Title: Reading between the sections of the Seiberg-Witten curve

Speakers: Cyril Closset

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Abstract: I will explore subtle aspects of rank-one 4d N=2 supersymmetric QFTs through their low-energy Coulomb-branch physics. The low-energy Lagrangian is famously encoded in "the Seiberg-Witten (SW) curve", which is a one-parameter family of elliptic curves. Here I will explain precisely how "global" aspects of the SQFT such as its spectrum of lines, which cannot be read off from the Lagrangian, are encoded into the SW curve -- more precisely, how they are encoded in its Mordell-Weil group of rational sections. In particular, I will
discuss in detail the difference between the pure SU(2) and the pure SO(3) N=2 SYM theories from this low-energy perspective. I will also comment on the global forms of rank-one 5d SCFTs compactified on a circle.

Zoom link https://pitp.zoom.us/j/94967350368?pwd=SHZZUnZNL3ZveFc3NElCaVc5YkY3Zz09

Reading between the sections of the Seiberg-Witten curve

Cyril Closset

University of Birmingham & Royal Society URF

Perimeter Institute, Waterloo, 12 December 2023

based on 2308.10225 with Horia Magureanu

Global structures in 4d QFT

Consider a QFT in four space-time dimensions.

Our question today: **What is the global form of the QFT?**

A QFT with a specific choice of global structure is called an absolute theory.

Example (gauge theory): pure $SU(2)$ versus pure $SO(3)$ Yang-Mills theory in 4d. The Lagrangian is the same in both cases:

$$S = \frac{1}{4g^2} \int d^4x \sqrt{g}\, \mathrm{tr} F_{\mu\nu} F^{\mu\nu}$$

and all local observables are the same, but the two theories differ through their spectrum of line operators.

Alternatively, the path integral over arbitrary $\mathcal{M}_4$ distinguishes between them.

Global structures in QFT

In general, we want to have a completely precise understanding of the symmetries of any given QFT. This is an ongoing task for the hep-th community. [cite here: everyone]

See: generalised symmetries, categorical symmetries,...

For QFTs in 4d, the global form of the theory is determined by a maximal set of mutually-local line operators. [Aharony, Seiberg, Tachikawa, 2013]

For instance, consider the pure $\mathfrak{su}(2)$ gauge theory:

- Wilson line in irrep $\mathfrak{R}$ of dimension $\lambda_e + 1$:

$$W = \mathrm{Tr}_{\mathfrak{R}}\, P e^{i \int A}$$

- 't Hooft line H corresponding to monopole of charge $\lambda_m \in \mathbb{Z}$.

Generic lines are not mutually local:

$$WH = (-1)^{\lambda_e \lambda_m} HW$$

4d $\mathcal{N}=2$ SQFTs: Global structures from the Coulomb branch

Let us consider **4d $\mathcal{N}=2$ supersymmetric theories** (8 supercharges).

4d $\mathcal{N}=2$ SQFT have a Coulomb branch (CB),

$$\mathcal{M}_C \cong \mathbb{C}^r \cong \mathrm{Spec}\,\mathbb{C}[u_1, \cdots, u_r] \ ,$$

with r photons (and its $\mathcal{N}=2$ superpartners) at low energy. The integer r is the rank of the SQFT. The CB parameters u are the VEVs of some specific half-BPS operators.

Due to supersymmetry, the low-energy physics is determined by a single locally holomorphic function, $\mathcal{F}(a)$, the prepotential. Here a denote the scalar partners to the low-energy photons.

[Seiberg, Witten, 1994] famously solved for the prepotential **exactly** – that is, non-perturbatively – for the pure $\mathcal{N}=2$ $SU(2)$ gauge theory (and for SQCD). This is the celebrated Seiberg-Witten solution.

In this talk, we wish to explain how the spectrum of lines is encoded into the Seiberg-Witten solution – **all in the special case $r=1$.**

Seiberg-Witten theory and the choice of gauge group

In some sense, we simply want to read the following papers together:

- Seiberg-Witten theory. [Seiberg, Witten, 1994]
- Reading between the lines. [Aharony, Seiberg, Tachikawa, 2013]
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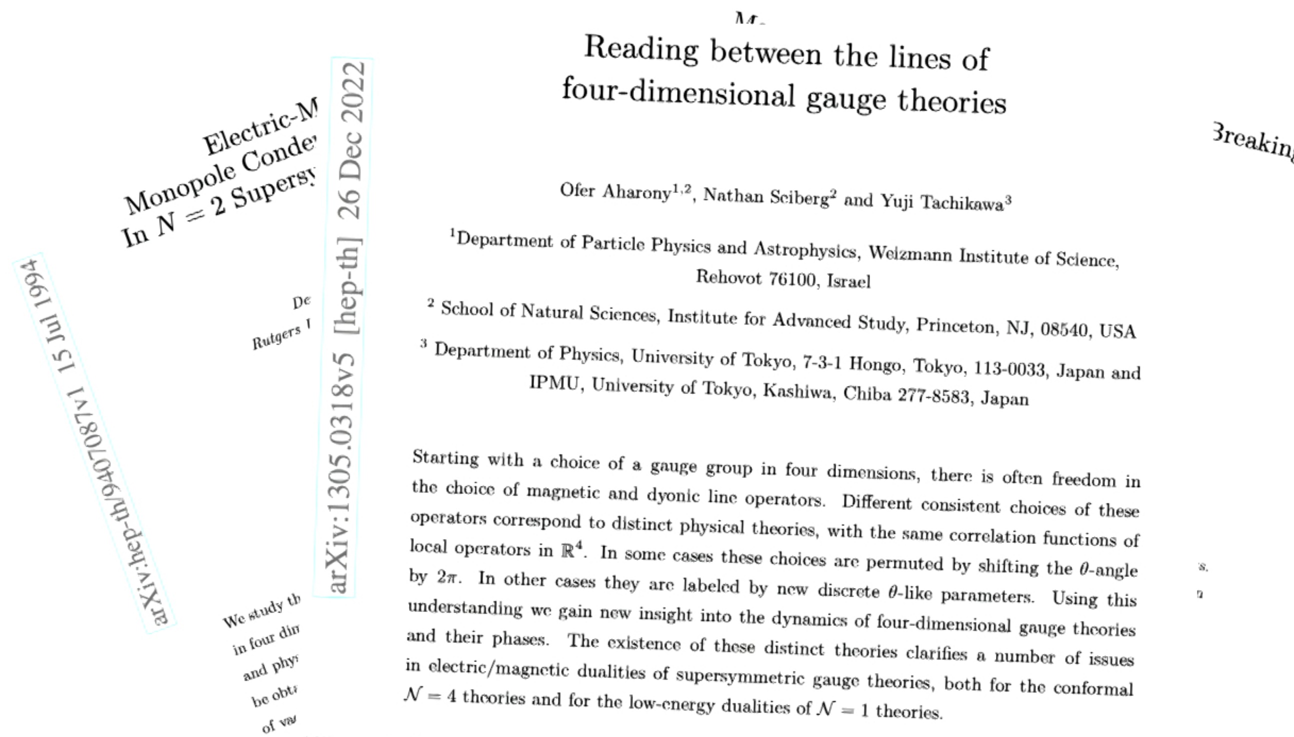


Reading between the lines of four-dimensional gauge theories

Ofer Aharony[1,2], Nathan Seiberg[2] and Yuji Tachikawa[3]

[1]Department of Particle Physics and Astrophysics, Weizmann Institute of Science, Rehovot 76100, Israel

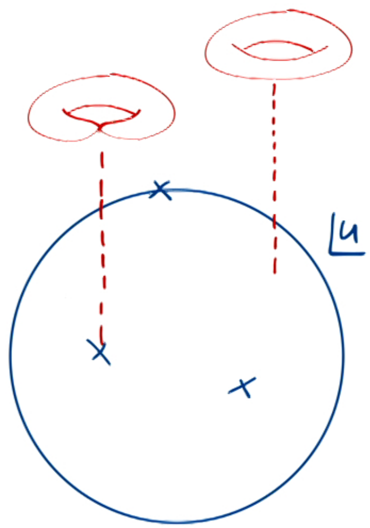
[2] School of Natural Sciences, Institute for Advanced Study, Princeton, NJ, 08540, USA

[3] Department of Physics, University of Tokyo, 7-3-1 Hongo, Tokyo, 113-0033, Japan and IPMU, University of Tokyo, Kashiwa, Chiba 277-8583, Japan

Starting with a choice of a gauge group in four dimensions, there is often freedom in the choice of magnetic and dyonic line operators. Different consistent choices of these operators correspond to distinct physical theories, with the same correlation functions of local operators in $\mathbb{R}^4$. In some cases these choices are permuted by shifting the θ-angle by 2π. In other cases they are labeled by new discrete θ-like parameters. Using this understanding we gain new insight into the dynamics of four-dimensional gauge theories and their phases. The existence of these distinct theories clarifies a number of issues in electric/magnetic dualities of supersymmetric gauge theories, both for the conformal $\mathcal{N}=4$ theories and for the low-energy dualities of $\mathcal{N}=1$ theories.

