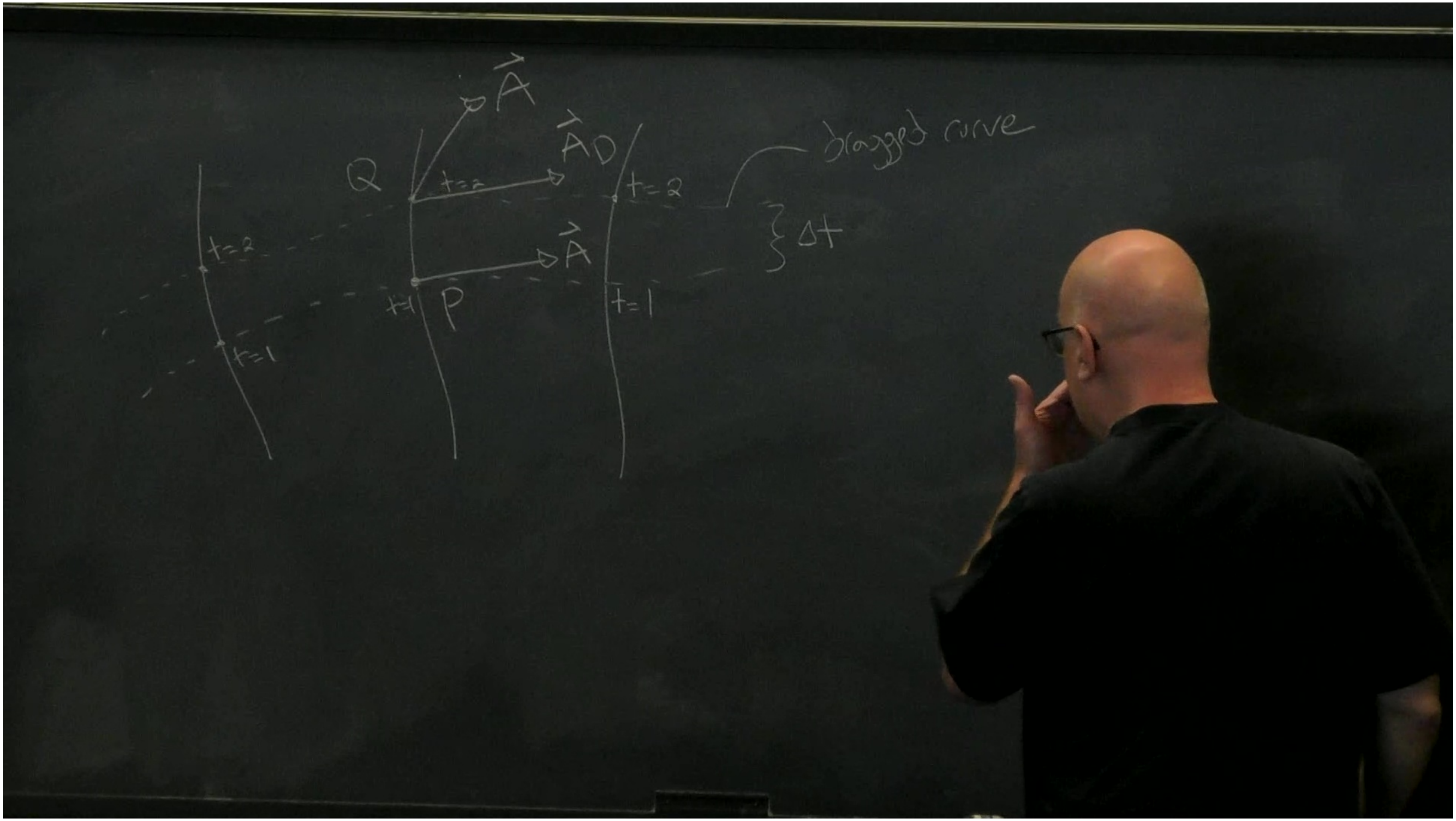
$$\frac{DV^\beta}{d\lambda} = t^\alpha(\nabla_\alpha V^\beta) = \text{cov deriv along curve.}$$

$$= \frac{dX^\alpha}{d\lambda} \nabla_\alpha V^\beta$$

$$= \frac{dX^\alpha}{d\lambda}\left(\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma\right) = \frac{dV^\beta}{d\lambda} +$$


Lie derivative
without P → more primitive.
two vector fields, $\vec{\xi}$, $\vec{A}$
$\vec{\xi}$
$t=2$
$\vec{A}$
Integral of $\vec{A}$ through P
P
$t=1$
t


Lie derivative → without Γ → more primitive.
→ two vector fields, $\vec{\xi}$, $\vec{A}$
Integral
$\vec{\xi}$
$\vec{\xi}$
t=2
Q
$\vec{A}$
$\vec{\xi}$
{ integral of $\vec{A}$ through P
t=1
s→
t=1
P
t=1
t
t
t


$\vec{A}$
$\vec{A}_D$
dragged curve
Q
t=2
t=2
t=2
Δt
$\vec{A}$
t=1
P
t=1
t=1

P → more primitive.

