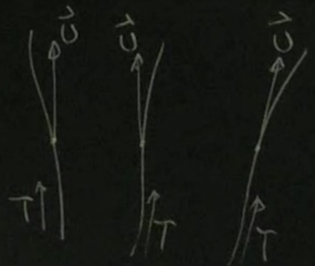
Title: Advanced General Relativity - 240131

Speakers: Eric Poisson

Collection: Advanced General Relativity (PHYS7840)

Date: January 31, 2024 - 10:30 AM

URL: https://pirsa.org/24010003



$$u_\alpha u^\alpha = -1$$

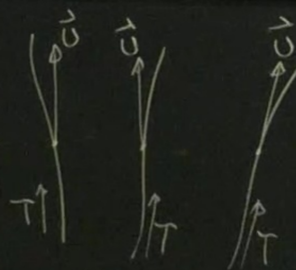
$$a^\alpha \equiv u^\beta \nabla_\beta u^\alpha \quad ; \quad u_\alpha a^\alpha = 0$$

$$P^\alpha{}_\beta = g^\alpha{}_\beta + u^\alpha u_\beta$$ — projector orthogonal to u^α

$$A^\alpha = \underbrace{A u^\alpha}_{\text{time}} + \underbrace{A_\perp^\alpha}_{\text{space}} \quad ; \quad u_\alpha A_\perp^\alpha = 0$$

$$A = -u_\alpha A^\alpha$$

$$A_\perp^\alpha = P^\alpha{}_\beta A^\beta$$



"comoving observer"

"time"

"spatial directions"

$$\boxed{u_\alpha u^\alpha = -1}$$

$$a^\alpha \equiv u^\beta \nabla_\beta u^\alpha \; ; \quad u_\alpha a^\alpha = 0$$

$P^\alpha{}_\beta = g^\alpha{}_\beta + u^\alpha u_\beta$ — projector orthogonal to u^α

$$\underset{4}{A^\alpha} = \underbrace{A\, u^\alpha}_{\text{time}} + \underbrace{A^\alpha_\perp}_{\text{space}} {}_{3} \; ; \quad u_\alpha A^\alpha_\perp = 0$$

$$A = -u_\alpha A^\alpha$$

$$A^\alpha_\perp = P^\alpha{}_\beta A^\beta$$

"time" $\quad A = -U_\alpha A^\alpha$

"spatial directions" $\quad A_\perp^\alpha = P^\alpha{}_\beta A^\beta$

Decomposition of tensor

$$\boxed{\begin{array}{ccccccccc} A^{\alpha\beta} & = & A\, U^\alpha U^\beta & + & U^\alpha B_\perp^\beta & + & C_\perp^\alpha U^\beta & + & D_{\perp\perp}^{\alpha\beta} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 16 & & 1 & & 3 & & 3 & & 9 \\ & & \text{time-time} & & \text{time-space} & & \text{space-time} & & \text{space-space} \end{array}}$$

$B_\perp^\alpha U_\alpha = 0$

$C_\perp^\alpha U_\alpha = 0$

$U_\alpha D_{\perp\perp}^{\alpha\beta} = 0 = D_{\perp\perp}^{\alpha\beta} U_\beta$

"time" $A = -U_\alpha A$

"spatial directions" $A_\perp^\alpha = P^\alpha{}_\beta A^\beta$

Decomposition of tensor

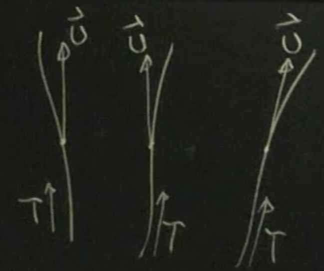
$$\boxed{\begin{array}{ccccccccc} A^{\alpha\beta} & = & A\, U^\alpha U^\beta & + & U^\alpha B_\perp^\beta & + & C_\perp^\alpha U^\beta & + & D_{\perp\perp}^{\alpha\beta} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 16 & & 1 & & 3 & & 3 & & 9 \\ & & \text{time-time} & & \text{time-space} & & \text{space-time} & & \text{space-space} \end{array}}$$

