Title: Partition counting, instantons and enumerative geometry

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Abstract: Counting partitions in diverse dimensions is a long-standing problem in enumerative combinatorics. It also plays a prominent role in the physics of instanton counting and in algebraic geometry through the computation of Donaldson-Thomas invariants. In this talk I will give an overview of these counting problems, and discuss how recent developments in the computation of instanton/Donaldson-Thomas partition functions clarify some open problems in the enumeration of higher-dimensional partitions.

Zoom link https://pitp.zoom.us/j/92547375606?pwd=VDBiTTV6QjBtWThnSjJPc0phVEI1dz09

Partition Counting, Instantons and Enumerative Geometry

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Introduction

- Counting problems of enumerative combinatorics arise in various areas of physics (gauge theory, statistical mechanics, ...) and mathematics (algebraic geometry, representations of ∞-D algebras, topology, ...)
- **In this talk:** Supersymmetric gauge theories in dimensions $2d = 2, 4, 6, 8$ and string theory through the lens of instanton counting
- Counting problems are well-understood for $d = 3$:
  - BPS states of D-branes in IIA string theory on toric Calabi–Yau 3-folds X, described as melting crystals
    (Aganagic *et al.* '03; Okounkov, Reshetikin & Vafa '03; Ooguri & Yamazaki '08; ...)
  - Instantons in 6D cohomological gauge theory on X
    (Iqbal *et al.* '03; Nekrasov '05; Jafferis '07; Cirafici, Sinkovics & Sz '08; Awata & Kanno '09; Del Zotto *et al.* '21; ...)
  - Donaldson–Thomas invariants (virtual numbers of sheaves on X)
    (Graber & Pandharipande '97; Maulik *et al.* '03; Okounkov '17; Fasola, Monavari & Ricolfi '20; ...)
- Until the last few years, generalizations to $d = 4$ were **not** well understood

Introduction

- For X a toric Calabi–Yau 4-fold, recent developments include:
  - Constructions of virtual cycles for moduli of sheaves on X (Cao & Leung '14; Borisov & Joyce '15; Cao & Kool '17; Oh & Thomas '20)
  - Instanton counting in 8D cohomological gauge theory on X (Nekrasov '17; Nekrasov & Piazzalunga '18; Bonelli *et al.* '20; Kimura '22; Sz & Tirelli '23; Kimura & Noshita '23; ...)
  - Donaldson–Thomas theory of X (Cao, Kool & Monavari '19; Liu '23; ...)
  - BPS D-branes on X/4D melting crystal (Galakhov & Li '23; Franco '23)
- From gauge theory perspective, predecessors are Donaldson–Witten and Vafa–Witten theories in $d = 2$, and by dimensional reduction vortex counting in $\mathcal{N} = (2,2)$ and $\mathcal{N} = (4,4)$ theories in $d = 1$ (Bonelli, Tanzini & Zhao '11; Yoshida '11; ...)
- **Goal:** Broad-brush overview and comparison of these developments in the various dimensionalities; see also (Sz & Tirelli [arXiv:2207.12862])

Outline

- Combinatorics: Partition counting
- Generalized instanton equations
- Moduli spaces
- Instanton counting: Enumerative geometry

$d = 1$: Counting Points

Count objects with no structure, paying attention only to total number:

	$\varnothing$	$\bullet$	$\bullet\bullet$	$\bullet\bullet\bullet$	$\cdots$
$n =$	0	1	2	3	$\cdots$

There is $p_1(n) = 1$ configuration for each n, weigh by q^n ($|q| < 1$)
Encode all counts at once in a generating (partition) function:

$$\mathcal{G}_1(q) := \sum_{n=0}^{\infty} p_1(n)\, q^n = \sum_{n=0}^{\infty} q^n$$

The geometric series was summed by Euclid (and others ...):

$$\mathcal{G}_1(q) = \frac{1}{1-q}$$

$d = 1$: **Counting Points**

Simple generating functions, such as $\mathcal{G}_1(q)$, can written as plethystic exponentials of even simpler functions ('single-particle partition functions'):

