Title: Advanced General Relativity - 240117

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Collection: Advanced General Relativity (PHYS7840)

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Lie derivative

$$\mathcal{L}_\xi A^\alpha = \xi^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \xi^\alpha$$

$$\mathcal{L}_\xi \omega_\alpha = \xi^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta$$

$$\mathcal{L}_\xi T^{\alpha\beta} = \xi^\gamma \partial_\gamma T^{\alpha\beta} - T^{\gamma\beta} \partial_\gamma \xi^\alpha - T^{\alpha\gamma} \partial_\gamma \xi^\beta$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

Killing vectors

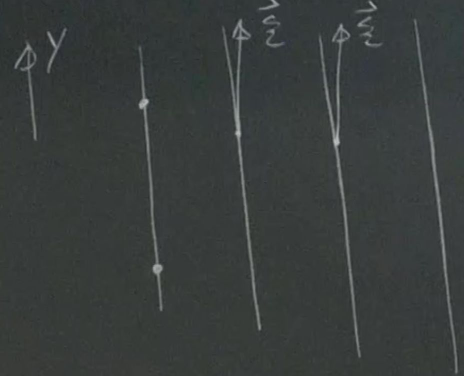
In some coordinates (t, x, y, z), suppose that $\partial_y g_{\mu\nu} \overset{*}{=} 0$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta}\partial_\alpha \xi^\gamma + T_{\alpha\gamma}\partial_\beta \xi^\gamma$$

Killing vectors

In some coordinates (t, x, y, z), suppose that $\partial_y g_{\mu\nu} \overset{*}{=} 0$



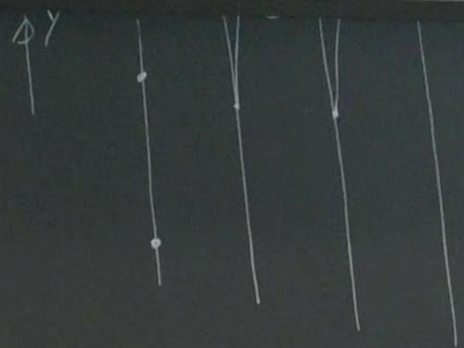
$$\xi^\mu \overset{*}{=} (0,0,1,0)$$

$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \partial_\lambda g_{\mu\nu}}_{\partial_y g_{\mu\nu} \overset{*}{=} 0} + g_{\lambda\nu}\underbrace{\partial_\mu \xi^\lambda}_{\overset{*}{=} 0} + g_{\mu\lambda}\underbrace{\partial_\nu \xi^\lambda}_{\overset{*}{=} 0}$$

$$\overset{*}{=} 0$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

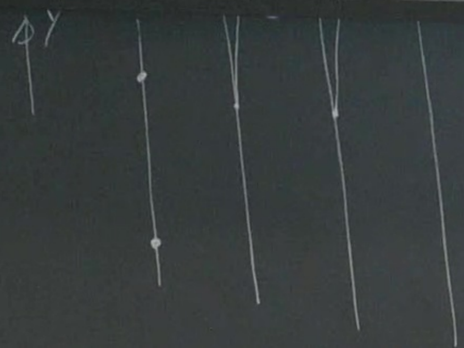


$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \partial_\lambda g_{\mu\nu}}_{\partial_y g_{\mu\nu} \overset{*}{=} 0} + g_{\lambda\nu} \underbrace{\partial_\mu \xi^\lambda}_{\overset{*}{=} 0} + g_{\mu\lambda} \underbrace{\partial_\nu \xi^\lambda}_{\overset{*}{=} 0}$$

$$\overset{*}{=} 0 \longrightarrow \boxed{\mathcal{L}_\xi g_{\alpha\beta} = 0}$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

