Title: Anomalous symmetries of spin chains

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Abstract: Several years ago, Nayak and Else argued that Symmetry Protected Topological phases in d dimensions can be classified using non-on-site actions of the symmetry group in d-1 dimensions. Such non-on-site actions can have an "anomaly", in the sense that the symmetry action cannot be consistently localized. This anomaly is similar but distinct from 't Hooft anomaly in QFT. Nayak and Else assumed that the symmetry group is finite and the non-on-site action is given by a finite-depth local unitary circuit. I will explain how to generalize the construction of the anomaly index in two directions: to Lie groups as well as to arbitrary actions which preserve locality. For simplicity, I will only discuss the one-dimensional case. One can prove that a nonzero anomaly index prohibits any invariant 1d Hamiltonian from having invariant ground states. This is similar to 't Hooft anomaly matching in QFT. Lieb-Schultz-Mattis-type theorems arise as a special case where the symmetry group involves translations. This is joint work with Nikita Sopenko.

Zoom link https://pitp.zoom.us/j/95661517248?pwd=SkMxUFJWTG56SG9hVlNiNS9yeEVrQT09

Anomalies for spin chains

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based on work in progress with Nikita Sopenko (IAS, Princeton)

Executive summary

- 't Hooft Anomalies on a lattice are not the same as 't Hooft anomalies in QFT
- But they have very similar dynamical consequences (at least in 1d): no gapped symmetric ground state
- The "right" version of group cohomology for Lie groups is "differentiable group cohomology"
- Simplicial sets and simplicial objects in categories are useful in physics

Outline

- 't Hooft anomalies in QFT
- Lieb-Schultz-Mattis theorem and anomalies
- Anomalies of non-on-site lattice symmetries (following Nayak and Else)
- New results (informal version)
- Some definitions

't Hooft anomalies in QFT

Consider a QFT with a global symmetry G. There may be obstructions to gauging G. These are often called 't Hooft anomalies.

Gerard 't Hooft did not invent them, but he was the 1st to realize that such obstructions provide non-perturbative constraints on the low-energy behavior of QFTs.

't Hooft anomaly matching:
't Hooft anomalies do not change under renormalization group flow. Thus 't Hooft anomaly of any QFT is the same as 't Hooft anomaly of its low-energy Effective Field Theory.

Corollary of 't Hooft anomaly matching

If G has nonzero 't Hooft anomalies, then a trivially gapped symmetric vacuum is impossible.

Where do 't Hooft anomalies take values?

Consider a QFT in d spatial dimensions with an internal symmetry G.
Usual answers:

- G compact connected Lie group: anomalies are in 1-1 correspondence with Chern-Simons terms for a G gauge field in $d+1$ spatial dimensions.
- G is finite: anomalies are elements of group cohomology $H^{d+2}(G, U(1))$.
- Better answer for G finite: anomalies are characters of the oriented bordism group $\Omega_{d+2}(BG)$.

BG is the classifying space of G bundles. For G finite, BG is any topological space X such that $\pi_1(X) = G$ and all other $\pi_i(X)$ vanish.

Time-reversal incorporated through unoriented bordism.
Fermions incorporated through Spin-bordism.

Bosonic QFT anomalies in low dimensions

Bordisms are better than group cohomology because they (supposedly) take into account "mixed" anomalies (anomalies which involve both G and diffeomorphisms).

For bosons in low dimensions this does not matter:
$\Omega_{d+2}(BG) = H^{sing}_{d+2}(BG)$ if $d \leq 1$, and its group of characters is
$Hom(H^{sing}_{d+2}(BG), U(1)) = H^{d+2}_{sing}(BG, U(1))$.
Also, for a finite G, $H^{d+2}_{sing}(BG, U(1)) = H^{d+2}(G, U(1))$. Hence

$d = 0$, G finite

Bosonic QFT anomalies take values in $H^2(G, U(1)) = H^2_{sing}(BG, U(1))$.

$d = 1$, G finite

Bosonic QFT anomalies take values in $H^3(G, U(1)) = H^3_{sing}(BG, U(1))$.