If $f(q) = \sum_{n=1}^{\infty} a_n q^n$ is an analytic function with $f(0) = 0$, then

$$\mathrm{PE}[f(q)] := \exp\left(\sum_{n=1}^{\infty} \frac{f(q^n)}{n}\right) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}}$$

or we may write $f(q) = \mathrm{PL}\left(\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}}\right)$ as a plethystic logarithm

For the geometric series:

$$\mathrm{PL}[\mathcal{G}_1(q)] = q$$

$d = 2$: Counting Partitions

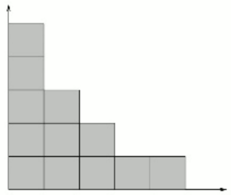
Now keep track of how a collection of n objects is composed of more elementary constituents, paying attention only to their total numbers:

$p_2(n)$:= number of ways of representing $n \in \mathbb{N}$ as a sum of natural numbers (up to permutations of summands)

Counts partitions $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots)$, $\lambda_i \in \mathbb{N}$ of size

$$n = |\boldsymbol{\lambda}| := \sum_i \lambda_i$$

Correspond to Young diagrams $Y_{\boldsymbol{\lambda}} \subset \mathbb{Z}^2_{\geqslant 0}$ with λ_i squares in i^{th} row:

E.g. For $n = 12$: $\boldsymbol{\lambda} = (5 \geqslant 3 \geqslant 2 \geqslant 1 \geqslant 1)$ $\quad Y_{\boldsymbol{\lambda}} =$ 

$d = 2$: **Counting Partitions**

Generating function: $\mathcal{G}_2(q) := \sum_{n=0}^{\infty} p_2(n)\, q^n = \sum_{\boldsymbol{\lambda}} q^{|\boldsymbol{\lambda}|}$

This series was summed by Euler:

$$\mathcal{G}_2(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

with plethystic logarithm

$$\mathrm{PL}[\mathcal{G}_2(q)] = \frac{q}{1-q} = \sum_{n=1}^{\infty} q^n$$

Modularity property ($q = \mathrm{e}^{2\pi\,\mathrm{i}\,\tau}$, $\mathrm{Im}(\tau) > 0$):

$$\mathcal{G}_2(\mathrm{e}^{2\pi\,\mathrm{i}\,\tau}) = \sqrt{\tfrac{\tau}{\mathrm{i}}}\ \mathrm{e}^{\frac{\pi\,\mathrm{i}}{12}\,(\tau+\frac{1}{2})}\ \mathcal{G}_2(\mathrm{e}^{-2\pi\,\mathrm{i}/\tau})$$

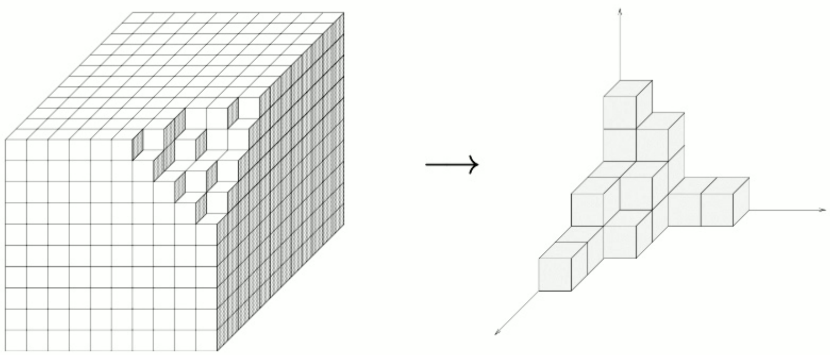
allows estimate of $p_2(n)$ for $n \gg 1$ to very good accuracy

$d = 3$: Counting Plane Partitions

Next we stack partitions to form plane partitions of integers:

$$\boldsymbol{\pi} \; = \; \{\pi_{i,j} \in \mathbb{N} \mid \pi_{i,j} \geqslant \pi_{i+1,j} \; , \; \pi_{i,j} \geqslant \pi_{i,j+1}\}$$