ector fields, $\vec{\xi}$, $\vec{A}$

Integral curves of $\vec{\xi}$, $X^\alpha(t)$

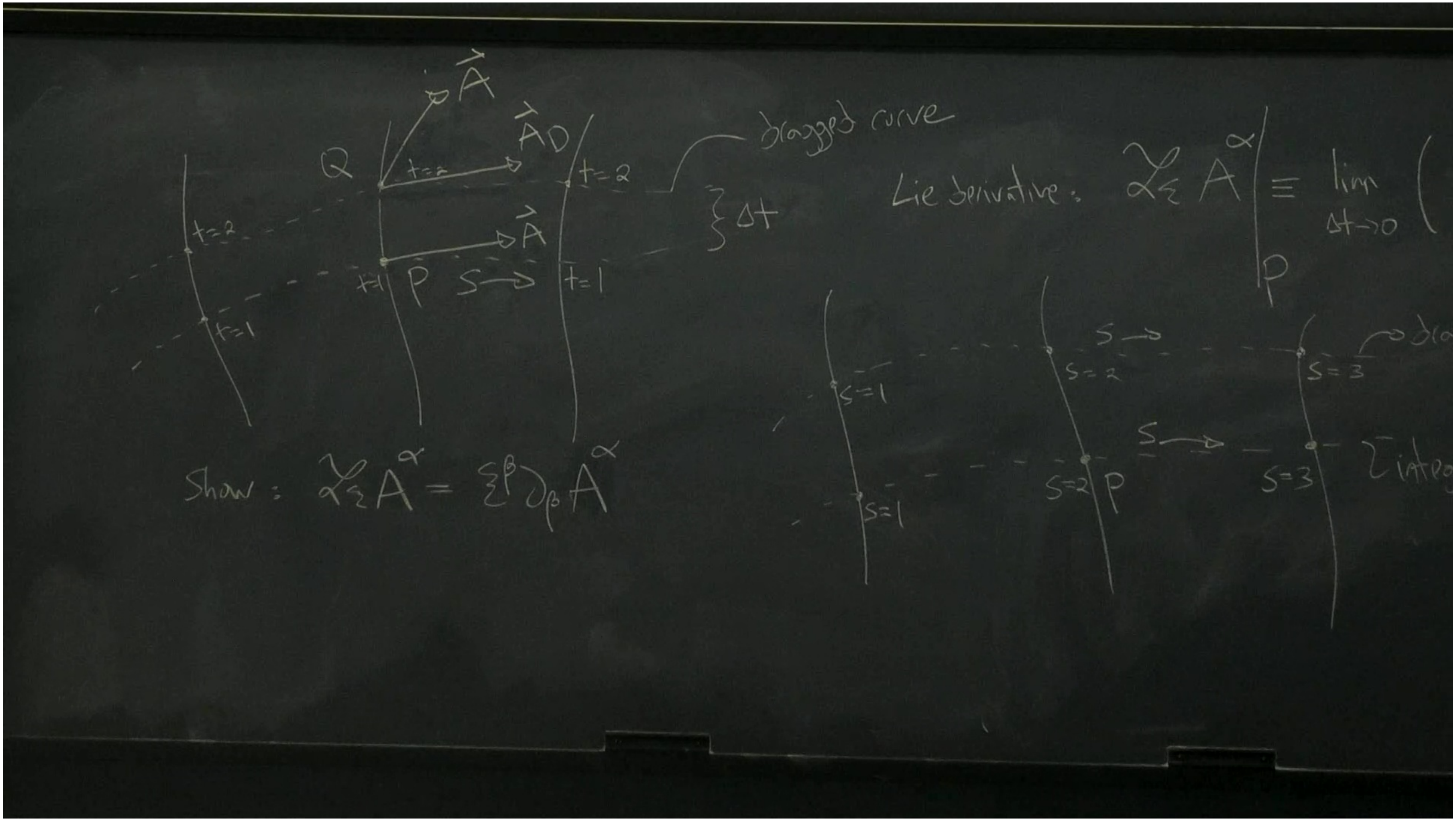
$\xi^\alpha = \partial X^\alpha / \partial t$