No "mixed" anomalies for bosonic QFTs if $d = 0, 1$.

Sanity check: degree-2 group cohomology

What is $H^n(G, U(1))$ and why is it relevant?

Consider $d = 0$ QFT, i.e. Quantum Mechanics with a Hilbert space $\mathcal{H}$.

Explicit description of $H^2(G, U(1))$

The abelian group of 2-cocycles of G modulo the subgroup of 2-coboundaries.

2-cocycle: function $\rho : G \times G \to U(1)$ such that $\forall g, h, k \in G$

$$\rho(g,h)\rho(gh,k) = \rho(g,hk)\rho(h,k).$$

2-coboundary: 2-cocycle of the form

$$\rho(g,h) = \frac{\psi(gh)}{\psi(g)\psi(h)}, \quad \psi : G \to U(1).$$

Projective representations

2-cocycles arise from projective representations of G:

$$\hat{U}(gh) = \rho(g,h)\hat{U}(g)\hat{U}(h), \quad \hat{U}(g) \in U(\mathcal{H}),\ g,h \in G$$

$g \mapsto \hat{U}(g)$ is a "twisted" homomorphisms from G to the group of unitaries $U(\mathcal{H})$, or an ordinary homomorphism from G to $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$.

Consequences of anomaly for $d = 0$

If $[\rho] \in H^2(G, U(1))$ is non-trivial, no G-invariant ray exists in $\mathcal{H}$.

Thus if $[\rho] \neq 0$, then

- The gauged Hilbert space is zero-dimensional.
- Ground states of any G-invariant Hamiltonian span a subspace of $\mathcal{H}$ of dimension $D > 1$.

Degree-3 group cohomology

$H^3(G, U(1))$ is the group of 3-cocycles modulo 3-coboundaries.

A 3-cocycle is a function $\omega : G \times G \times G \to U(1)$ such that for all $g, h, k, l \in G$

$$\omega(g,h,k)\omega(h,k,l)\omega(g,hk,l) = \omega(gh,k,l)\omega(g,h,kl).$$

A 3-coboundary is a 3-cocycle ω of the form

$$\omega(g,h,k) = \frac{\psi(g,h)\psi(gh,k)}{\psi(h,k)\psi(g,hk)},$$

for some $\psi : G \times G \to U(1)$.

The case of Lie groups

Lie groups cause problems, because they are uncountably infinite. Then $card(C^n(G, U(1)) > card(\mathbb{R})$. Group cohomology is very hard to compute, essentially unmanageable.

Chen, Gu, Liu, Wen, 2013 proposed to use measurable cocycles instead. But the physical meaning of such cocycles is unclear.

For compact connected Lie groups, like $U(1)$ or $SO(3)$, can use the Chern-Simons prescription. But what about disconnected Lie groups groups such as $O(2) = U(1) \rtimes \mathbb{Z}_2$?

Lieb-Schultz-Mattis theorem

The statement of the LSM theorem is reminiscent of 't Hooft anomaly matching:

Lieb-Schultz-Mattis 1961

Consider a 1d spin chain Hamiltonian invariant under translations $\mathbb{Z}$ and an on-site $SO(3)$. If the on-site Hilbert space is a projective representation of $SO(3)$, then the ground state in the thermodynamic limit cannot be unique, gapped, and $\mathbb{Z} \times SO(3)$-invariant.

There is a version of LSM where $SO(3)$ is replaced with an arbitrary finite symmetry group G (Ogata, Tachikawa, Tasaki, 2020).

Several authors suggested that the LSM theorem and its relatives are a manifestation of a QFT anomaly. But which anomaly?

LSM vs QFT anomalies

Since both translations and $SO(3)$ are involved, one might think that this is a "mixed" anomaly between $SO(3)$ and diffeomorphisms. But there are no such anomalies for $d = 1$.

Instead, Cheng and Seiberg, 2021, proposed that the low-energy EFT has an "emanant" $\mathbb{Z}_2^C$, and the anomaly is a mixed $SO(3) \times \mathbb{Z}_2^C$ anomaly.