Seiberg-Witten theory and the choice of gauge group

The Seiberg-Witten solution packages the low-energy effective action of the $SU(2)$ $\mathcal{N}=2$ SYM theory in terms the **Seiberg-Witten curve**, an auxiliary mathematical object.



Basic questions:

- The $SU(2)$ gauge theory has a one-form symmetry under which Wilson lines are charged:

$$\mathbb{Z}_2^{[1]} \quad : \quad W \to (-1)^{\lambda_e} W$$

  Can we see this from the SW curve?

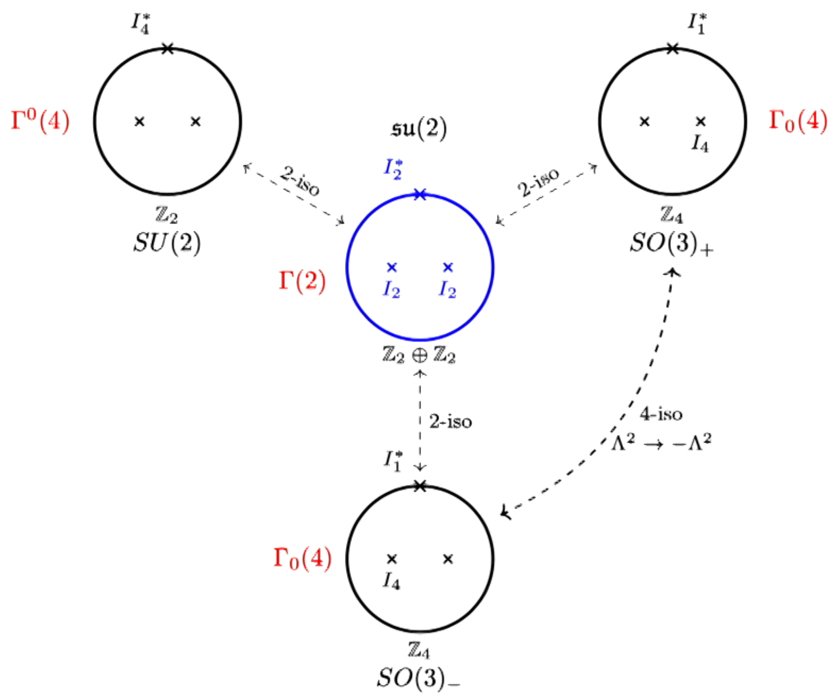
- If we gauge $\mathbb{Z}_2^{[1]}$, we obtain the gauge group:

$$SO(3) = SU(2)/\mathbb{Z}_2$$

How do gauge $\mathbb{Z}_2^{[1]}$ at the level of the SW curve?

Seiberg-Witten theory and the choice of gauge group

Short answer: 'consider the rational sections'!



We will need to distinguish between relative and **absolute** SW curves. This clarifies the status of some of the curves first discussed in [Seiberg, Witten, 1994].

Reading between the sections in pure $\mathfrak{su}(2)$ $\mathcal{N}=2$ SYM

Reading between the lines

Consider the case of $\mathfrak{su}(2)$ gauge theories.

[Gaiotto, Moore, Neitzke, 2010; Aharony, Seiberg, Tachikawa, 2013]

Given the $\mathfrak{su}(2)$ Lie algebra, we have two possible compact Lie groups, $SU(2)$ or $SO(3)$. They are related as:

$$\mathbb{Z}_2 \to SU(2) \to SO(3)$$

where $\mathbb{Z}_2 = Z(SU(2))$, the center. This is simply $\{\mathbf{1}, -\mathbf{1}\}$.

For any 4d gauge theory with Lie algebra $\mathfrak{g}$, consider the possible dyonic lines. They have electromagnetic charges determined by magnetic and electric weights:

$$\boxed{(\lambda_m, \lambda_e) \in \Lambda_{\mathrm{mw}} \oplus \Lambda_{\mathrm{w}}}$$

For $\mathfrak{g} = \mathfrak{su}(2)$, we have the electric weight lattices:

$$\Lambda_{\mathrm{w}}^{SO(3)} \subset \Lambda_{\mathrm{w}}^{SU(2)} \equiv \Lambda_{\mathrm{w}}^{\mathfrak{su}(2)} \subset \mathfrak{t}^*$$

and the magnetic weight lattices:

$$\mathfrak{t} \supset \Lambda_{\mathrm{mw}}^{SO(3)} \equiv \Lambda_{\mathrm{mw}}^{\mathfrak{su}(2)} \supset \Lambda_{\mathrm{mw}}^{SU(2)}$$

Here $\mathfrak{t} \cong \mathfrak{u}(1)$ is the Cartan subalgebra.

Reading between the lines

To have a consistent QFT, we need to have a spectrum of lines that are mutually local:

$$\big\langle(\lambda_m, \lambda_e), (\lambda'_m, \lambda'_e)\big\rangle = 0 \bmod 2$$

There are three solutions, corresponding to a choice of **gauge group G**:

$$\begin{aligned} SU(2) &\quad : \quad \lambda_m \in 2\mathbb{Z}\ ,\ \lambda_e \in \mathbb{Z}\ , \\ SO(3)_+ &\quad : \quad \lambda_m \in \mathbb{Z}\ ,\ \lambda_e \in 2\mathbb{Z}\ , \\ SO(3)_- &\quad : \quad \lambda_m, \lambda_e \in \mathbb{Z}\ ,\ \lambda_m + \lambda_e \in 2\mathbb{Z}\ . \end{aligned}$$

The $SO(3)_\pm$ theories differ only by a shift of the θ angle:

$$SO(3)_-^\theta = SO(3)_+^{\theta+2\pi}$$

In summary, a choice of lines is the same as a choice for G. This is the global structure.

Reading between the lines

In pure gauge theories (or with adjoint matter only, like $\mathcal{N}=2$ SYM), we can consider the defect group $\mathbb{D}$ of $\mathfrak{su}(2)$ lines that cannot be screened by dynamical particles. These are essentially the charges λ mod 2:

$$(z_m, z_e) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \equiv \mathbb{D}$$

Choosing a consistent set of lines is equivalent to choosing a 'Lagrangian subgroup'

$$\mathbb{Z}_2 \subset \mathbb{D}$$

Here we have the possible $\mathbb{Z}_2$'s generated by $(0,1)$, $(1,0)$ or $(1,1)$, respectively.

These unscreened lines are charged under a one-form symmetry $\mathbb{Z}_2^{[1]}$:

$SU(2)$	:	Wilson line $(0,1)$, electric $\mathbb{Z}_2^{[1]}$	$\leftarrow$ 'center symmetry'
$SO(3)_+$	:	't Hooft line $(1,0)$, magnetic $\mathbb{Z}_2^{[1]}$	$\leftarrow$ 'magnetic one-form symmetry'
$SO(3)_-$	:	dyonic line $(1,1)$, diagonal $\mathbb{Z}_2^{[1]}$.	

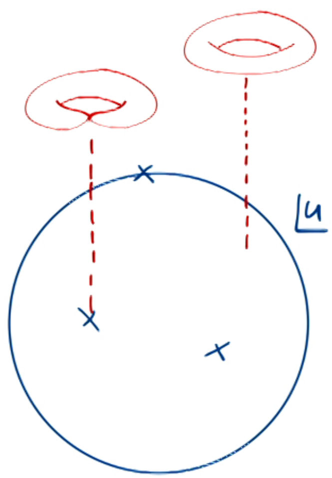
Seiberg-Witten curves

In SW theory, we are interested in the LEEA on the Coulomb branch parameterised by

$$u = \langle \mathrm{Tr}(\Phi^2) \rangle$$

The low-energy dynamics is written in terms of a low-energy photon a and its magnetic dual a_D. Together they determine **the infrared gauge coupling** τ as a function of u:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} , \qquad \tau = \frac{\partial a_D}{\partial a} = \frac{\partial^2 \mathcal{F}}{\partial a^2} .$$



- The SW solution identifies the 'physical periods' (a_D, a) as periods of an elliptic curve E_u:

$$a_D = \int_{\gamma_B} \lambda_{\mathrm{SW}} , \qquad a = \int_{\gamma_A} \lambda_{\mathrm{SW}}$$

The SW differential λ_{SW} satisfies $\frac{d\lambda_{\mathrm{SW}}}{du} = \frac{1}{2\pi}\boldsymbol{\omega}$.