$B_\perp^\alpha U_\alpha = 0$

$C_\perp^\alpha U_\alpha = 0$

$U_\alpha D_{\perp\perp}^{\alpha\beta} = 0 = D_{\perp\perp}^{\alpha\beta} U_\beta$

$$\boxed{\begin{aligned} A &= U_\mu U_\nu A^{\mu\nu} \\ B_\perp^\beta &= -U_\mu P^\beta{}_\nu A^{\mu\nu} \\ C_\perp^\alpha &= -P^\alpha{}_\mu U_\nu A^{\mu\nu} \\ D_{\perp\perp}^{\alpha\beta} &= P^\alpha{}_\mu P^\beta{}_\nu A^{\mu\nu} \end{aligned}}$$



"comoving observer"

"time"

"spatial directions"

$$\boxed{U_\alpha U^\alpha = -1}$$

$$a^\alpha = U^\beta \nabla_\beta U^\alpha \;;\; U_\alpha a^\alpha = 0$$

$P^\alpha{}_\beta = g^\alpha{}_\beta + U^\alpha U_\beta$ — projector orthogonal to U^α

$$\underbrace{A^\alpha}_{4} = \underbrace{A U^\alpha}_{\text{time}} + \underbrace{A_\perp^\alpha}_{\text{space} \to 3} \;;\; U_\alpha A_\perp^\alpha = 0$$

$$A = -U_\alpha A^\alpha$$

$$A_\perp^\alpha = P^\alpha{}_\beta A^\beta$$

side calculation:

$$U^\alpha \nabla_\beta U_\alpha = \frac{1}{2} U^\alpha \nabla_\beta U_\alpha + \frac{1}{2} U_\alpha \nabla_\beta U^\alpha$$

$$= \frac{1}{2} \nabla_\beta \left(U^\alpha U_\alpha \right)$$

$$C_\perp^\alpha = -P^\alpha{}_\mu U_\nu A^{\mu\nu}$$

$$D_{\perp\perp}^{\alpha\beta} = P^\alpha{}_\mu P^\beta{}_\nu A^{\mu\nu}$$

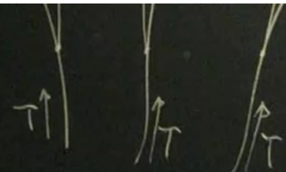
$$C^{\perp}_{\alpha} = - P_{\alpha}{}^{\rho} U^{\nu} \nabla_{\rho} U_{\nu} = 0$$

$$D^{\parallel}_{\alpha\beta} = P_{\alpha}{}^{\rho} P_{\beta}{}^{\nu} \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho})(g_{\beta}{}^{\nu} + \cancel{U_{\beta} U^{\nu}}) \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho}) \nabla_{\rho} U_{\beta} = \nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta}$$

$$\boxed{\nabla_{\alpha} U_{\beta} = - \underbrace{U_{\alpha} a_{\beta}}_{\text{time-space}} + \big(\underbrace{\nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta}}_{\text{space-space}} \big)} \longrightarrow \text{analogous to Newtonian } \partial_j V_k$$

Further decomposition (trace, symmetric tracefree, antisymmetric)

$$\nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta} =$$



$a^\alpha = u^\beta \nabla_\beta u^\alpha$; $u_\alpha a^\alpha = 0$

$P^\alpha{}_\beta = g^\alpha{}_\beta + u^\alpha u_\beta$ — projector orthogonal to u^α

$\underbrace{A^\alpha}_{4} = \underbrace{A\, u^\alpha}_{\text{time}} + \underbrace{A^\alpha_\perp}_{\text{space}\ \to 3}$; $u_\alpha A^\alpha_\perp = 0$

"comoving observer"

"time" $\qquad A = -u_\alpha A^\alpha$

"spatial directions" $\qquad A^\alpha_\perp = P^\alpha{}_\beta A^\beta$

↳ $u^\alpha \overset{*}{=} (1,0,0,0)$, $u_\alpha \overset{*}{=} (-1,0,0,0)$

$$P^\alpha{}_\beta \overset{*}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \delta_{jk}$$

side calculation:

$$u^\alpha \nabla_\beta u_\alpha = \frac{1}{2} u^\alpha \nabla_\beta u_\alpha + \frac{1}{2} u_\alpha \nabla_\beta u^\alpha$$