Not completely clear how to check this since QFT anomalies for disconnected Lie groups like $SO(3) \times \mathbb{Z}_2$ are confusing.

Also, this is satisfactory only if the low-energy EFT has "emanant" $\mathbb{Z}_2^C$.

On the lattice $\mathbb{Z}_2^C$ is not present, there is only the translation symmetry $\mathbb{Z}$.

One of our goals is to deduce LSM from a more general result about anomalies of symmetries of lattice systems.

(The same problem was recently studied by Seifnashri but his approach is different.)

Anomalies and Symmetry Protected Topological phases

Consider an SPT in $d+1$ dimensions protected by a unitary on-site symmetry G.
Nayak and Else, 2015, proposed that any symmetric edge of such an SPT carries an "anomalous" action of G. Assuming this action is local but not on-site, argued that anomalies are labeled by elements of $H^{d+2}(G, U(1))$.

They further proposed that SPTs in the bulk are in 1-1 correspondence with anomalies of the edge degrees of freedom.

- Nayak and Else assumed that the edge is a lattice system, not a QFT.
- They assumed that the action of G on the edge system is by unitary circuits.
- For $d = 1$, their definition of the anomaly "index" is mathematically rigorous and purely kinematic.
- The precise meaning of the anomaly is also clear.

Lattice anomalies and gapped symmetric ground states

Nayak and Else also argued that a nonzero anomaly obstructs the existence of symmetric Short-Range Entangled states. More precisely:

Nayak-Else, 2015 (non-rigorous)

If the anomaly is nonzero, then a state which can be obtained from a trivial state by a circuit is never G-invariant.

This result is similar to LSM, but

- it does not put constraints on gapped states
- assumes action by circuits, so does not allow for translation symmetry
- problems for uncountable G, since $H^3(G, U(1))$ is unmanageable

Symmetries of 1d systems

A 1d lattice system is specified by a choice of an "on-site" Hilbert space V. The basic object is the algebra of local observables

$$\mathscr{A}^l = \otimes_{j\in\mathbb{Z}}\mathscr{A}_j,$$

where $\mathscr{A}_j \simeq B(V)$, the algebra of bounded operators on V. $\mathscr{A}^l$ is a normed $*$-algebra, its completion $\mathscr{A}^{ql}$ is a C^*-algebra.

Symmetries act by automorphisms of $\mathscr{A}^l$ (or perhaps $\mathscr{A}^{ql}$): bijective linear maps $\alpha : \mathscr{A}^l \to \mathscr{A}^l$ such that

$$\alpha(\mathcal{A}\mathcal{B}) = \alpha(\mathcal{A})\alpha(\mathcal{B}), \quad \alpha(\mathcal{A}^*) = \alpha(\mathcal{A})^*, \quad \forall \mathcal{A}, \mathcal{B} \in \mathscr{A}^l.$$

Gates and Circuits

A unitary gate is an automorphism of the form

$$\alpha = \mathrm{Ad}_{\mathcal{U}} : \mathscr{A}^l \to \mathscr{A}^l, \quad \mathcal{A} \mapsto \mathcal{U}\mathcal{A}\mathcal{U}^{-1}, \quad \mathcal{U}^*\mathcal{U} = 1,\ \mathcal{U} \in \mathscr{A}^l.$$

A block-partitioned unitary is a composition of an infinite number of gates with non-overlapping supports:

$$\alpha = \prod_{n\in\mathbb{Z}} \mathrm{Ad}_{\mathcal{U}_n}, \quad \mathcal{U}_n \in \mathscr{U}^l \subset \mathscr{A}^l.$$

Typically, one assumes that the gates have a bounded diameter.

A circuit is a composition of a finite number of block-partitioned unitaries.

Circuits form a subgroup $\mathcal{G}^{circ}$ in the group of all automorphisms.

Anomaly according to Nayak-Else

Let $\alpha : G \mapsto \mathcal{G}^{circ}$ be a homomorphism. For each $g \in G$ split $\alpha(g)$ into a product $\alpha_-(g) \circ \alpha_+(g)$ where $\alpha_+(g)$ (resp. $\alpha_-(g)$) acts trivially on local observables supported sufficiently far to the left (resp. right).