- Hence, the IR gauge coupling τ is identified with the modular parameter of E_u:

$$\boxed{\tau = \frac{\partial a_D}{\partial a} = \frac{\omega_D}{\omega_a} .} \qquad \left(\frac{da_D}{du} = \frac{\omega_D}{2\pi} , \; \frac{da}{du} = \frac{\omega_a}{2\pi} , \right)$$

Mathematical intermezzo: Elliptic curves

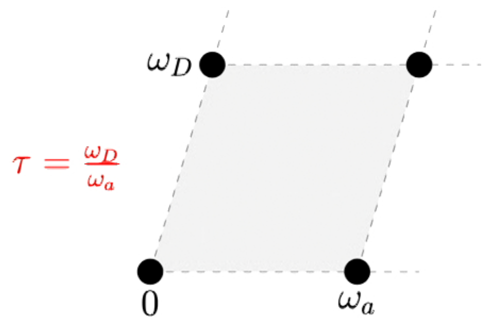
The SW geometry is the total space of the elliptic fibration:

$$E \to \mathcal{S} \to \mathbb{P}^1 \cong \{u\}$$

It is a rational elliptic surface (RES), which we can write in Weierstrass normal form:

$$\boxed{y^2 = 4x^3 - g_2(u)x - g_3(u)\ ,} \qquad g_2, g_3 \in \mathbb{C}(u)\ , \qquad \boldsymbol{\omega} = \frac{dx}{y}$$

The generic fiber is a smooth elliptic curve E_u:



$$E \cong \mathbb{C}/\Lambda\ , \qquad \Lambda \cong \omega_a\mathbb{Z} + \omega_D\mathbb{Z}$$

For $z \in \mathbb{C}$, we have a point on the elliptic curve:

$$(x, y) = (\wp(z), \wp'(z))$$

The RES must also have singular fibers at points on $\mathbb{P}^1$ such that:

$$\Delta(u) \equiv g_2(u)^3 - 27g_3(u)^2 = 0$$

Mathematical intermezzo: Elliptic curves

Allowed singular fibers of RES follow the Kodaira classification. There are 20 possibilities: [Persson, 1990; Miranda, 1990]

$$I_1, \cdots, I_9 \ , \quad I_0^*, \cdots, I_4^* \ , \quad , II \ , \ III \ , \ IV \ , \ II^* \ , \ III^* \ , \ IV^* \ .$$

In the Weierstrass model, these are ADE singularities associated to subgroups of E_8:

$$\begin{aligned} I_n \quad &\leftrightarrow \quad A_{n-1} = \mathfrak{su}(n) \ , \\ I_n^* \quad &\leftrightarrow \quad D_{k+4} = \mathfrak{so}(2k+8) \ , \\ &\vdots \\ II^* \quad &\leftrightarrow \quad \mathfrak{e}_8 \ . \end{aligned}$$

We can focus here on the I_n singularities. Physically, an I_n singularity on the CB corresponds to a low-energy $U(1)$ gauge theory with massless hypermultiplets and β-function coefficient:

$$b_0 \equiv \sum_i Q_i^2 = n \ .$$

Two examples:

$U(1) \oplus n$ electrons of charge 1 **or** $U(1) \oplus 1$ electrons of charge $\sqrt{n}$,

Fully deformable versus undeformable singularity. This will be important below.

Mathematical intermezzo: Rational sections

A classic problem in the theory of elliptic curves is to find the *rational points*. E.g.:

$$y^2 = 4x^3 - 8 , \qquad (x, y) = (3, 10) , \left(\frac{129}{100}, -\frac{383}{500}\right) , \left(\frac{164323}{29241}, -\frac{132469670}{5000211}\right) , \cdots$$

Here, we have $g_2, g_3 \in \mathbb{C}(u)$, so we are looking for solutions:

$$P = (x(u), y(u)) \in \mathbb{C}(u)^2$$

These are the rational sections of the RES. The Mordell-Weil theorem tells us that the rational points form a finitely generated abelian group:

$$\Phi(\mathcal{S}) \equiv E(\mathbb{C}(u)) \cong \mathbb{Z}^{\mathrm{rank}(\Phi)} \oplus \mathbb{Z}_{N_1} \oplus \cdots \mathbb{Z}_{N_k} \ .$$

Today we will focus our attention on the torsion subgroup:

$$\boxed{\Phi_{\mathrm{tor}} \cong \mathbb{Z}_{N_1} \oplus \cdots \mathbb{Z}_{N_k}}$$

generated by points of finite order. For a N-torsion section:

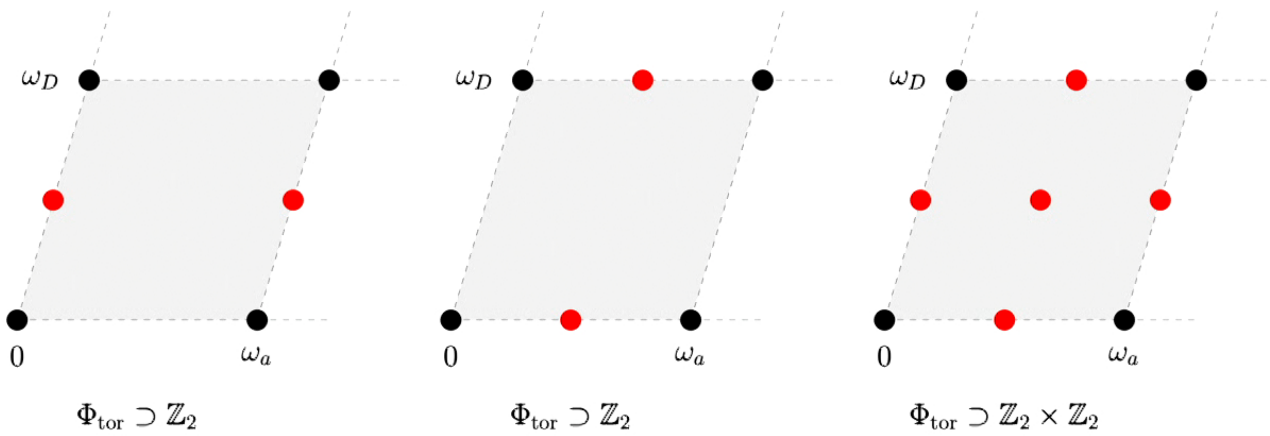
$$NP = \underbrace{P + \cdots + P}_{\text{N times}} = \mathcal{O} \equiv 0$$

Mathematical intermezzo: Rational sections

For most of what follows, we can focus on 2-torsion sections, $2P = 0$. These are:

$$P = (x_0, 0)\ , \qquad x = x_0 \quad \text{solves} \quad f(x) \equiv 4x^3 - g_2 x - g_3 = 4\prod_{i=1}^{3}(x - x_i) = 0$$

and there are either zero, one or three rational solutions. On the smooth fiber, they are the obvious order-2 points:



Mathematical intermezzo: Isogenies between elliptic curves

An *isogeny* is an homomorphism of elliptic curves:

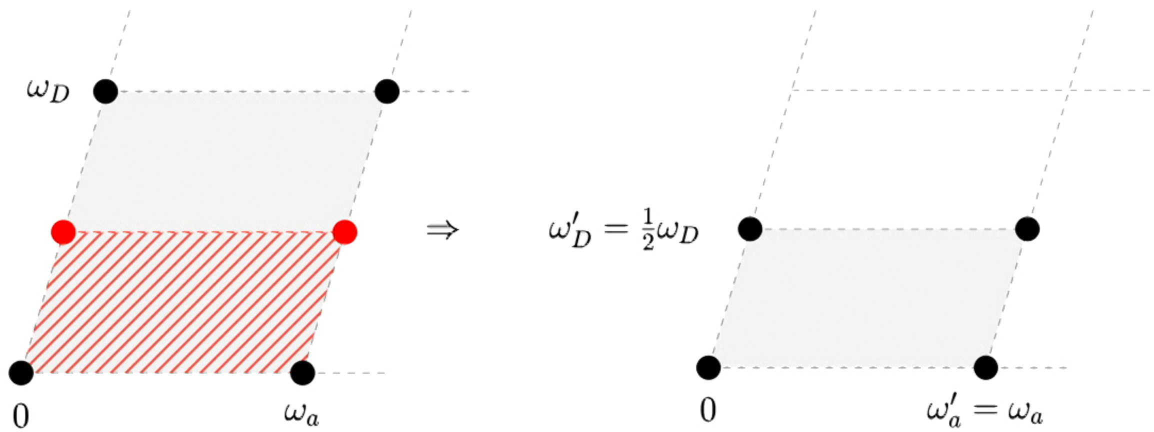
$$\psi_\alpha \;:\; E \to E'$$

The kernel of ψ_α is finite. At the level of complex tori, $\mathbb{C}/\Lambda$, we have $\psi_\alpha(0) = 0$, hence:

$$\psi_\alpha(z + L) = \alpha z + L' \;, \qquad \ker(\psi_\alpha) = L'/\alpha L$$

Here $\alpha \in \mathbb{C}^*$ gives a homothety (rescaling).