Piling $\pi_{i,j}$ cubes vertically at $(i,j,0)$ gives 3D Young diagram in $\mathbb{Z}^3_{\geqslant 0}$, obtained from a melting crystal model:



Unit cube at $(I,J,K) \in \mathbb{Z}^3_{\geqslant 0}$ evaporates
$\Longleftrightarrow$ all $(i \leqslant I, j \leqslant J, k \leqslant K)$ already evaporated

Removing each atom from corner of crystal contributes q to Boltzmann weight

$d = 3$: Counting Plane Partitions

$p_3(n)$:= number of plane partitions $\boldsymbol{\pi}$ of size (total number of boxes)

$$n = |\boldsymbol{\pi}| := \sum_{i,j} \pi_{i,j}$$

Generating function: $\mathcal{G}_3(q) = \sum_{n=0}^{\infty} p_3(n)\, q^n = \sum_{\boldsymbol{\pi}} q^{|\boldsymbol{\pi}|}$

This series was summed by MacMahon:

$$\mathcal{G}_3(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

with plethystic logarithm

$$\mathrm{PL}[\mathcal{G}_3(q)] = \frac{q}{(1-q)^2} = \sum_{n=1}^{\infty} n\, q^n$$

$d = 3$: Calabi–Yau Crystals

Trivalent planar graph Γ with:

(1) Plane partition $\boldsymbol{\pi}_v$ at each vertex v

(2) Ordinary partition $\boldsymbol{\lambda}_e$ at each edge e; $\boldsymbol{\lambda}_e = \varnothing$ for each external edge (asymptotics of $\boldsymbol{\pi}_v$)

"Topological string" partition function:

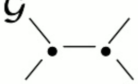
$$\mathcal{G}_\Gamma(q,\vec{x}) = \sum_{\substack{\text{Young diagrams}\\ \boldsymbol{\lambda}_e}} \prod_{\text{edges } e} x_e^{|\boldsymbol{\lambda}_e|} \prod_{\substack{\text{vertices}\\ v=(e_1,e_2,e_3)}} \mathcal{G}_{\boldsymbol{\lambda}_{e_1},\boldsymbol{\lambda}_{e_2},\boldsymbol{\lambda}_{e_3}}(q)$$

Generating function for plane partitions $\boldsymbol{\pi}$ with boundaries $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$:

$$\mathcal{G}_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}(q) = \sum_{\boldsymbol{\pi}\,:\,\partial\boldsymbol{\pi}=(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})} q^{|\boldsymbol{\pi}|}$$

$d = 3$: **Calabi–Yau Crystals**

Examples: $\mathcal{G}$  $(q) = \mathcal{G}_3(q)$

$\mathcal{G}$  (q,x)

$$= \sum_{\boldsymbol{\lambda}} \mathcal{G}_{0,0,\boldsymbol{\lambda}}(q)\, \mathcal{G}_{0,0,\boldsymbol{\lambda}}(q)\; x^{|\boldsymbol{\lambda}|}$$

$$= \sum_{\boldsymbol{\lambda}} \sum_{\boldsymbol{\pi}_1,\boldsymbol{\pi}_2} q^{|\boldsymbol{\pi}_1|+|\boldsymbol{\pi}_2|+\sum_{(i,j)\in\boldsymbol{\lambda}}(i+j+1)}\; x^{|\boldsymbol{\lambda}|}$$

$$= \frac{\mathcal{G}_3(q)^2}{\mathcal{G}_3(q,x)}$$

$$\mathcal{G}_3(q,x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x\, q^n)^n}$$ counts weighted plane partitions

$d = 4$: Counting Solid Partitions

Stack plane partitions to solid partitions of integers:

$$\boldsymbol{\sigma} \;=\; \{\sigma_{i,j,k} \in \mathbb{N} \mid \sigma_{i,j,k} \geqslant \sigma_{i+1,j,k} \;,\; \sigma_{i,j,k} \geqslant \sigma_{i,j+1,k} \;,\; \sigma_{i,j,k} \geqslant \sigma_{i,j,k+1}\}$$