$\alpha_+ : G \to \mathcal{G}^{circ}$ is "almost a homomorphism": $\exists \mathcal{V} : G \times G \to \mathscr{U}^l$ such that

$$\alpha_+(g)\alpha_+(h) = \mathrm{Ad}_{\mathcal{V}(g,h)} \circ \alpha_+(gh).$$

These equations imply that

$$\omega(g,h,k) = \mathcal{V}(g,h)\mathcal{V}(gh,k)\mathcal{V}(g,hk)^{-1}\alpha_+(g)\left(\mathcal{V}(h,k)\right)^{-1}$$

is proportional to 1 and thus is an element of $U(1)$.

Finally, Nayak and Else prove that $\omega : G^3 \to U(1)$ is a 3-cocycle whose cohomology class does not depend on the choices made.

Quantum Cellular Automata

Turns out a translation of a 1d spin chain is not a circuit. But it is a QCA.

QCAs

A QCA of radius R an automorphism of $\mathscr{A}^l$ which maps any $\mathcal{A} \in \mathscr{A}_j$ to $\alpha(\mathcal{A}) \in \mathscr{A}^l_{[j-R,j+R]}$.

Translation by one site is a QCA of radius 1. QCAs form a subgroup $\mathcal{G}^{qca}$ of all automorphisms of $\mathscr{A}^l$. Every circuit is a QCA, but not all QCAs are circuits:

Gross-Nesme-Vogts-Werner, 2012

Let $\{p_i\}_{i\in J}$ be the set of all primes which divide dim V. There is a homomorphism $\mathcal{G}^{qca} \to \mathbb{Z}[\{\log p_i\}_{i\in J}]$ whose kernel is $\mathcal{G}^{circ}$.

I will call this homomorphism the GNVW index.

Results: anomaly for an abstract symmetry G

Let $\mathcal{G}^{lpa}$ denote the group of all Locality Preserving Automorphisms of $\mathscr{A}^{ql}$. It contains all QCAs and a lot more.

Anomaly index for an abstract symmetry G

For any homomorphism $\alpha : G \to \mathcal{G}^{lpa}$, there is a well-defined element of $H^3(G, U(1))$. If the GNVW index of $\alpha(g)$ vanishes for all $g \in G$, the anomaly index is an obstruction to writing $\alpha(g) = \alpha_-(g) \circ \alpha_+(g)$, where $\alpha_\pm : G \to \mathcal{G}^{lpa}$ are commuting homomorphisms such that α_- is approximately localized on $(-\infty, 0)$ and α_+ is approximately localized on $(0, +\infty)$.

Here we use an important and difficult result by Ranard-Walter-Witteveen, 2022, who proved that the GNVW index is well-defined not only for QCAs, but also for LPAs.

Results: anomaly and gapped symmetric ground states

We also prove a theorem which contains LSM-type results and Nayak-Else as special cases.

Anomaly as an obstruction to gapped and symmetric ground states

Let $\alpha : G \to \mathcal{G}^{lpa}$ be a homomorphism. Suppose there exists a Hamiltonian whose ground state is G-invariant and gapped. Then the anomaly index for α vanishes.

Relation to LSM: suppose $G = G_0 \times \mathbb{Z}$ where G_0 acts on V projectively, with a 2-cocycle $\rho : G \times G \to U(1)$, and $\mathbb{Z}$ acts by translations.

For any G_0, $H^3(G_0 \times \mathbb{Z}, U(1)) = H^3(G_0, U(1)) \oplus H^2(G_0, U(1))$.

In the above situation, the component of the anomaly index in $H^3(G_0, U(1))$ vanishes, while the component in $H^2(G_0, U(1))$ is equal to $[\rho]$.

Computation of the LSM anomaly, I

The LSM anomaly is a mixed anomaly between on-site G_0 and $\mathbb{Z}$. Let's see how it is computed.