Fact: **Any N-torsion section induces an isogeny, which extends to the full RES $\mathcal{S}$,** including a well-understood action on its singular fibers. For a $\mathbb{Z}_2$ torsion, for instance:



BPS states and central charge

Back to physics. The last bit that we need to recall is that the Coulomb-branch theory admits massive one-particle excitations with dyonic charges. For the $SU(2)$ gauge theory:

$$\gamma = (m, q) \in \widehat{\Gamma}_{SU(2)} \subsetneq \Gamma \cong \mathbb{Z}^2$$

Here, we can identify Γ with the the homology lattice of the generic fiber. Identifying γ with a one-cycle

$$[\gamma] = m\gamma_B + q\gamma_A \ , \qquad \Gamma \cong H_1(E, \mathbb{Z}) \ ,$$

the **Dirac pairing** is identify exactly with the homology pairing:

$$\langle \gamma_1, \gamma_2 \rangle = m_1 q_2 - q_1 m_2 = [\gamma_1] \cdot [\gamma_2] \ .$$

This is the statement that the SW curve is principally polarised. [Argyres, Martone, Ray, 2022]

The central charge of the BPS state is given by:

$$Z_\gamma(u) = m\, a_D(u) + q\, a(u)$$

$SU(2)$ curve and one-form symmetry

The $SU(2)$ curve is given explicitly by: [Seiberg, Witten, 1994]

$$g_2^{SU(2)} = \frac{4u^2}{3} - \Lambda^4 \, , \qquad g_3^{SU(2)} = -\frac{8u^3}{27} + \frac{u\Lambda^4}{3} \, ,$$

with the discriminant:

$$\Delta^{SU(2)} = \Lambda^8 (u - \Lambda^2)(u + \Lambda^2)$$

At the singularities at $u = \pm\Lambda^2$, the monopole and dyon become massless. We encode the Dirac pairing of these light states in a BPS quiver: [Alim *et al*, 2011]

$\gamma_M = (1, 0)$

$\gamma_D = (-1, 2)$

The full BPS spectrum is built out of bound states of these two hypermultiplet states. [SW, 1994; Bilal, Ferrari, 1996] In particular, we see that all one-particle states are of the form:

$$\gamma = (n, 2m) \in \widehat{\Gamma}_{SU(2)} \, , \qquad n, m \in \mathbb{Z}$$

Let us call $\widehat{\Gamma}_{SU(2)}$ the lattice of "allowed charges".

$SU(2)$ curve and one-form symmetry

The dyonic charges $\gamma = (m, q)$ are related to the $\mathfrak{su}(2)$ weights above as

$$\gamma = (m, q) = \left(\frac{\lambda_m}{2}, \lambda_e\right) \in \Lambda_{\mathrm{mw}}^{SU(2)} \oplus \Lambda_{\mathrm{w}}^{SU(2)}$$

This is the statement that the minimal $SU(2)$ 't Hooft line corresponds to $\lambda_m = 2$.

The only BPS line that cannot be screened is the fundamental Wilson line

$$\gamma_L = (0, 1)$$

It is charged under the one-form symmetry $\mathbb{Z}_2^{(1)}$.

[del Zotto, Garcia Etxebarria, 2022]

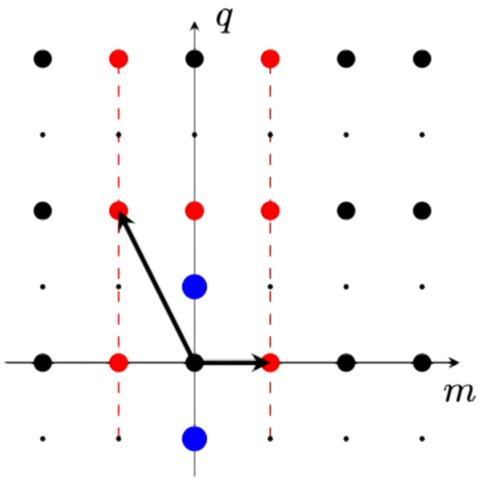
$\mathbb{Z}_2^{(1)}$ can be identified with MW group of the $SU(2)$ SW curve:

$$\boxed{\Phi = \Phi_{\mathrm{tor}} = \mathbb{Z}_2 \cong \mathbb{Z}_2^{(1)}\ ,}$$

$$P_{\mathbb{Z}_2} = \left(\frac{u}{3}, 0\right)$$

[CC, Magureanu, 2021]

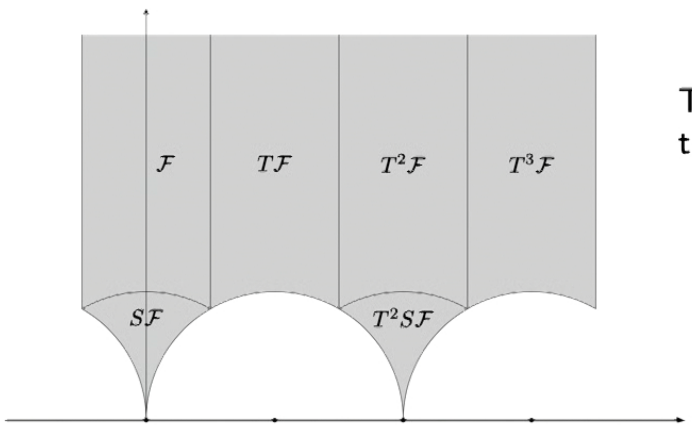
[Cecotti, Caorsi, 2017]



Aside: Coulomb branch as a modular curve

The Coulomb branch of pure $SU(2)$ is famously a modular curve for the congruence subgroup $\Gamma^0(4)$ of $\mathrm{PSL}(2,\mathbb{Z})$.

I.e. there exists a biholomorphism from the u-plane to the τ-plane (mod $\Gamma^0(4)$).



$$\frac{u(\tau)}{\Lambda^2} = \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{2\vartheta_2(\tau)^2\vartheta_3(\tau)^2}$$

The CB monodromies can be read off from the cusps:

$$\mathbb{M}_{u=1} = STS^{-1} \ ,$$
$$\mathbb{M}_{u=-1} = (T^2S)T(T^2S)^{-1} \ ,$$
$$\mathbb{M}_{\infty} = PT^4 \ .$$

The two SW singularities are 'of width 1'. These are singular fiber 'of type I_1'.

One-form symmetry from rational sections: General claim (I)

Jumping ahead, and extrapolating from this example, we assert the following:

(1) The SW geometry of any **absolute** rank-one 4d $\mathcal{N}=2$ theory $\mathcal{T}$ is given by a **principally polarised RES** $\mathcal{S}$. **The line lattice of $\mathcal{T}$ is identified with the homology lattice of E_u.**

(2) Given such a theory, its discrete one-form symmetry $\Gamma^{[1]}$ is isomorphic to a subgroup of the torsion part of the MW group of $\mathcal{S}$:

$$\Gamma^{[1]} \subset \Phi_{\rm tor}(\mathcal{S})$$

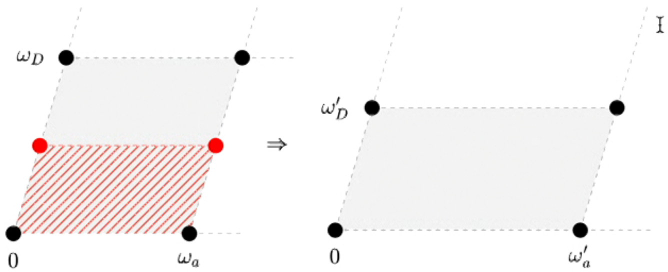
- **Absolute 4d QFT** means a QFT with a maximal consistent choice of lines.
- Simple heuristic argument: Put the theory on $\mathbb{R}^3 \times S^1$. Then $\Gamma^{[1]}$ gives rise to a 0-form symmetry $\Gamma^{[0]}$ spontaneously broken on the 3d $\mathcal{N}=4$ CB. The latter is essentially the SW geometry itself. [Seiberg, Witten, 1996; Gaiotto, Moore, Neitzke, 2010]
- Can be motivated by IIB geometric engineering/mirror symmetry (SW curve as part of IIB background/ LG model).
- We haven't specified how to identify the subgroup, yet! Not all torsion sections are born equal.