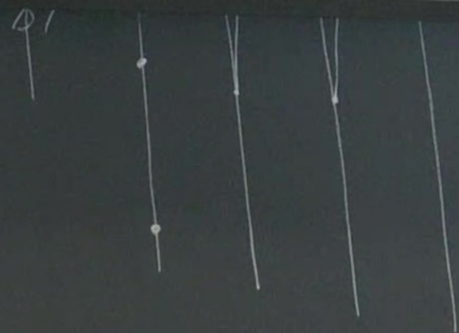


$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \partial_\lambda g_{\mu\nu}}_{\partial_y g_{\mu\nu} \overset{*}{=} 0} + g_{\lambda\nu} \underbrace{\partial_\mu \xi^\lambda}_{\overset{*}{=} 0} + g_{\mu\lambda} \underbrace{\partial_\nu \xi^\lambda}_{\overset{*}{=} 0}$$

$$\overset{*}{=} 0 \longrightarrow \boxed{\mathcal{L}_\xi g_{\alpha\beta} = 0}$$ in all coordinates

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta}\, \partial_\alpha \xi^\gamma + T_{\alpha\gamma}\, \partial_\beta \xi^\gamma$$



$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \partial_\lambda g_{\mu\nu}}_{\partial_\gamma g_{\mu\nu} \overset{*}{=} 0} + g_{\lambda\nu} \underbrace{\partial_\mu \xi^\lambda}_{\overset{*}{=} 0} + g_{\mu\lambda} \underbrace{\partial_\nu \xi^\lambda}_{\overset{*}{=} 0}$$

$$\overset{*}{=} 0 \longrightarrow \boxed{\mathcal{L}_\xi g_{\alpha\beta} = 0}$$

in all coordinates

$$\boxed{\xi^\alpha \equiv \text{killing vector}}$$

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

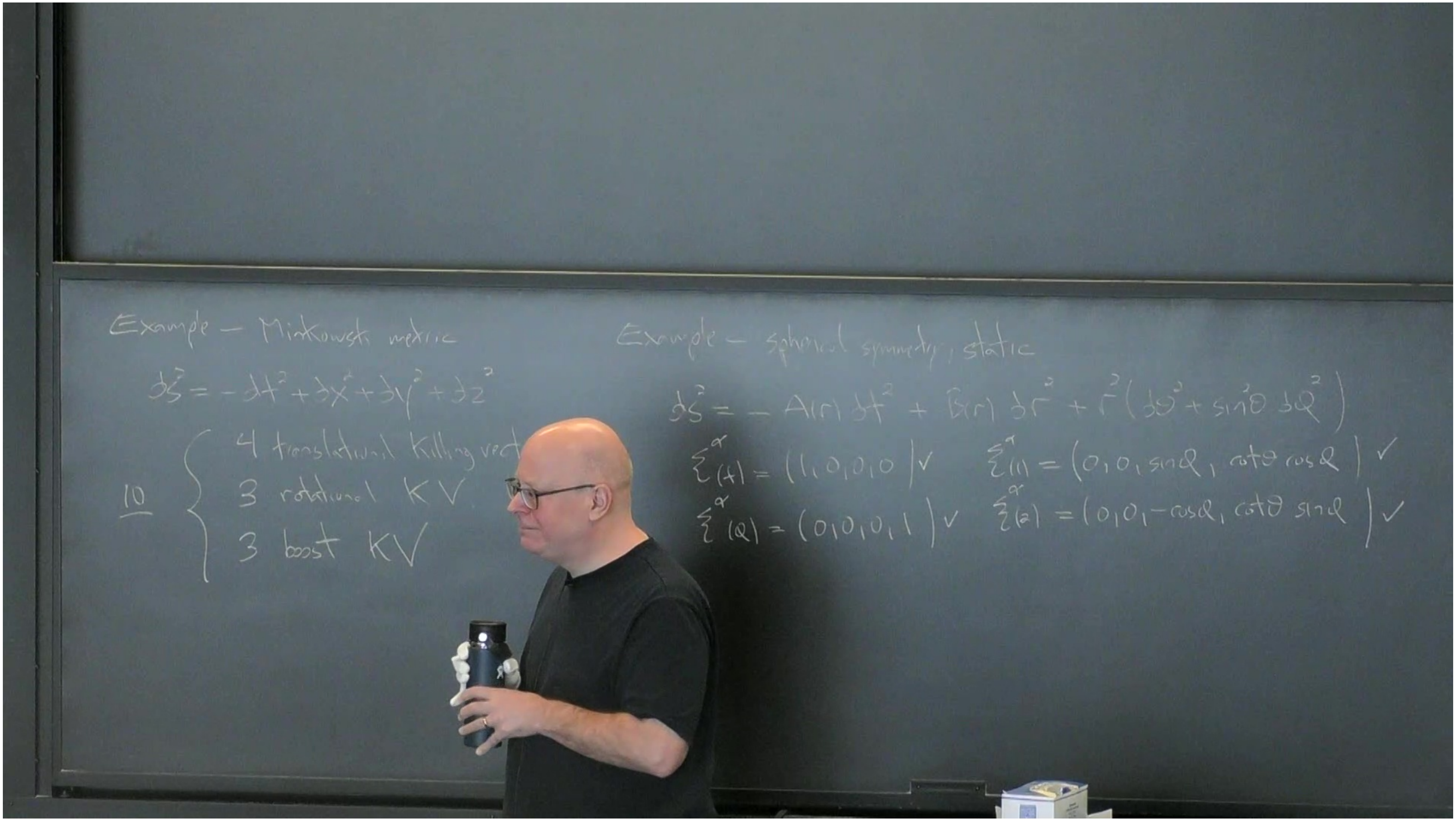
$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