Let G_0 act on V via a projective representation $\pi : G_0 \to U(V)$. $\forall g \in G_0$, the corresponding automorphism of $\mathscr{A}^l$ is a circuit:

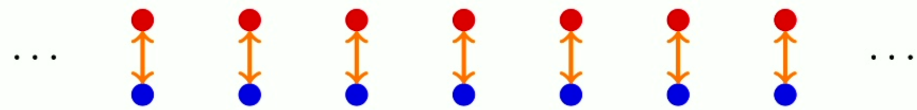
$$\alpha(g) = \prod_{j \in \mathbb{Z}} \mathrm{Ad}_{\pi_j(g)}.$$

Translations are QCAs which are not circuits. Let's stack with an identical spin chain on which G_0 acts trivially, but translations act "oppositely".

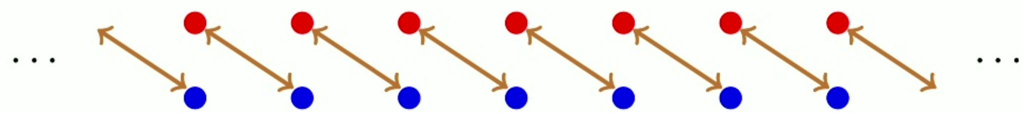
$G_0 \times \mathbb{Z}$ acts on the composite system by circuits. Indeed, since the GNVW index is additive under composition, the GNVW index of the $\mathbb{Z}$-action on the composite is zero.

Computation of the LSM anomaly, II

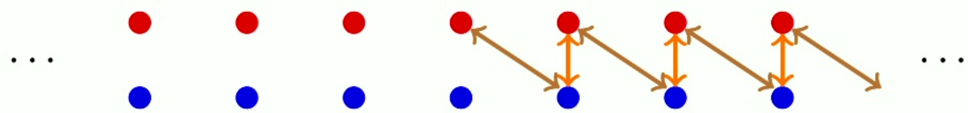
Explicitly, the generator of $\mathbb{Z}$ acts as $S' \circ S$ where S is a swap:



and S' is another swap:



A restriction of $S' \circ S$ to the right half-chain is a circuit τ_+:



A restriction of $\alpha(g)$ to the right half-chain is $\alpha_+(g) = \prod_{j=1}^{\infty} \mathrm{Ad}_{\pi_j(g)}$.

Computation of the LSM anomaly, III

Following Nayak and Else, we compare $\alpha_+(g) \circ \tau_+ \alpha_+(h)$ and $\tau_+ \alpha_+(gh)$. They differ by a conjugation with a unitary observable $\pi_1(g) \in \mathscr{U}_1 \subset \mathscr{A}^l$ which acts only on the 1st site. Thus $\mathcal{V}(g, 0; h, 1) = \pi_1(g)$.

On the other hand, $\mathcal{V}(g, 0; h, 0) = 1$, reflecting the fact that G_0 acts on-site, and thus $\alpha_+(g) \circ \alpha_+(h) = \alpha_+(gh)\ \forall g, h \in G_0$.

Finally, we evaluate the 3-cocycle ω on $G_0 \times \mathbb{Z}$ following Nayak and Else. The result is complicated in general, but it is sufficient to evaluate the component of the anomaly index in $H^2(G_0, U(1))$.

This amounts to taking the "slant product" of $[\omega] \in H^3(G_0 \times \mathbb{Z}, U(1))$ with the generator of the group homology $H_1(\mathbb{Z}, \mathbb{Z})$ and yields the 2-cocycle

$$\pi_1(gh)\pi_1(h)^{-1}\pi_1(g)^{-1} = \rho(g, h).$$

Lie group symmetries

A Lie group is simultaneously a group and a smooth manifold. An action of a Lie group G on a Hilbert space $\mathcal{H}$ is a smooth homomorphism $\alpha : G \to U(\mathcal{H})$.

It is clear what is meant by a smooth on-site action of G. But what is a smooth homomorphism of G into $\mathcal{G}^{qca}$ or $\mathcal{G}^{lpa}$? One needs to introduce some manifold structure on Locality-Preserving Automorphisms.