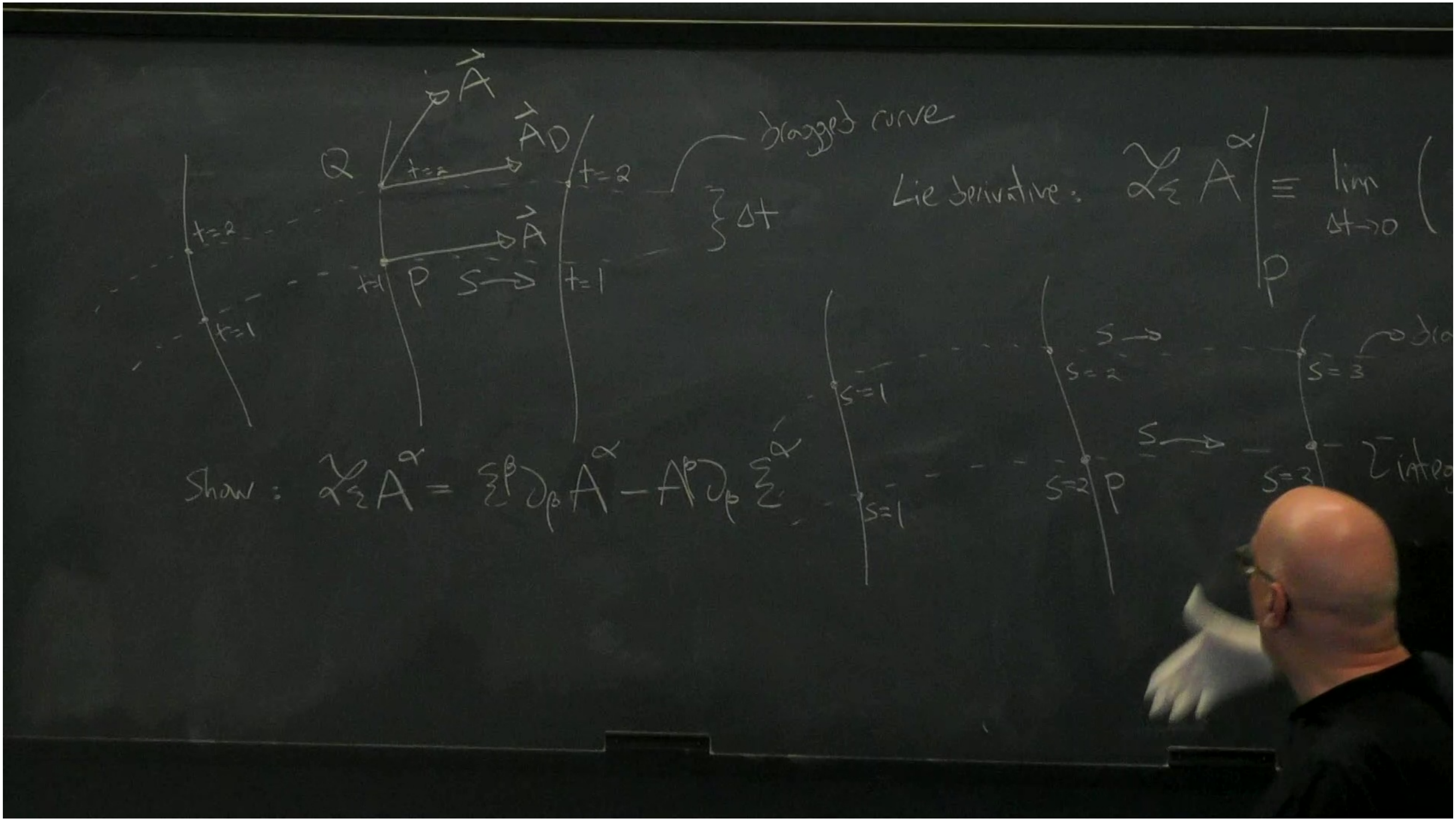
$\vec{\xi}$

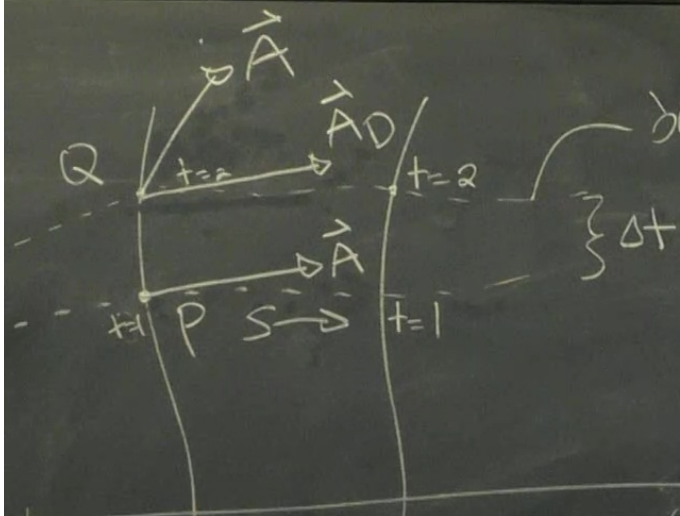
$t=1$

Integral of $\vec{A}$ through P — $Z^\alpha(s)$, $A^\alpha = \underline{\partial Z^\alpha / \partial s}$

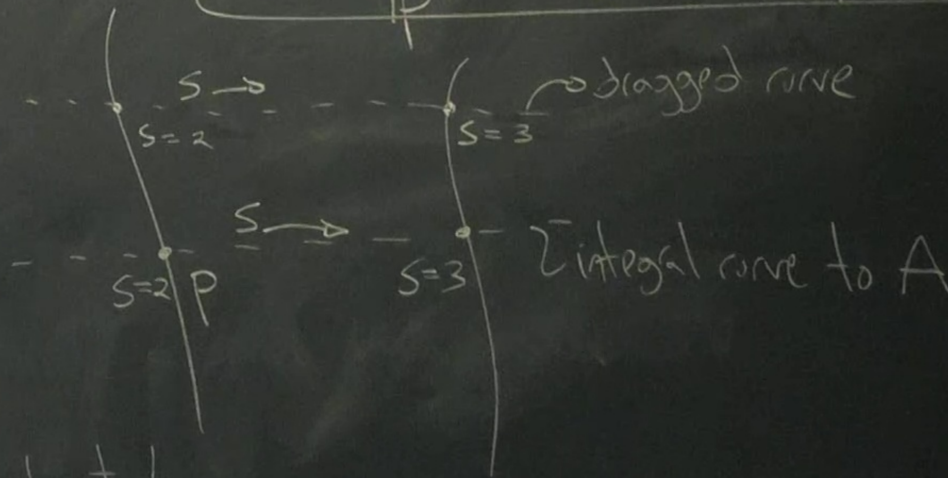
t


dragged curve
$\{\Delta t$
Lie derivative: $\mathcal{L}_\xi A^\alpha \big|_P \equiv \lim_{\Delta t \to 0}$ (
Show: $\mathcal{L}_\xi A^\alpha = \xi^\beta \partial_\beta A^\alpha$
$\vec{A}$
$\vec{A}_D$
Q
P
t=1
t=2
s=1
s=2
s=3


dragged curve
Lie derivative: $\mathcal{L}_\xi A^\alpha \big|_P \equiv \lim_{\Delta t \to 0}$ (
Δt
t=2
t=1
Q
P
s=1
s=2
s=3
Show: $\mathcal{L}_\xi A^\alpha = \xi^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \xi^\alpha$