$$0 = \mathcal{L}_\xi g_{\alpha\beta} = \xi^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{0} + \underbrace{g_{\gamma\beta} \nabla_\alpha \xi^\gamma}_{\nabla_\alpha (g_{\gamma\beta} \xi^\gamma)} + \underbrace{g_{\alpha\gamma} \nabla_\beta \xi^\gamma}_{\nabla_\beta (g_{\alpha\gamma} \xi^\gamma)}$$

$$\rightarrow \boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0}$$ killing's equation

$$\nabla_{(\alpha} \xi_{\beta)} = 0$$

$\nabla_\alpha \xi_\beta =$ antisymmetric


Example – Minkowski metric
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
4 translational Killing vect
10
3 rotational KV
3 boost KV
Example – spherical symmetry, static
$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
$\xi^\alpha_{(t)} = (1,0,0,0)$ ✓
$\xi^\alpha_{(\varphi)} = (0,0,0,1)$ ✓
$\xi^\alpha_{(1)} = (0,0,\sin\varphi, \cot\theta\cos\varphi)$ ✓
$\xi^\alpha_{(2)} = (0,0,-\cos\varphi, \cot\theta\sin\varphi)$ ✓

$$\mathcal{L}_\xi T^\alpha{}_\beta = \xi^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \xi^\alpha + T^\alpha{}_\gamma \partial_\beta \xi^\gamma$$