The relative $\mathfrak{su}(2)$ curve

Next, we would like to understand the gauging

$$SO(3) = SU(2)/\mathbb{Z}_2^{[1]}$$

A natural guess is to look at the isogeny generated by $P_{\mathbb{Z}_2}$ for the $SU(2)$ theory.



$$\psi_{\alpha=\sqrt{2}} : E^{SU(2)} \to E^{\mathfrak{su}(2)}$$

with

$$\boxed{E^{\mathfrak{su}(2)} \equiv \sqrt{2} \circ E^{SU(2)}/t_{P_{\mathbb{Z}_2}}}$$

- The periods and coupling are related by:

$$a'_D = \frac{\sqrt{2}}{2} a \ , \qquad a' = \sqrt{2} a \ , \qquad \tau' = \frac{\tau}{2}$$

  and then the charges are:

$$(m', q') = \left(\sqrt{2} m, \frac{1}{\sqrt{2}} q \right)$$

- The rescaling by $\alpha = \sqrt{2}$ is crucial to preserve the Dirac pairing. As a result of this rescaling, we end up with a non-principally polarised curve.

The relative $\mathfrak{su}(2)$ curve

The SW curve we obtain from this isogeny is what we call the **relative curve.** It reads:

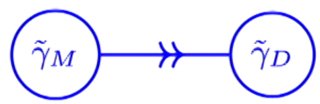
$$g_2^{\mathfrak{su}(2)} = \frac{u^2}{3} + \Lambda^4 \ , \qquad g_3^{\mathfrak{su}(2)} = -\frac{u^3}{27} + \frac{u\Lambda^4}{3} \ ,$$

with the discriminant:

$$\Delta^{\mathfrak{su}(2)} = \Lambda^4 \left(u^2 - \Lambda^4\right)^2 \ .$$

Now we have to singularities with the massless particles of charge ('type I_2'):

$$\tilde{\gamma}_M = \sqrt{2}(1,0)$$



$$\tilde{\gamma}_D = \sqrt{2}(-1,1)$$

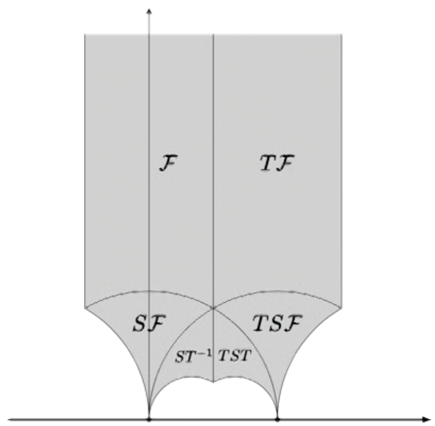
Here the homology lattice and the charge lattice are related by a factor of $\sqrt{2}$:

$$\tilde{\gamma} = \sqrt{2}\widehat{\gamma} \ , \qquad \widehat{\gamma} = \left(\frac{\lambda_m}{2}, \frac{\lambda_e}{2}\right) \in \widehat{\Gamma} \ .$$

It is only $\widehat{\Gamma}$ which is identified with the homology lattice.

The relative $\mathfrak{su}(2)$ curve and its modular group

Up to the (crucial) rescaling by $\sqrt{2}$, the relative curve is the $\Gamma(2)$ curve of the first SW paper. Namely, the CB is a modular curve for $\Gamma(2)$. [SW, 1994]



This is the τ' plane. The modular function is simply obtained from the $\Gamma^0(4)$ function using:

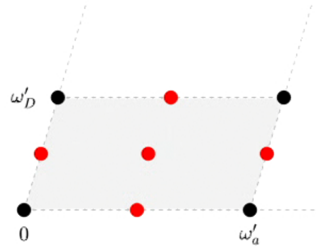
$$\tau = 2\tau'$$

The CB monodromies can be read off from the cusps:

$$\mathbb{M}_{u=1} = ST^2S^{-1}\ ,$$
$$\mathbb{M}_{u=-1} = (T^{-1}S)T^2(T^{-1}S)^{-1}\ ,$$
$$\mathbb{M}_{\infty} = PT^2\ .$$

Note for the experts: This is the same curve as for $SU(2)$ $N_f = 2$ but the I_2 singularities here are undeformable. [Aspman, Furrer, Manschot, 2021; CC, Magureanu, 2021]

Rational sections of the relative $\mathfrak{su}(2)$ curve

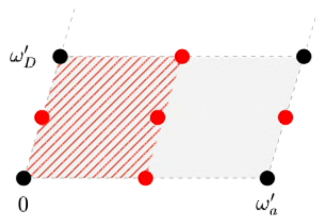


$$\boxed{\Phi_{\rm tor} = \mathbb{Z}_2 \oplus \mathbb{Z}_2}$$

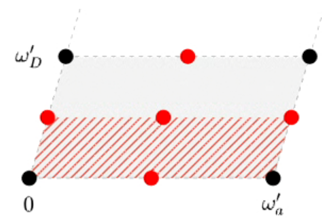
generated by $P_i = (x_i, 0)$ with

$$x_1 = -\frac{u}{3}\,, \qquad x_2 = \frac{u}{6} + \frac{\Lambda^2}{2}\,, \qquad x_3 = \frac{u}{6} - \frac{\Lambda^2}{2}\,.$$

Performing isogenies along any $\mathbb{Z}_2 \subset \Phi_{\rm tor}$, we recover absolute curves:



$SU(2)$



$SO(3)_+$

General claim (II): The MW torsion of a (mass-deformed) relative curve is identified with the defect group of the theory:

$$\Phi(\mathcal{S}_{\rm mass}) \cong \mathbb{D}$$

The absolute curves are then obtained through isogenies. ('Reading between the sections.')

The $SO(3)_{\pm}$ curves

Performing the isogeny explicitly, we obtain the curves:

$$E^{SO(3)_+} = \sqrt{2} \circ E^{\mathfrak{su}(2)}/t_{P_2} \ , \qquad E^{SO(3)_-} = \sqrt{2} \circ E^{\mathfrak{su}(2)}/t_{P_3} \ ,$$

They read:

$$\boxed{g_2^{SO(3)_{\pm}} = \frac{u^2}{12} \pm \frac{5\Lambda^2 u}{2} + \frac{11\Lambda^4}{4} \ , \qquad g_3^{SO(3)_{\pm}} = -\frac{u^3}{216} \pm \frac{7u^2\Lambda^2}{24} + \frac{29u\Lambda^4}{24} \pm \frac{7\Lambda^6}{8} \ ,}$$

with the discriminant:

$$\Delta^{SO(3)_{\pm}} = \frac{1}{8}\Lambda^2 \left(u \pm \Lambda^2\right) \left(u \mp \Lambda^2\right)^4 \ .$$

Now we have an I_1 and an I_4 singularity. The two curves (and the two singularities) are exchanged by a shift of the θ-angle,

$$\theta \to \theta + 2\pi \qquad \leftrightarrow \qquad \Lambda^2 \to -\Lambda^2$$

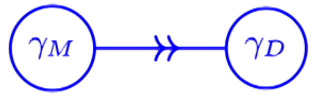
Due to total rescaling $\sqrt{2} \circ \sqrt{2} = 2$, the SW differentials are related as:

$$\lambda_{SO(3)_{\pm}} = 2\lambda_{SU(2)} \ .$$

The $SO(3)_\pm$ curves

For $SO(3)_+$, the light BPS states are:

$\gamma_M = (2,0)$



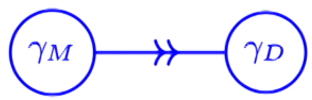
$\gamma_D = \sqrt{2}(-2,1)$

The monopole has now charge 2 in the $SO(3)_+$ normalisation.

$$(m_+, q_+)_{SO(3)_+} = \left(2m, \frac{q}{2}\right)_{SU(2)} .$$

Similarly, for $SO(3)_-$, we have:

$\gamma_M = (2,-1)$



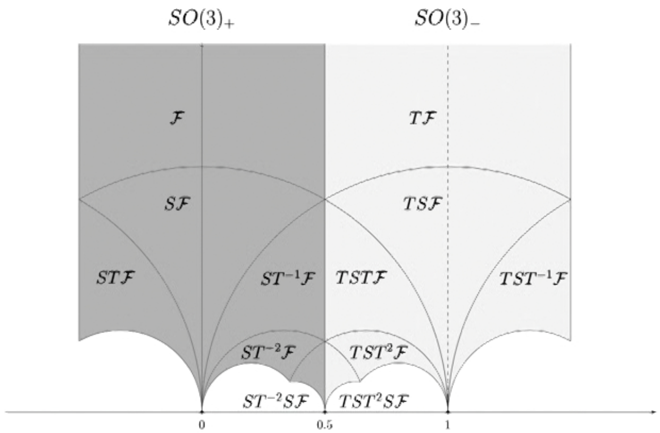
$\gamma_D = \sqrt{2}(-2,2)$

and here the dyon has charge 2 in the $SO(3)_-$ normalisation:

$$(m_-, q_-)_{SO(3)_-} = \left(2m\ ,\ \frac{q}{2} - m\right)_{SU(2)} = \left(m_+\ ,\ q_+ - \frac{m_+}{2}\right)_{SO(3)_+} .$$

The $SO(3)_\pm$ curves

The $SO(3)_\pm$ Coulomb branch is a modular curve for $\Gamma_0(4)$.



The modular function is obtained from the $\Gamma^0(4)$ function using:

$$\tau_{SO(3)_+} = \frac{1}{4}\tau_{SU(2)} \ ,$$

$$\tau_{SO(3)_-} = \tau_{SO(3)_+} + \frac{1}{2} \ .$$

Note for the experts: This is the same curve as for $SU(2)$ $N_f = 3$ but the I_4 is undeformable.

The $SO(3)_\pm$ curves and their torsion sections

The $SO(3)_\pm$ curves have an interesting Mordell-Weil group:

$$\Phi = \Phi_{\rm tor} = \mathbb{Z}_4$$

For $SO(3)_+$, they are generated by:

$$P_{\mathbb{Z}_4} = \left(\frac{u}{12} - \frac{3\Lambda^2}{4}, \frac{iu\Lambda}{\sqrt{2}} - \frac{i\Lambda^3}{\sqrt{2}}\right) , \qquad 2P_{\mathbb{Z}_4} = P_{\mathbb{Z}_2^{[1](m)}} = \left(-\frac{u}{6} - \frac{\Lambda^2}{2}, 0\right)$$

Writing this as a group extension

$$0 \to \mathbb{Z}_2^{[1](m)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

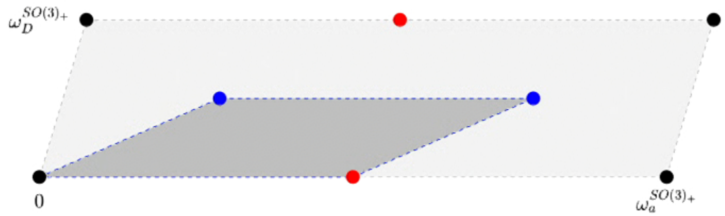
the $\mathbb{Z}_2^{[1](m)} \subset \mathbb{Z}_4$ subgroup is identified with the magnetic one-form symmetry of $SO(3)_+$. So this is another example of our **general claim (I):**

$$\Gamma^{[1]} \subset \Phi_{\rm tor}(\mathcal{S})$$

Coming from the relative curve, the $\Gamma^{[1]}$ can be identified as the generator of the *dual isogeny*.

The $SO(3)_\pm$ curves and their torsion sections

But what about the $\mathbb{Z}_4$ generator here? In fact, we simply have:



with a rescaling by $\alpha = \sqrt{4} = 2$, so that:

$$\boxed{E^{SO(3)_+}/t_{P_{\mathbb{Z}_4}} = E^{SO(3)_-}}$$

There is an interesting physical interpretation of this mathematical fact.

The $SO(3)_\pm$ curves and non-invertible symmetry

The $SU(2)$ theory has a $\mathbb{Z}_8^{(R)}$ R-symmetry, which is spontaneously broken to $\mathbb{Z}_4$ on the CB branch. The spontaneously broken $\mathbb{Z}_2$ exchanges the two SW singularities.

In the $\mathcal{N}=1$ deformation of the theory, we have two physically equivalent confining vacua, exchanged by the $\mathbb{Z}_2$ R-symmetry. [SW, 1994]

Moreover, the $\mathbb{Z}_2 \subset \mathbb{Z}_8^{(R)}$ has a mixed anomaly with the $\mathbb{Z}_2^{[1]}$ one-form symmetry: [Cordova, Dumitrescu, 2018]

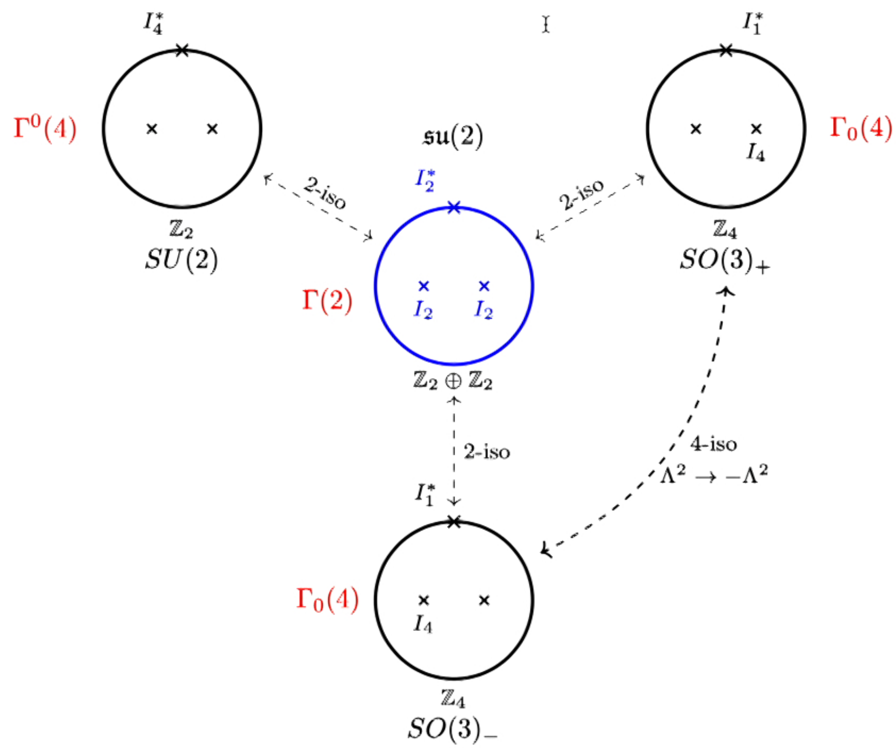
$$\frac{\pi}{2}\int_{\mathcal{M}_5} C^{[1]} \cup \mathcal{P}(B^{[2]})$$

Gauging $\mathbb{Z}_2^{[1]}$, we loose the $\mathbb{Z}_2$ one-form symmetry, hence the two SW singularities are *not* physically equivalent. Indeed, upon mass deformation to $\mathcal{N}=1$ $SO(3)$ SYM one vacuum confines trivially and the other has an IR TQFT. [Aharony, Seiberg, Tachikawa, 2013]

The $\mathbb{Z}_2$ 0-form symmetry of the $SU(2)$ gauge theory becomes a non-invertible symmetry (NIS) $\mathcal{N}$ in the $SO(3)_\pm$ theory. [Kaidi, Ohmori, Zheng, 2021]

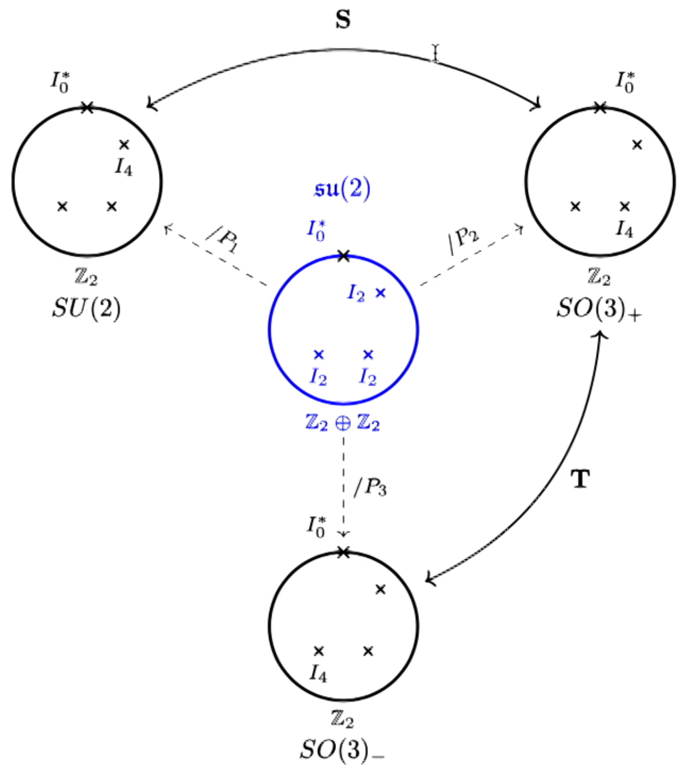
Claim (III): The 4-isogeny above composed with the shift $\theta \to \theta + 2\pi$ implements the action of the NIS $\mathcal{N}$ on the $SO(3)$ SW curve.