Leaving technicalities aside, there is a rather natural choice of a subgroup $\mathcal{G}^{ual} \subset \mathcal{G}^{lpa}$ which is a sort of infinite-dimensional manifold. $\mathcal{G}^{ual}$ includes smooth on-site Lie group actions, translations, and time evolutions by finite-range Hamiltonians.

Results: Lie symmetry G

For a Lie group G, there is a version of group cohomology which takes into account that both G and $U(1)$ are smooth manifolds. It was defined by J-L. Brylinsky and is denoted $H^n_{diff}(G, U(1))$. More generally, $H^n_{diff}(G, A)$ is defined for any Lie group G and any abelian Lie group A.

Anomaly index for a Lie group action

For any smooth homomorphism $\alpha : G \to \mathcal{G}^{ual}$ there is a well-defined element in $H^3_{diff}(G, U(1))$ which we call the anomaly index. When all LPAs in the image of α have a vanishing GNVW index, the anomaly index obstructs writing $\alpha = \alpha_- \circ \alpha_+$ for smooth commuting homomorphisms $\alpha_\pm : G \to \mathcal{G}^{ual}$ approximately localized on the two half-chains.

One can also prove that a nonzero anomaly index does not allow gapped symmetric ground states to exist.

Anomaly index and LSM

This allows one to include the original LSM, where $G = SO(3) \times \mathbb{Z}$, $SO(3)$ acts on-site and $\mathbb{Z}$ acts by translations.

One can show that for any Lie group G_0,
$H^3_{diff}(G_0 \times \mathbb{Z}, U(1)) = H^3_{diff}(G_0, U(1)) \oplus H^2_{diff}(G_0, U(1))$.

Also, if G_0 acts on-site, only the component in $H^2_{diff}(G_0, U(1))$ measuring a mixed anomaly between G_0 and $\mathbb{Z}$ can be nonzero.

$H^2_{diff}(G_0, U(1))$ labels equivalence classes of smooth projective representations of G_0, i.e. smooth homomorphisms $\pi : G_0 \to U(V)/U(1)$, where V is a Hilbert space. In our case, π is the on-site action of G_0.

The group $H^2_{diff}(G_0, U(1))$ is easily computed. For example, $H^2_{diff}(SO(3), U(1)) = \mathbb{Z}_2$.

Results: lattice anomalies vs QFT anomalies

In 1d QFT there can be no mixed anomaly between internal symmetry and translations. Lattice anomalies are different from QFT anomalies.

Another result highlighting the difference:

No anomalies for connected Lie group symmetries

Let G be a connected Lie group which acts on a 1d system by a smooth homomorphism $G \to \mathcal{G}^{ual}$. Then the anomaly index in $H^3_{diff}(G, U(1))$ vanishes.

One can show that for such a G, $H^3_{diff}(G, U(1)) \simeq H^4(BG, \mathbb{Z})$. This group labels Chern-Simons couplings for G in $d = 2$, i.e. QFT anomalies for G for $d = 1$.

$H^4(BG, \mathbb{Z})$ is nonzero for all compact connected Lie groups. For example, for $G = U(1)$, it is $\mathbb{Z}$. Its elements label the $U(1)$ Kac-Moody level of the 1+1d CFT. No spin chain can have such an anomaly.

Differentiable cohomology of Lie groups

I will only explain (very roughly) how to define $H^2_{diff}(G, A)$, where G is a Lie group and A is an abelian Lie group.

Choose "nice" open covers of G: $\bigsqcup_{i \in I_1} U_1^{(i)} \to G$ and $G \times G$: $\bigsqcup_{i \in I_2} U_2^{(i)} \to G \times G$, and $G \times G \times G$, etc.
We also need double overlaps $U_1^{(ij)} = U_1^{(i)} \cap U_1^{(j)}$, $i, j \in I_1$ of elements of U_1, double overlaps $U_2^{(ij)}$ of elements of U_2, and triple overlaps $U_1^{(ijk)}$ of elements of U_1.