Lie derivative: $\boxed{\mathcal{L}_\xi A^\alpha\Big|_P \equiv \lim_{\Delta t\to 0} \left(\frac{A^\alpha(Q) - A_D^\alpha(Q)}{\Delta t} \right)}$



$$\boxed{\begin{aligned}\mathcal{L}_\xi A^\alpha &= \xi^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \xi^\alpha \\ &= \xi^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta \xi^\alpha \\ &= -\mathcal{L}_A \xi^\alpha\end{aligned}}$$

↳ Γ cancel out !

$$= -\mathcal{L}_A \xi^\alpha$$

Γ cancel out!

Covectors

$$\mathcal{L}_\xi(\omega_\alpha V^\alpha) = \xi^\beta\partial_\beta(\omega_\alpha V^\alpha) = (\xi^\beta\partial_\beta\omega_\alpha)V^\alpha + \omega_\alpha(\xi^\beta\partial_\beta V^\alpha)$$

$$\mathcal{L}_\xi(\omega_\alpha V^\alpha) = (\mathcal{L}_\xi\omega_\alpha)V^\alpha + \omega_\alpha(\mathcal{L}_\xi V^\alpha) = (\mathcal{L}_\xi\omega_\alpha)V^\alpha + \omega_\alpha(\xi^\beta\partial_\beta V^\alpha - V^\beta\partial_\beta\xi^\alpha)$$

$$(\mathcal{L}_\xi\omega_\alpha)V^\alpha = -\omega_\alpha\xi^\beta\partial_\beta V^\alpha + \omega_\alpha V^\beta\partial_\beta\xi^\alpha + (\xi^\beta\partial_\beta\omega_\alpha)V^\alpha + \omega_\alpha\xi^\beta\partial$$

$$= -\mathcal{L}_A \xi^\alpha$$

Γ cancel out!

$$\mathcal{L}_\xi(\omega_\alpha V^\alpha) = (\mathcal{L}_\xi \omega_\alpha) V^\alpha + \omega_\alpha(\mathcal{L}_\xi V^\alpha) = (\mathcal{L}_\xi \omega_\alpha) V^\alpha + \omega_\alpha(\xi^\beta \partial_\beta V^\alpha - V^\beta \partial_\beta \xi^\alpha)$$

$$(\mathcal{L}_\xi \omega_\alpha) V^\alpha = -\omega_\alpha \cancel{\xi^\beta} \partial_\beta V^\alpha + \omega_\alpha V^\beta \partial_\beta \xi^\alpha + (\xi^\beta \partial_\beta \omega_\alpha) V^\alpha + \omega_\alpha \cancel{\xi^\beta} \partial_\beta V^\alpha$$