Summary: SW curves for pure $\mathfrak{su}(2)$ $\mathcal{N}=2$ SYM



Reading between the lines $\Leftrightarrow$ Reading between the rational sections

Another example: SW curves for the $\mathfrak{su}(2)$ $\mathcal{N} = 2^*$ theory



Here, S-duality acts non-trivially on the UV coupling $\tau_{\rm uv}$ (and on the mass), and there is a unique absolute curve. We recover the pure $\mathfrak{su}(2)$ in the large mass limit.

5d SCFTs and global structures of KK theories

Global structures of 5d SCFTs on S^1

Consider a 5d SCFT. It is a strongly coupled conformal field theory with 8 Poincaré supercharges. There is a conjectured classification of rank-one 5d SCFTs, indexed by their flavour symmetry algebra: [Seiberg, 1996; ... ; Bhardwaj, 2019]

$$E_0 = \{\} , \quad E_1 = A_1 , \quad \tilde{E}_1 = \mathfrak{u}(1) , \quad E_2 , \quad \cdots , E_8 , \quad \mathfrak{u}(1)_B .$$

Only two of these theories have a non-trivial one-form (electric) symmetry:

$$E_0 \ : \qquad \Gamma^{[1]}_{\rm 5d} = \mathbb{Z}_3$$
$$E_1 \ : \qquad \Gamma^{[1]}_{\rm 5d} = \mathbb{Z}_2$$

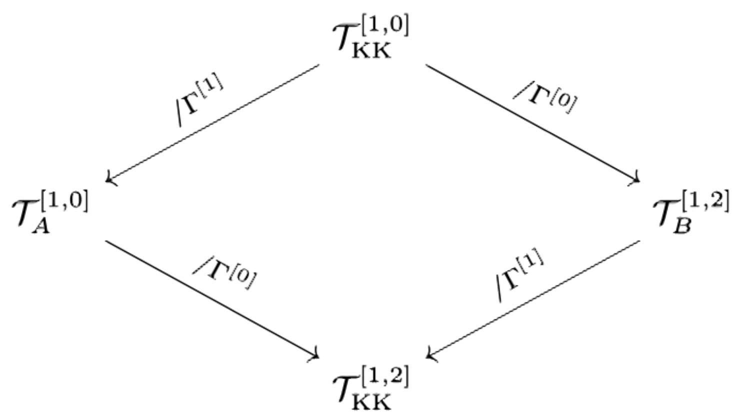
We consider these theories on $\mathbb{R}^4 \times S^1$, giving us rank-one 4d $\mathcal{N} = 2$ KK theories with symmetry:

$$\Gamma^{[1]}_{\rm 5d} = \mathbb{Z}_N \quad \to \quad \Gamma^{[1]} = \mathbb{Z}_N , \quad \Gamma^{[0]} = \mathbb{Z}_N .$$

Global structures of 5d SCFTs on S^1

In 5d, gauging a one-form symmetry gives us a dual two-form symmetry. (In M-theory on a local Calabi-Yau, this corresponds to either M2-brane or M5-brane defects.)

In 4d, we can gauge the zero-form and one-form symmetries separately. Thus, we expect to have a number of global structures in 4d:

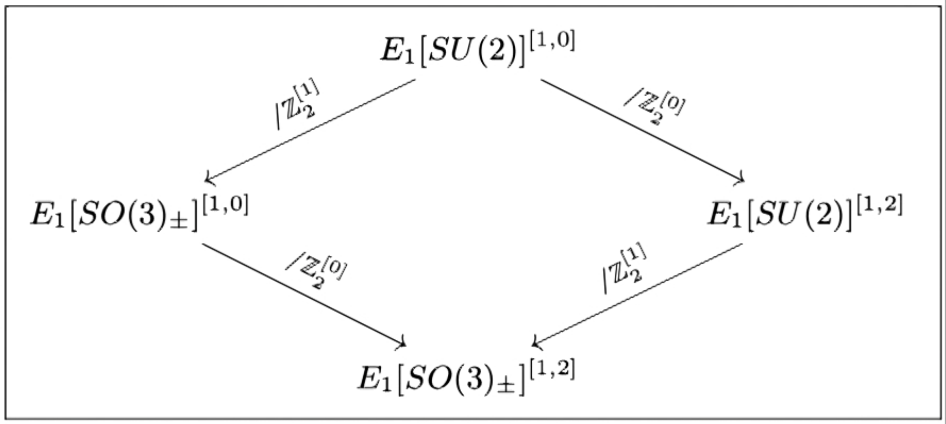


That is, *assuming 't Hooft anomalies vanish*. In fact, $\Gamma_{5\text{d}}^{[1]}$ of E_1 is non-anomalous, while E_0 has an anomaly $\sim B^3$. [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki, 2021]
The analysis of the SW curves bears this out.

Global structures for E_1

The E_1 theory can be mass-deformed to the 5d $\mathcal{N} = 1$ $\mathfrak{su}(2)$ gauge theory. Its SW curves depends on a "mass parameter" λ, with $\lambda = 1$ the SCFT limit.

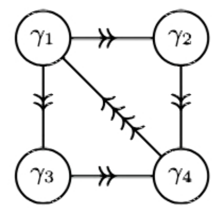
We find the KK theories:



For the $E_1[SU(2)]^{[1,2]}$ theory, we have four I_1 singularities if $\lambda \neq 1$,

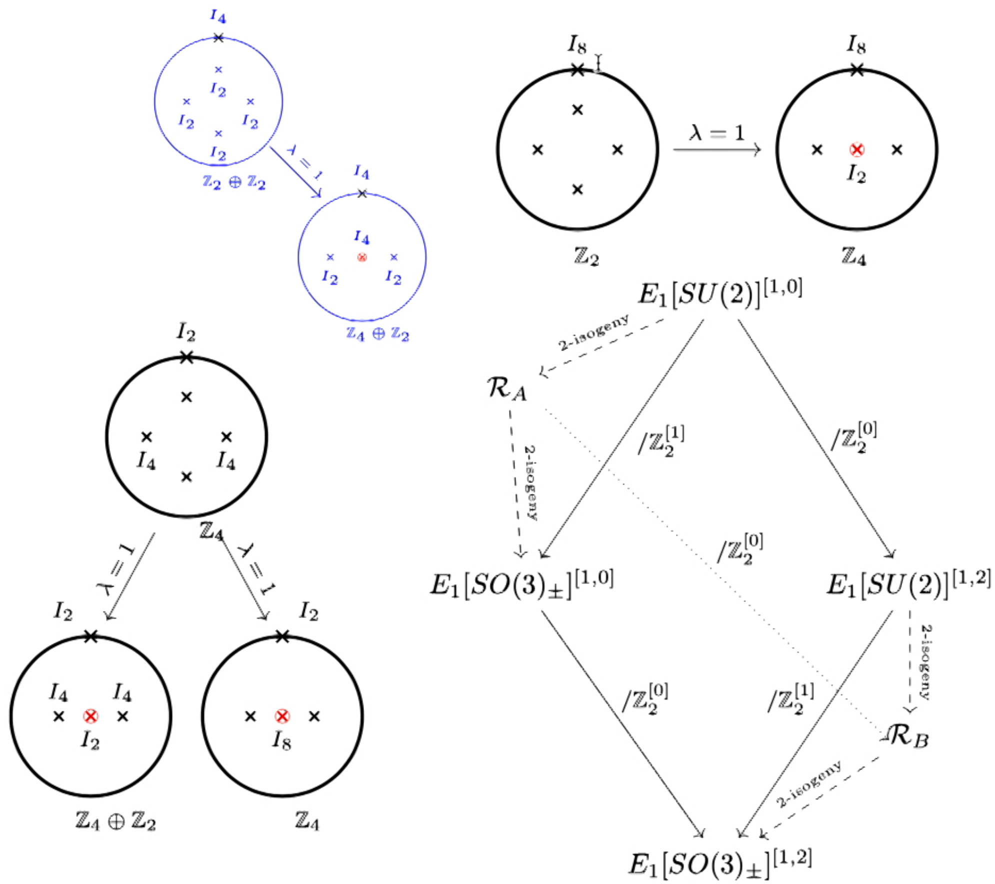
$$\gamma_1 = (1,0) \ ,$$
$$\gamma_2 = \gamma_3 = (-1,2) \ ,$$
$$\gamma_4 = (1,-4) \ ,$$

with BPS quiver:



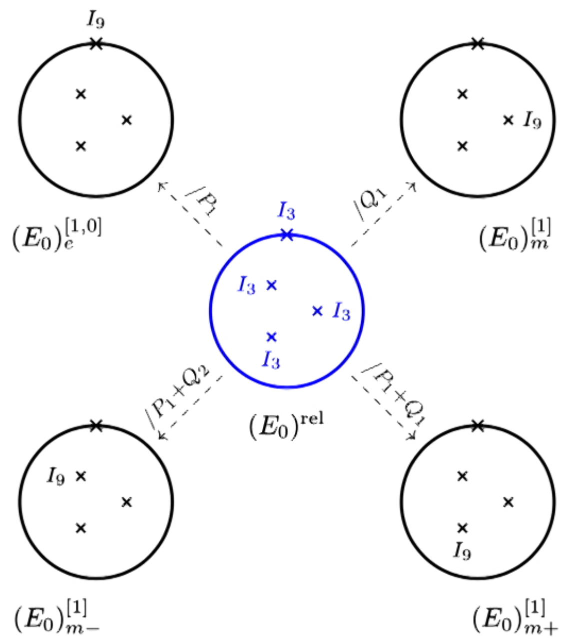
see also [Jia, Yi, 2022]

Global structures for E_1: with $[1,0]$-form symmetry



Global structures for E_0

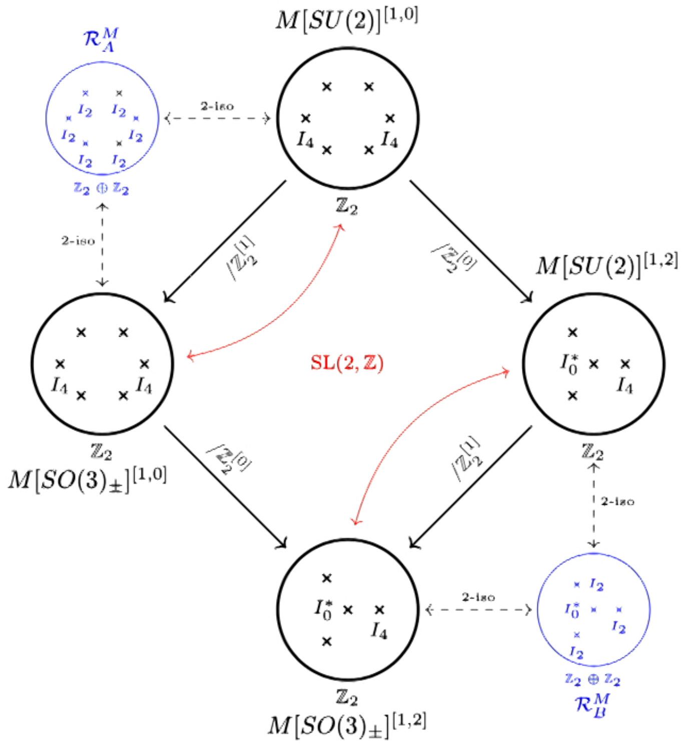
For E_0, we similarly have relative and absolute curves for the $\mathbb{Z}_3^{[1]}$ structure:



The cubic anomaly for $\mathbb{Z}_3^{[1]}$ in 5d gives a mixed anomaly in 4d, so that upon gauging $\mathbb{Z}_3^{[1]}$ is 4d we loose the $\mathbb{Z}_3^{[0]}$ apparent in the $(E_0)_e^{[1,0]}$ theory.

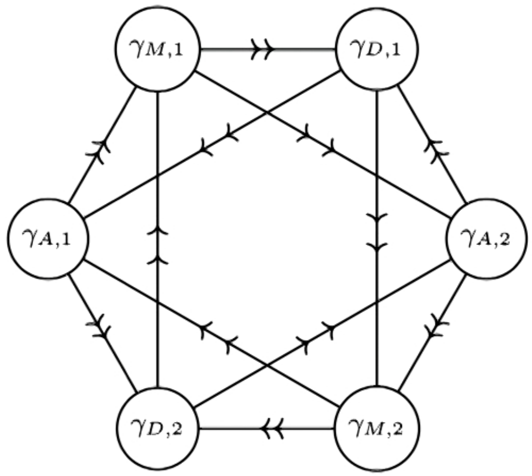
Absolute SW curves for the M-string

We have a similar story for the M-string theory – that is, the 6d $\mathcal{N} = (2,0)$ A_1 theory on $\mathbb{R}^4 \times T^2$ with a flat connection on T^2 ('adjoint mass' in the 4d limit):



A 6d BPS quiver

Finally, analysing the M-string theory Coulomb branch, we discovered a simple 6d BPS quiver for the M-string theory on $\mathbb{R}^4 \times T^2$:



This passes many consistency checks, but deserves further study...

[CC, del Zotto, Grossutti, Magureanu, to appear]

Summary: Reading between the rational sections

We developped a systematic understanding of the global structure of rank-one 4d $\mathcal{N} = 2$ SQFTs from their Seiberg-Witten geometry. We summarise this in three conjectures:

Conjecture I. **(Defect group.)** *The Seiberg-Witten curve of any* *<u>relative</u>* *rank-one theory* $\mathcal{T}_{\rm rel}$ *is given by a non-principally-polarised elliptic curve. In this case, the defect line group of the theory is isomorphic to the torsion part of the MW group of the mass-deformed curve:*

$$\mathbb{D} \cong \Phi_{\rm tor}(\mathcal{S}_{\rm mass}) \ .$$

Conjecture II. **(One-form symmetry.)** *The Seiberg-Witten curve of any* *<u>absolute</u>* *rank-one theory* $\mathcal{T}$ *must be a principally-polarised elliptic curve. The line lattice is identified with the homology lattice. Then, the one-form symmetry* $\Gamma^{[1]}$ *of* $\mathcal{T}$ *is isomorphic to a subgroup of the torsion part of the Mordell-Weil group of the mass-deformed curve:*

$$\Gamma^{[1]} \subseteq \Phi_{\rm tor}(\mathcal{S}_{\rm mass}) \ .$$

Conjecture III. **(Gauging $\Gamma^{[1]}$.)** *At the level of the rank-one SW geometry, the gauging of a one-form symmetry* $\Gamma^{[1]} = \mathbb{Z}_N^{[1]}$ *is the composition of two* N*-isogenies:*

$$E(\mathcal{T}) \xrightarrow{N\text{-isogeny}} E_{\rm rel} \xrightarrow{N\text{-isogeny}} E(\mathcal{T}/\Gamma^{[1]})$$

Summary: Reading between the rational sections

- We also hinted at further structures encoded in the SW geometry: non-invertible symmetries, 2-groups [CC, Magureanu, 2021], etc. This deserves further study.
- We clarified aspects of the global structures of 5d and 6d SCFTs compactified to $\mathbb{R}^4 \times T^{d-4}$. A rich structure emerged, combining one-form symmetries and discrete gauging.

Outlook:

- We focussed on rank-one theories. We need new tools to go to higher ranks.
- The reduction to 3d $\mathcal{N} = 4$ is extemely rich [Seiberg, Witten, 1996; Gaiotto, Moore, Neitzke, 2010]. This will be an important area for future work.
- We would like a more direct physics proof of our conjectures. For instance, through geometric engineering in string theory.
- Relatedly, our global structures for 5d theories on $\mathbb{R}^4 \times S^1$ should be understood in terms of branes (D-branes, NS5-branes and F1) in Type IIA.

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