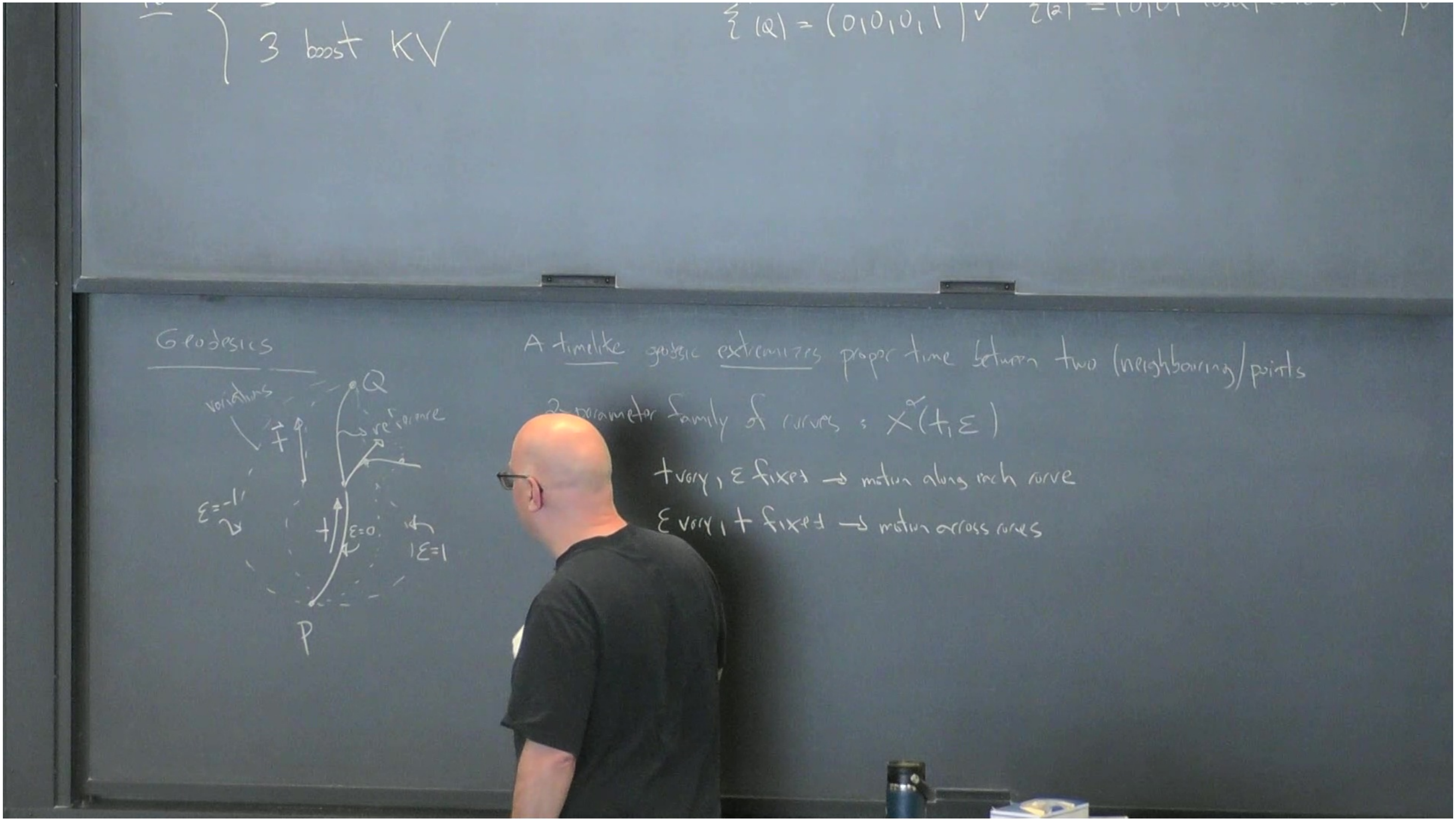
$$\mathcal{L}_\xi T_{\alpha\beta} = \xi^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \xi^\gamma + T_{\alpha\gamma} \partial_\beta \xi^\gamma$$

$$0 = \mathcal{L}_\xi g_{\alpha\beta} = \xi^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{0} + \underbrace{g_{\gamma\beta} \nabla_\alpha \xi^\gamma}_{\nabla_\alpha (g_{\gamma\beta} \xi^\gamma)} + \underbrace{g_{\alpha\gamma} \nabla_\beta \xi^\gamma}_{\nabla_\beta (g_{\alpha\gamma} \xi^\gamma)}$$

$$\rightarrow \boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0}$$ killing's equation

$$\nabla_{(\alpha} \xi_{\beta)} = 0$$

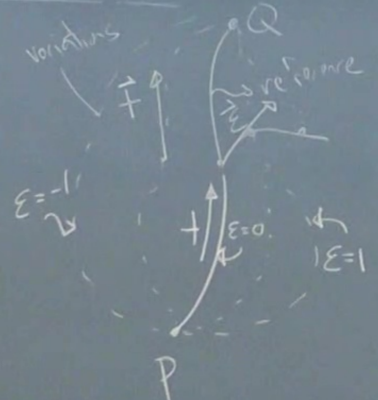
$$\nabla_\alpha \xi_\beta = \text{antisymmetric}$$