$$= \frac{1}{2} \nabla_\beta \big(\underbrace{u^\alpha u_\alpha}_{-1} \big)$$

$$= 0$$

$$A = u_\mu u_\nu A^{\mu\nu}$$

$$B^\beta_\perp = -u_\mu P^\beta{}_\nu A^{\mu\nu}$$

$$C^\alpha_\perp = -P^\alpha{}_\mu u_\nu A^{\mu\nu}$$

$$D^{\alpha\beta}_{\perp\perp} = P^\alpha{}_\mu P^\beta{}_\nu A^{\mu\nu}$$

$$C^{\perp}_{\alpha} = - P_{\alpha}{}^{\rho} U^{\nu} \nabla_{\rho} U_{\nu} = 0$$

$$D^{\perp\perp}_{\alpha\beta} = P_{\alpha}{}^{\rho} P_{\beta}{}^{\nu} \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho})(g_{\beta}{}^{\nu} + \cancel{U_{\beta} U^{\nu}}) \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho}) \nabla_{\rho} U_{\beta} = \nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta}$$

$$\boxed{\nabla_{\alpha} U_{\beta} = \underbrace{- U_{\alpha} a_{\beta}}_{\text{time-space}} + \underbrace{(\nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta})}_{\text{space-space}}}$$

analogous to Newtonian $\partial_j V_K$

Further decomposition (trace, symmetric tracefree, antisymmetric)

$$\boxed{\nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta} = \frac{1}{3} \underbrace{\Theta}_{\text{rate of expansion}} P_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{rate of shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rate of rotation}}} \;;$$

$\sigma_{\alpha\beta} P^{\alpha\beta} = 0$; $\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)}$

$\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$

$\sigma_{\alpha\beta} U^{\beta} = 0$, $\omega_{\alpha\beta} U^{\beta} = 0$

"time" $\quad A = -U_\alpha A^\alpha$

"spatial directions" $\quad A^\alpha_\perp = P^\alpha{}_\beta A^\beta$

$\rightarrow \quad U^\alpha \overset{*}{=} (1,0,0,0)\,, \quad U_\alpha \overset{*}{=} (-1,0,0,0)$

$$P^\alpha{}_\beta \overset{*}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \delta_{jk}$$

$$U^\beta \nabla_\beta U_\alpha = \frac{1}{2} U^\beta \nabla_\beta U_\alpha + \frac{1}{2} U_\alpha \nabla_\beta U^\beta$$

$$= \frac{1}{2} \nabla_\beta \big(\underbrace{U^\alpha U_\alpha}_{-1}\big)$$

$$= 0$$

$$\nabla_{(\alpha} U_{\beta)} + U_{(\alpha} a_{\beta)} = \frac{1}{3} \Theta P_{\alpha\beta} + \sigma_{\alpha\beta}$$

$$\underbrace{P^{\alpha\beta}\left(\nabla_{(\alpha} U_{\beta)} + U_{(\alpha} a_{\beta)}\right)} = \frac{1}{3} \Theta \underbrace{P^{\alpha\beta} P_{\alpha\beta}}_{3} = \Theta$$

$$\left(g^{\alpha\beta} + U^\alpha U^\beta\right)\left(\nabla_\alpha U_\beta + U_\alpha a_\beta\right) = \nabla_\alpha U^\alpha$$

$$\boxed{\Theta = \nabla_\alpha U^\alpha}$$

$$\boxed{\sigma_{\alpha\beta} = \nabla_{(\alpha} U_{\beta)} + U_{(\alpha} a_{\beta)} - \frac{1}{3} \Theta P_{\alpha\beta}}$$

$$\boxed{\omega_{\alpha\beta} = \nabla_{[\alpha} U_{\beta]} + U_{[\alpha} a_{\beta]}}$$

$$C^{\perp}_{\alpha} = -P_{\alpha}{}^{\rho} U^{\nu} \nabla_{\rho} U_{\nu} = 0$$

$$D^{\parallel}_{\alpha\beta} = P_{\alpha}{}^{\rho} P_{\beta}{}^{\nu} \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho})(g_{\beta}{}^{\nu} + \cancel{U_{\beta} U^{\nu}}) \nabla_{\rho} U_{\nu} = (g_{\alpha}{}^{\rho} + U_{\alpha} U^{\rho}) \nabla_{\rho} U_{\beta} = \nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta}$$

$$\boxed{\nabla_{\alpha} U_{\beta} = \underbrace{- U_{\alpha} a_{\beta}}_{\text{time-space}} + (\underbrace{\nabla_{\alpha} U_{\beta} + U_{\alpha} a_{\beta}}_{\text{space-space}})}$$