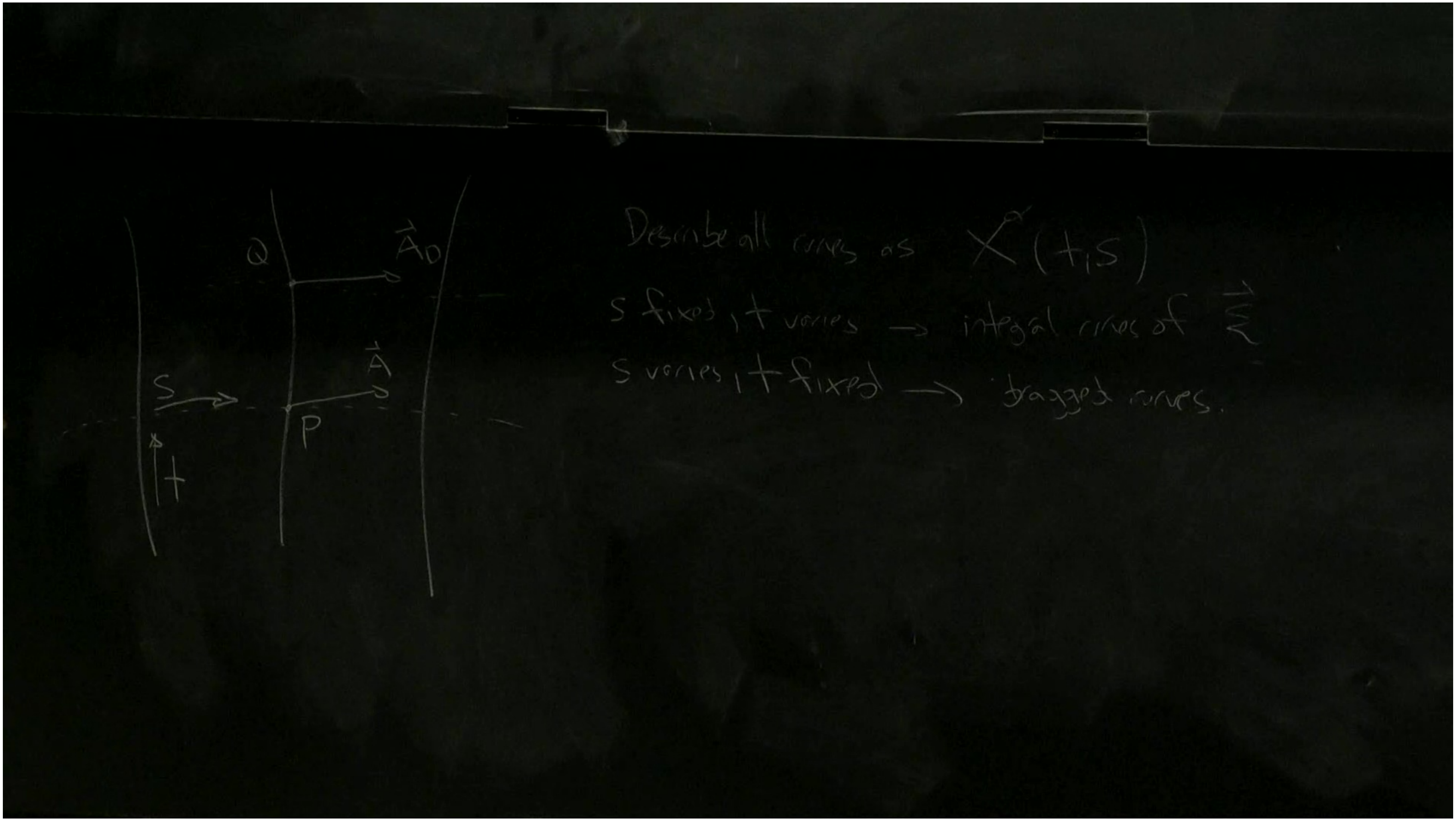
$$= (\xi^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta) V^\alpha \to \boxed{\begin{aligned} \mathcal{L}_\xi \omega_\alpha &= \xi^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta \\ &= \xi^\beta \nabla_\beta \omega_\alpha + \omega_\beta \nabla_\alpha \xi^\beta \end{aligned}}$$

(Γ's cancel out!)

$$\mathcal{L}_\xi T^{\alpha\beta} = \xi^\gamma \partial_\gamma T^{\alpha\beta} - T^{\gamma\beta}\partial_\gamma \xi^\alpha - T^{\alpha\gamma}\partial_\gamma \xi^\beta$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta}\partial_\alpha \xi^\beta + T_{\alpha\gamma}\partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$


Q
$\vec{A}_Q$
$\vec{A}$
s
P
t
Describe all curves as $X^\alpha(t,s)$
s fixed, t varies → integral curves of $\vec{\xi}$
s varies, t fixed → dragged curves.