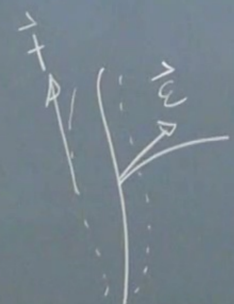


3 boost KV
ξ(Q) = (0,0,0,1)
Geodesics
A timelike geodesic extremizes proper time between two (neighbouring) points
2-parameter family of curves : x^α(t, ε)
t vary, ε fixed → motion along each curve
ε vary, t fixed → motion across curves
Q
variations
reference curve
ε=-1
ε=0
ε=1
P

3 boost KV

$\xi(Q) = (0,0,0,1)$





2-parameter family of curves: $x^\alpha(t,\varepsilon)$

t vary, ε fixed → motion along each curve → $t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_\varepsilon$

ε vary, t fixed → motion across curves → $\varepsilon^\alpha = \left(\frac{\partial x^\alpha}{\partial \varepsilon}\right)_t$

Show: $\mathcal{L}_\varepsilon t^\alpha = 0 = \mathcal{L}_t \varepsilon^\alpha$

$$\boxed{\varepsilon^\beta \nabla_\beta t^\alpha = t^\beta \nabla_\beta \varepsilon^\alpha}$$

$\partial_\gamma g_{\mu\nu} \overset{*}{=} 0$

$\overset{*}{=} 0$

$\overset{*}{=} 0$

$\overset{*}{=} 0 \longrightarrow \boxed{\mathcal{L}_\xi g_{\alpha\beta} = 0}$ in all coordinates

$\xi^\alpha \equiv$ killing vector

Lie derivative

$$\mathcal{L}_\xi A^\alpha = \xi^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \xi^\alpha$$

$$\begin{aligned}
\mathcal{L}_\varepsilon t^\alpha &= \varepsilon^\beta \partial_\beta t^\alpha - t^\beta \partial_\beta \varepsilon^\alpha \\
&= \frac{\partial}{\partial \varepsilon} t^\alpha - \frac{\partial}{\partial t} \varepsilon^\alpha \\
&= \frac{\partial}{\partial \varepsilon} \frac{\partial x^\alpha}{\partial t} - \frac{\partial}{\partial t} \frac{\partial x^\alpha}{\partial \varepsilon} \\
&= 0 \checkmark
\end{aligned}$$

$= 0$ ✓

Q

P

proper time : $d\tau^2 = -ds^2 = -g_{\alpha\beta}\,dx^\alpha dx^\beta$

$$d\tau = \sqrt{-g_{\alpha\beta}\,dx^\alpha dx^\beta} = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}\;dt$$

$$= \sqrt{-g_{\alpha\beta}t^\alpha t^\beta}\;dt$$

$$\Delta\tau = \int_P^Q \sqrt{-g_{\alpha\beta}t^\alpha t^\beta}\;dt$$

$= 0$ ✓

$$= \sqrt{-g_{\alpha\beta} t^\alpha t^\beta}\, dt$$

$$\Delta T = \int_P^Q \sqrt{-g_{\alpha\beta} t^\alpha t^\beta}\, dt$$

P

$$\boxed{S = -\int_0^1 \sqrt{-g_{\alpha\beta} t^\alpha t^\beta}\, dt}$$

P

$$\boxed{S = -\int_0^1 \sqrt{-g_{\alpha\beta} t^\alpha t^\beta}\, dt} \qquad \boxed{L = -\sqrt{-g_{\alpha\beta} t^\alpha t^\beta}}$$

$$\delta S = S(\varepsilon) - S(0) = \varepsilon \left.\frac{\partial S}{\partial \varepsilon}\right|_{\varepsilon=0} = \varepsilon \int_0^1 \frac{\partial L}{\partial \varepsilon}\, dt$$