$\to$ analagous to Newtonian $\partial_j V_K$

rate of expansion, rate of shear, rate of rotation

$\sigma_{\alpha\beta} U^{\beta} = 0$, $\omega_{\alpha\beta} U^{\beta} = 0$

Final decomposition:

$$\boxed{\underbrace{\nabla_{\alpha} U_{\beta}}_{12} = - U_{\alpha} \underset{3}{a_{\beta}} + \frac{1}{3} \underset{1}{\Theta} P_{\alpha\beta} + \underset{5}{\sigma_{\alpha\beta}} + \underset{3}{\omega_{\alpha\beta}}}$$

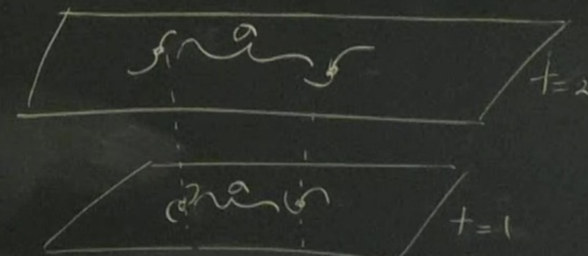
$$\boxed{\Theta = \nabla_{\alpha} U^{\alpha}}$$

Example 1 LFRW cosmology

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

cosmological flow: $U^\alpha = (1,0,0,0)$

$U_\alpha = (-1,0,0,0)$



world lines of galaxies: $\begin{cases} x = \text{const} \\ y = \text{const} \\ z = \text{const} \end{cases}$

$$\boxed{\nabla_\alpha U_\beta + U_\alpha a_\beta = \tfrac{1}{3}\,\underbrace{\Theta}_{\text{rate of expansion}}\,P_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{rate of shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rate of rotation}}} \; ; \qquad \omega_{\alpha\beta} = \omega_{[\alpha\beta]}$$

$$\sigma_{\alpha\beta} U^\beta = 0 \, , \quad \omega_{\alpha\beta} U^\beta = 0$$

$$\nabla_\alpha U_\beta = \partial_\alpha U_\beta - \Gamma^\gamma_{\alpha\beta} U_\gamma = -\Gamma^\gamma_{\alpha\beta} U_\gamma = +\Gamma^t_{\alpha\beta}$$

$$\Gamma^t_{\alpha\beta} = \frac{1}{2} g^{t\mu}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\right) = -\frac{1}{2}\left(\cancel{\partial_\alpha g_{t\beta}} + \cancel{\partial_\beta g_{t\alpha}} - \partial_t g_{\alpha\beta}\right)$$

$$= \frac{1}{2}\partial_t g_{\alpha\beta} = a\dot{a}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\nabla_\alpha U_\beta = \frac{\dot{a}}{a} P_{\alpha\beta}}$$

$$\Theta = 3\frac{\dot{a}}{a}$$

$$= \frac{1}{2}\partial_t g_{\alpha\beta} = a\dot{a}\begin{pmatrix} \cdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

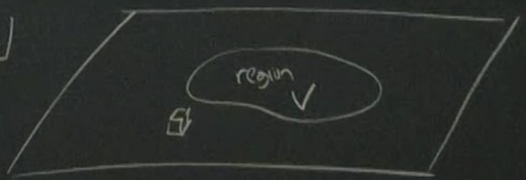
$$P_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\nabla_\alpha U_\beta = \frac{\dot{a}}{a} P_{\alpha\beta}}$$

$$\Theta = 3\frac{\dot{a}}{a}$$

$$\Theta = 3\frac{\dot{a}}{a} = \frac{1}{a^3}\frac{d}{dt}a^3 = \frac{1}{\text{Volume}}\frac{d}{dt}(\text{volume})$$