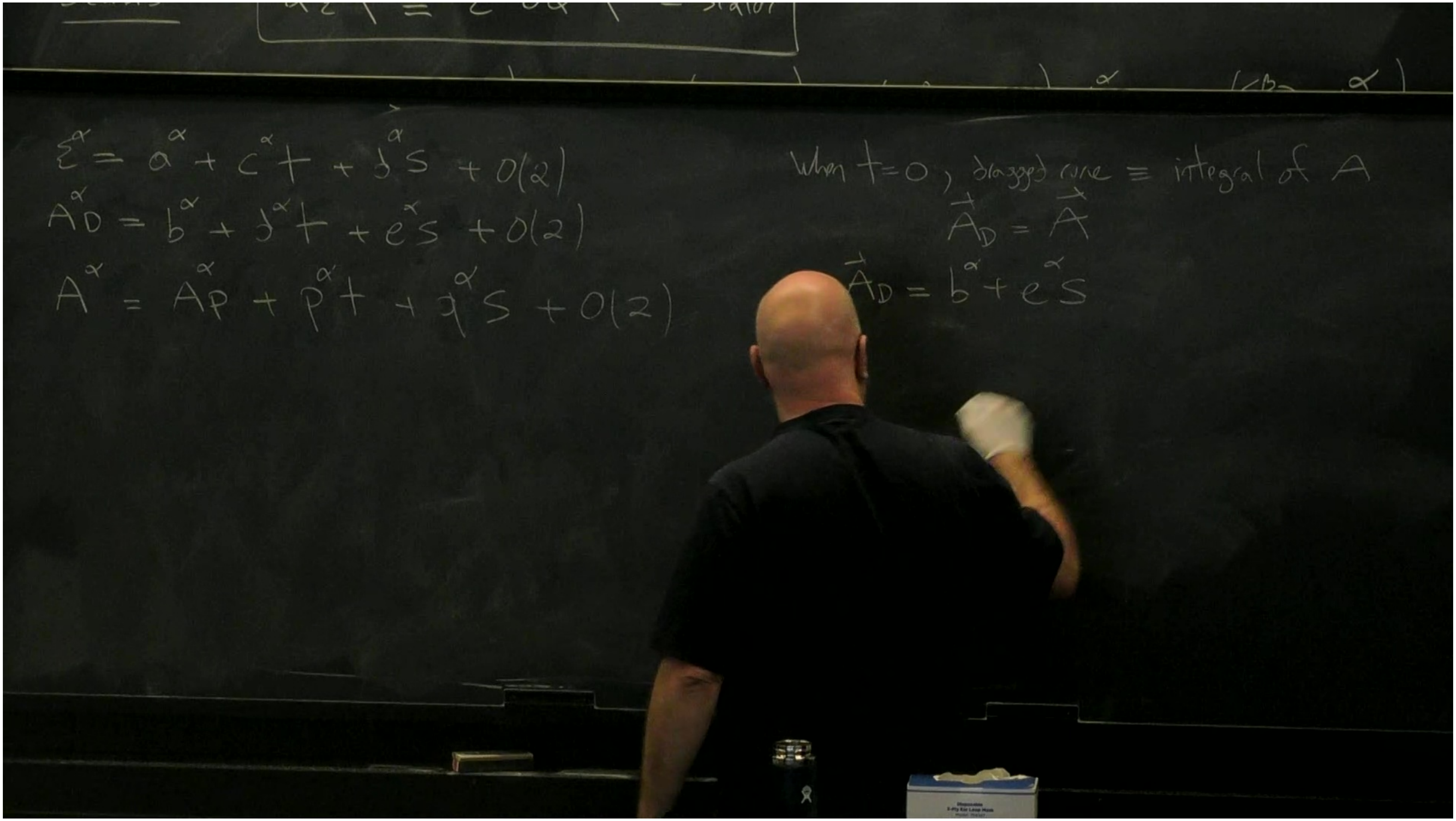
Describe all curves as $X^\alpha(t,s)$

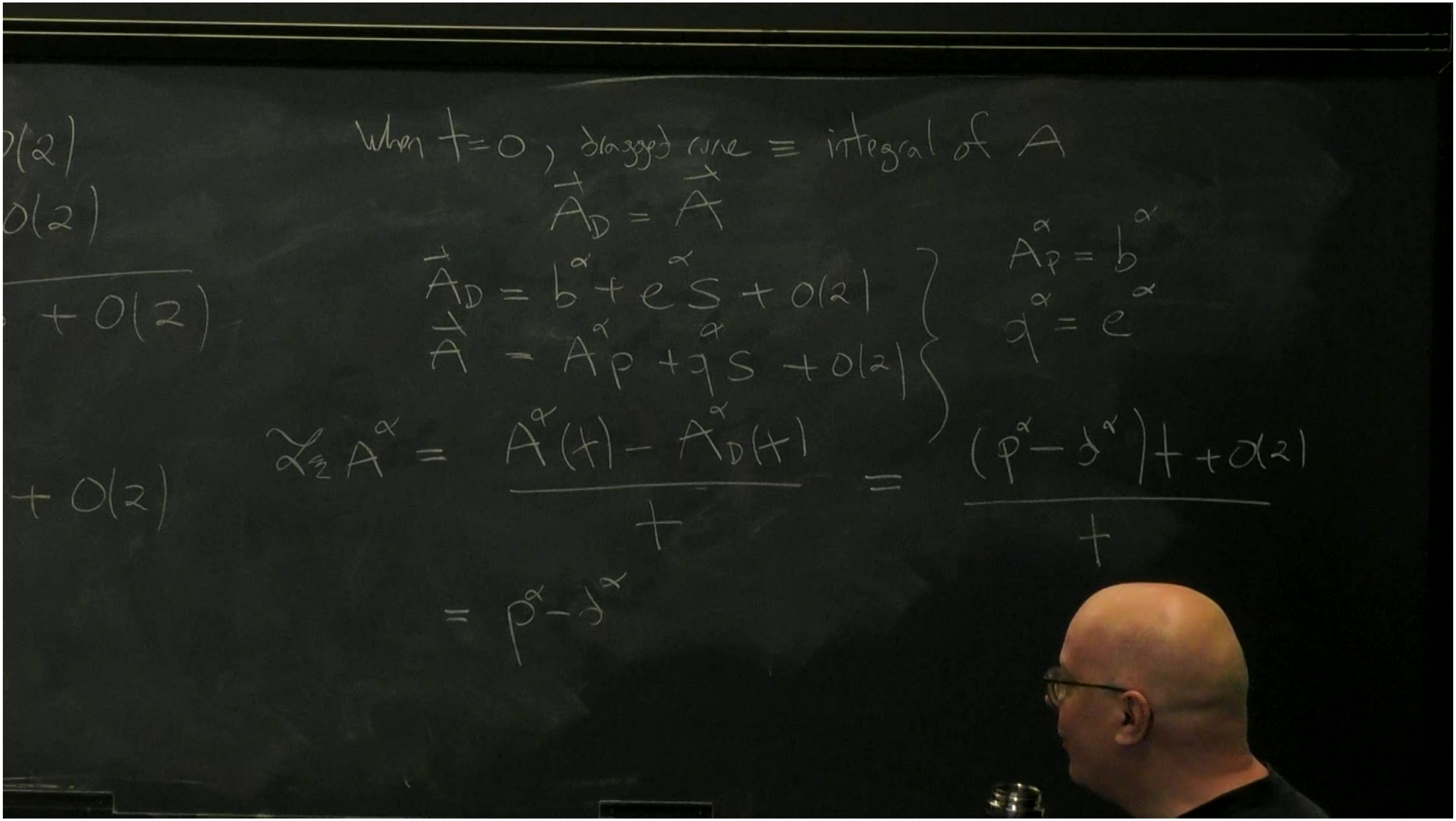
s fixed, t varies $\rightarrow$ integral curves of $\vec{\xi}$ $\rightarrow$ $\xi^\alpha = \left(\frac{\partial X^\alpha}{\partial t}\right)_s$

s varies, t fixed $\rightarrow$ dragged curves. $\rightarrow$ $A^\alpha_D = \left(\frac{\partial X^\alpha}{\partial s}\right)_t$