$$\frac{\partial L}{\partial \varepsilon} = \varepsilon^\mu \partial_\mu L$$

$$= \varepsilon^\mu \nabla_\mu L = \varepsilon^\mu \left(\frac{\partial L}{\partial g_{\alpha\beta}} \cancel{\nabla_\mu g_{\alpha\beta}} + \frac{\partial L}{\partial t^\alpha} \nabla_\mu t^\alpha \right)$$

$$= \frac{\partial L}{\partial t^\alpha} \varepsilon^\mu \nabla_\mu t^\alpha$$

p

$$\boxed{S = -\int_0^1 \sqrt{-g_{\alpha\beta} t^\alpha t^\beta}\, dt} \qquad \boxed{L = -\sqrt{-g_{\alpha\beta} t^\alpha t^\beta}}$$

$$\delta S = S(\varepsilon) - S(0) = \varepsilon \left.\frac{\partial S}{\partial \varepsilon}\right|_{\varepsilon=0} = \varepsilon \int_0^1 \frac{\partial L}{\partial \varepsilon}\, dt$$

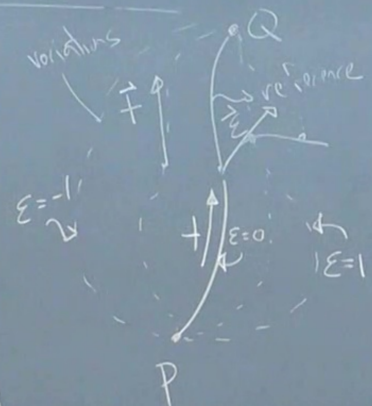
$$\frac{\partial L}{\partial \varepsilon} = \varepsilon^\mu \partial_\mu L$$

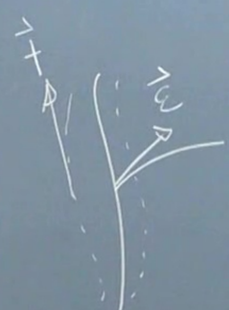
$$= \varepsilon^\mu \nabla_\mu L = \varepsilon^\mu \left(\frac{\partial L}{\partial g_{\alpha\beta}} \cancel{\nabla_\mu g_{\alpha\beta}} + \frac{\partial L}{\partial t^\alpha} \nabla_\mu t^\alpha \right)$$

$$= \underbrace{\frac{\partial L}{\partial t^\alpha}}_{\equiv p_\alpha} \underbrace{\varepsilon^\mu \nabla_\mu t^\alpha}_{t^\mu \nabla_\mu \varepsilon^\alpha}$$

3 boost KV

$\xi(Q) = (0,0,0,1)$





For all curves,

$t(P) = 0$

$t(Q) = 1$

2-parameter family of curves : $x^\alpha(t, \varepsilon)$

t vary, ε fixed $\rightarrow$ motion along each curve $\longrightarrow$ $t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_\varepsilon$

ε vary, t fixed $\rightarrow$ motion across curves $\longrightarrow$ $\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial \varepsilon}\right)_t$

Show: $\mathcal{L}_\xi t^\alpha = 0 = \mathcal{L}_t \xi^\alpha$

$$\boxed{\xi^\beta \nabla_\beta t^\alpha = t^\beta \nabla_\beta \xi^\alpha} \checkmark$$

$$\boxed{p_\alpha \equiv \frac{\partial L}{\partial t^\alpha}} \qquad = \underbrace{\frac{\partial L}{\partial t^\alpha}}_{\equiv p_\alpha} \underbrace{\varepsilon^\mu \nabla_\mu t^\alpha}_{t^\mu \nabla_\mu \varepsilon^\alpha}$$

$$\frac{\partial L}{\partial \varepsilon} = p_\alpha t^\mu \nabla_\mu \varepsilon^\alpha = p_\alpha \frac{D\varepsilon^\alpha}{dt}$$

$$\varepsilon^{-1}\delta S = \int_0^1 \underbrace{p_\alpha \frac{D\varepsilon^\alpha}{dt}}_{\frac{d}{dt}(p_\alpha \varepsilon^\alpha) - \varepsilon^\alpha \frac{Dp_\alpha}{dt}} dt$$

$$= p_\alpha \varepsilon^\alpha \Big|_0^1 - \int_0^1 \varepsilon^\alpha \frac{Dp_\alpha}{dt}\, dt$$