A differentiable 2-cochain is a pair $\left(\{\rho^{(i)}\}_{i \in I_2}, \{\sigma^{(ij)}\}_{i,j \in I_1}\right)$, where $\rho^{(i)} \in C^\infty(U_2^{(i)}, A)$ and $\sigma^{(ij)} \in C^\infty(U_1^{(ij)}, A)$.

A 2-cocycle is a 2-cochain whose coboundary vanishes. The coboundary map takes a 2-cochain to a triple $\left(\{\omega^{(i)}\}_{i \in I_3}, \{\upsilon^{(ij)}\}_{i,j \in I_2}, \{\eta^{(ijk)}\}_{i,j,k \in I_1}\right)$, where $\omega^{(i)} \in C^\infty(U_3^{(i)}, A)$, $\upsilon^{(ij)} \in C^\infty(U_2^{(ij)}, A)$, $\eta^{(ijk)} \in C^\infty(U_1^{(ijk)}, A)$.

Locality-Preserving Automorphisms

Recall that $\mathscr{A}^{ql}$ is the norm-closure of $\mathscr{A}^{l}$. $\forall X \subset \mathbb{Z}$ and $\forall r > 0$, let $B_X(r)$ be the r-thickening of X. $\forall X \subset \mathbb{Z}$, let $\mathscr{A}_X$ denote the algebra of local observables localized on X. Let $g(r)$ be a monotonicall decreasing positive function such that $\lim_{r\to\infty} g(r) = 0$.

Automorphisms with tails according to Ranard-Walter-Witteveen

An automorphism $\alpha : \mathscr{A}^{ql} \to \mathscr{A}^{ql}$ has $g(r)$-tails if $\forall X \subset \mathbb{Z}$, $\forall \mathcal{A} \in \mathscr{A}^{l}_{X}$ and $\forall r > 0$ there is $\mathcal{B} \in \mathscr{A}^{l}_{B_X(r)}$ such that $\|\alpha(\mathcal{A}) - \mathcal{B}\| \leq \|\mathcal{A}\| g(r)$.

Definition of a Locality-Preserving Automorphism

An LPA is an automorphism which has $g(r)$-tails for some $g(r)$.

One can show that LPAs form a group $\mathcal{G}^{lpa}$.

A QCA of radius R is an LPA with $g(r)$ which vanishes for $r > R$. Time-evolution with a finite-range Hamiltonian is an LPA with $g(r)$ which decays exponentially.

Uniformly Almost Local automorphisms

Definition of UAL automorphisms

A UAL automorphism is an LPA with $g(r) = \mathcal{O}(r^{-\infty})$.

UAL automorphisms form a group $\mathcal{G}^{ual} \subset \mathcal{G}^{lpa}$. This group is a sort of infinite-dimensional Lie group. More precisely, it is a diffeological group, in the sense that the notion of a smooth map $M \to \mathcal{G}^{ual}$ is defined, where M is any manifold.

The Lie algebra of $\mathcal{G}^{lpa}$ is the Lie algebra of local Hamiltonians with $\mathcal{O}(r^{-\infty})$-decaying interactions. Any element in the identity component of $\mathcal{G}^{ual}$ can be reached by an evolution with a time-dependent Hamiltonian of this kind.

Components of $\mathcal{G}^{ual}$ are labeled by the GNVW index which takes values in $\mathbb{Z}[\{\log p_i\}_{i \in J}]$. Thus every UAL automorphism is a composition of some translations and evolution with a time-dependent Hamiltonian with $\mathcal{O}(r^{-\infty})$-decaying interactions.

The split property

The proof of the main theorem relies on the fact that ground states of gapped Hamiltonians satisfy the split property: the ground state $\omega : \mathscr{A} \to \mathbb{C}$ is unitarily equivalent to $\omega_+ \otimes \omega_-$, where $\omega_\pm$ are some pure states of left and right half-chains.

The split property follows (Matsui, 2013) from the area law for gapped 1d states (Hastings, 2007).

Split property holds for other types of states too, and the main theorem applies to them.