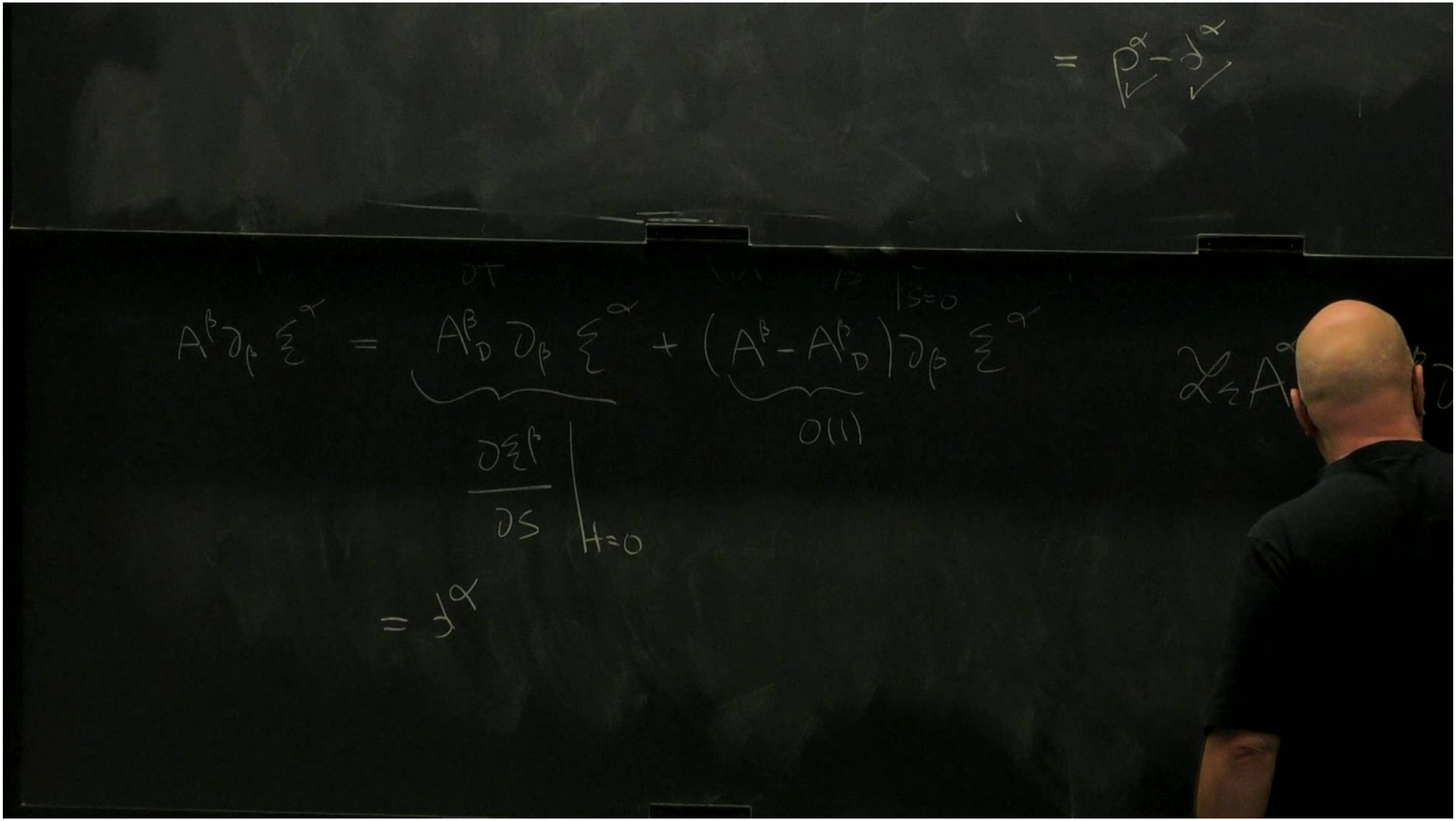
At P, $X^\alpha = 0$, $t = 0$, $s = 0$

$$X^\alpha(t,s) = a^\alpha t + b^\alpha s + \frac{1}{2}c^\alpha t^2 + d^\alpha st + \frac{1}{2}e^\alpha s^2 + O(3)$$