↳ fractional rate of volume change



$u_\alpha = (-1, 0, 0, 0)$

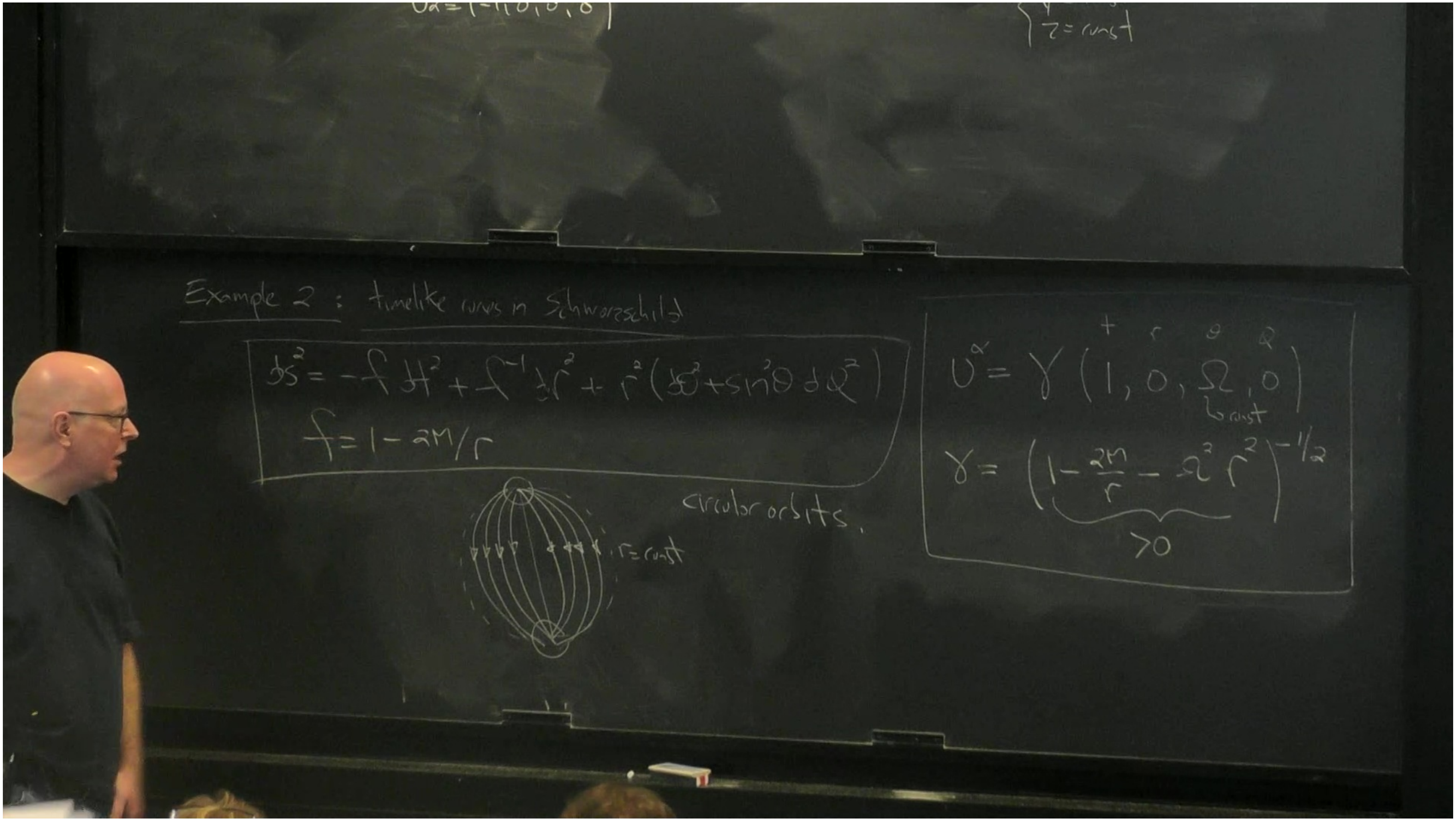
$z = \text{const}$

Example 2 : timelike curves in Schwarzschild

$$\boxed{\begin{gathered} ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \\ f = 1 - 2M/r \end{gathered}}$$

$$\begin{array}{cccccc} & t & r & \theta & \varphi & \\ u^\alpha = \gamma\,(& 1, & 0, & \underbrace{\Omega}_{\text{const}}, & 0 &) \end{array}$$

$$\gamma = \Big(\underbrace{1 - \frac{2M}{r} - \Omega^2 r^2}_{>0}\Big)^{-1/2}$$



U_α = (−1, 0, 0, 0)
r = const
Example 2 : timelike curves in Schwarzschild
ds² = −f dt² + f⁻¹ dr² + r²(dθ² + sin²θ dφ²)
f = 1 − 2M/r
circular orbits.
r = const
t r θ φ
u^α = γ(1, 0, Ω, 0)
const
γ = (1 − 2M/r − Ω² r²)^(−1/2)
>0