3 boost KV

$\xi(Q) = (0,0,0,1)$

Rules of variation — ε^α is arbitrary between P and Q, but <u>zero</u> at P and Q

$$\delta S = 0 \Rightarrow \int_0^1 \varepsilon^\alpha \frac{Dp_\alpha}{dt}\, dt \longrightarrow \boxed{\frac{Dp_\alpha}{dt} = 0}$$ (geodesic equation)

$$p_\alpha = \frac{\partial L}{\partial t^\alpha} \;;\; L = -\sqrt{-g_{\alpha\beta}\, t^\alpha t^\beta}$$

$$p_\alpha = \frac{\partial L}{\partial t^\alpha} \; ; \; L = -\sqrt{-g_{\alpha\beta} t^\alpha t^\beta}$$

$$L = -\left(-g_{\mu\nu} t^\mu t^\nu\right)^{1/2}$$

$$p_\alpha = \frac{\partial L}{\partial t^\alpha} = -\frac{1}{2}\left(-g_{\mu\nu} t^\mu t^\nu\right)^{-1/2} \left(-g_{\mu\nu}\right)\left(\underbrace{\frac{\partial t^\mu}{\partial t^\alpha}}_{\delta^\mu_\alpha} t^\nu + t^\mu \underbrace{\frac{\partial t^\nu}{\partial t^\alpha}}_{\delta^\nu_\alpha} \right)$$

$$= \frac{1}{2}\frac{1}{(-L)} \left(g_{\alpha\nu} t^\nu + g_{\mu\alpha} t^\mu \right)$$

$$\boxed{p_\alpha = \frac{1}{(-L)} t_\alpha} \quad \boxed{p^\alpha = \frac{1}{(-L)} t^\alpha}$$

$$\boxed{p^\alpha \equiv \frac{\partial L}{\partial t^\alpha}} \qquad = \underbrace{\frac{\partial L}{\partial t^\alpha}}_{\equiv p_\alpha} \underbrace{\epsilon^\mu \nabla_\mu t^\alpha}_{t^\mu \nabla_\mu \epsilon^\alpha}$$

$$\frac{\partial L}{\partial \epsilon} = p_\alpha t^\mu \nabla_\mu \epsilon^\alpha = p_\alpha \frac{D\epsilon^\alpha}{dt}$$

$$\frac{Dp^\alpha}{dt} = 0 \quad \to \quad \frac{D}{dt}\left(\frac{t^\alpha}{(-L)}\right) = \frac{1}{(-L)}\frac{Dt^\alpha}{dt} - \frac{1}{(-L)^2}\frac{d(-L)}{dt}\, t^\alpha$$

$$\boxed{\frac{Dt^\alpha}{dt} = k\, t^\alpha} \qquad \to \qquad \boxed{\frac{Dt^\alpha}{dt} = \underbrace{\frac{1}{(-L)}\frac{d(-L)}{dt}}_{k}\, t^\alpha}$$

$$= \frac{1}{2}\frac{1}{(-L)}\left(g_{\alpha\nu} t^{\nu} + g_{\mu\alpha} t^{\mu} \right)$$

$$\boxed{p_{\alpha} = \frac{1}{(-L)} t_{\alpha}} \qquad \boxed{p^{\alpha} = \frac{1}{(-L)} t^{\alpha}}$$

After variation — choose $t \equiv$ proper time τ.

$$L = -\sqrt{g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}} = -\sqrt{\frac{d\tau^2}{d\tau^2}} = -$$

$L = -1$

$K = 0$

$$\rightarrow \boxed{\frac{D}{d\tau}\left(\frac{dx^{\alpha}}{d\tau}\right) = 0}$$

geodesic eqn (proper time parametrization)