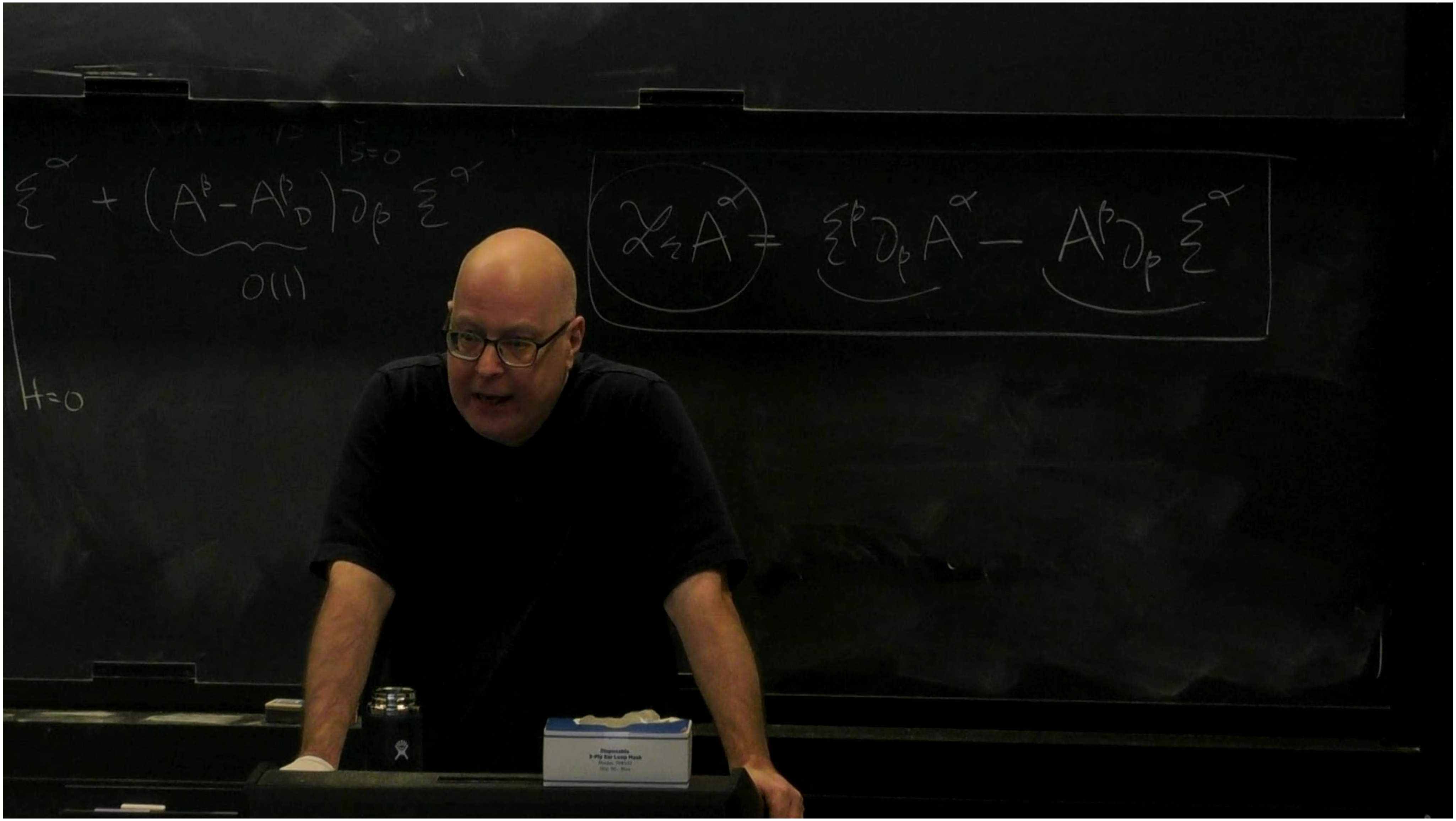

$\xi^\alpha = a^\alpha + c^\alpha t + d^\alpha s + O(2)$
$A_D^\alpha = b^\alpha + d^\alpha t + e^\alpha s + O(2)$
$A^\alpha = A_P^\alpha + p^\alpha t + q^\alpha s + O(2)$
When $t=0$, dragged curve $\equiv$ integral of A
$\vec{A}_D = \vec{A}$
$\vec{A}_D = b^\alpha + e^\alpha s$



$O(2)$
$O(2)$
$+ O(2)$
$+ O(2)$
When $t=0$, dragged curve $\equiv$ integral of A
$\vec{A}_D = \vec{A}$
$\vec{A}_D = b^\alpha + e^\alpha s + O(2)$
$\vec{A} = A_p^\alpha + q^\alpha s + O(2)$
$A_p^\alpha = b^\alpha$
$q^\alpha = e^\alpha$
$\mathcal{L}_\xi A^\alpha = \frac{A^\alpha(t) - A_D^\alpha(t)}{t} = \frac{(p^\alpha - \lambda^\alpha)t + O(2)}{t}$
$= p^\alpha - \lambda^\alpha$



$= \rho^\alpha - \lambda^\alpha$
$A^\beta \partial_\beta \xi^\alpha = \underbrace{A^\beta_D \partial_\beta \xi^\alpha}_{\left.\frac{\partial \xi^\beta}{\partial s}\right|_{t=0}} + (\underbrace{A^\beta - A^\beta_D}_{O(1)}) \partial_\beta \xi^\alpha$
$= \lambda^\alpha$
$\mathcal{L}_\xi A^\alpha$



$|_{s=0}$
$\xi^\alpha + (\underbrace{A^\beta - A^\beta_D}_{O(1)})\partial_\beta \xi^\alpha$
$H=0$
$\boxed{\mathcal{L}_\xi A^\alpha = \xi^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \xi^\alpha}$