Represent as 4D Young diagrams, equivalently 4D melting crystals, in $\mathbb{Z}^4_{\geqslant 0}$

For the (naive) generating function $\mathcal{G}_4(q) \;=\; \sum_{\boldsymbol{\sigma}} q^{|\boldsymbol{\sigma}|}$, MacMahon conjectured

the formula $\mathcal{G}_4(q) \;=\; \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\frac{n\,(n+1)}{2}}}$ (with $\mathrm{PL}[\mathcal{G}_4(q)] \;=\; \frac{q}{(1-q)^3}$)

This guess famously fails at order q^6, where it begins to overcount

The actual plethystic logarithm complicates matters:
its coefficients start to become non-monotonic at q^{13} and negative at q^{15}

This problem was clarified recently by Nekrasov using the geometry of instanton moduli spaces to adopt a non-trivial measure in the counting function, giving a closed formula for the generating function of solid partitions with a physically meaningful plethystic logarithm

Generalized Instanton Equations

M_d = connected oriented Riemannian manifold of dimension $2d \geqslant 4$ with Hodge operator $*$; consider a vector bundle over M_d with connection

Σ-self-duality equations: $\Sigma \wedge F = *F$

$F = \mathrm{d}A + A \wedge A$ = curvature of connection 1-form A

$\Sigma \in \Omega^{2d-4}(M_d)$ closed and singlet of holonomy group $H \subseteq SO(2d)$

Since $\mathrm{d}\Sigma = 0$, solutions labelled by instanton number $n = \mathrm{ch}_d \in \mathbb{Z}$ and satisfy Yang–Mills equations $\mathrm{d}_A * F = \mathrm{d} * F + A \wedge * F = 0$

Introduce complex structure J making (M_d, ω, J) a Kähler d-fold; reduces holonomy to $H \subseteq U(d)$

Instanton equations with $\mathrm{ch}_1 = 0$ under suitable conditions are equivalent to Donaldson–Uhlenbeck–Yau equations:

$$F^{0,2} = 0 \quad , \quad \omega^{d-1} \wedge F^{1,1} = 0$$

for $F = F^{2,0} + F^{1,1} + F^{0,2}$; these equations describe stable holomorphic vector bundles over M_d with finite characteristic classes

Generalized Instanton Equations

$d=2$: $\Sigma F = *F$ and $*^2 = \mathbb{1}$ imply $\Sigma = \pm 1$ with $H = SO(4)$, complex structure J reduces H to $U(2)$

$d=1$: Dimensional reduction of $F = -*F$ on $M_4 = M_2 \times S^2$ (with holonomy $SO(2)\times SO(2)$) gives vortex equations for Yang–Mills–Higgs theory on M_2 for a connection A and section ϕ on M_2, with holonomy $U(1)$

$d=3$: Take $\Sigma = \omega$, $H = SU(3) \subset SO(6)$

$d=4$: Σ is a $Spin(7)$-structure on M_4, $*\circ(\Sigma\wedge -)$ acts on 2-forms with eigenvalues $-3, 1$

J breaks holonomy to $SU(4)\subset Spin(7)\subset SO(8)$ with $SU(4)$-structure $\Omega\in\Omega^{4,0}(M_4)$ and compatible $Spin(7)$-structure $\Sigma = \frac{1}{2}\,\omega\wedge\omega - \mathrm{Re}(\Omega)$

$Spin(7)$-instanton equation $*(\Sigma\wedge F) = F$ reduces along $SU(4)\subset Spin(7)$ to

$$F^{0,2}_- = 0 \quad , \quad \omega\wedge\omega\wedge\omega\wedge F^{1,1} = 0$$

where $F^{0,2}_- = \frac{1}{2}\,(F^{0,2} - *_\Omega\, F^{0,2})$ in -1-eigenspace of antilinear involution $*_\Omega : \Omega^{0,2}(M_4) \longrightarrow \Omega^{0,2}(M_4)$ induced by Ω

Instanton Moduli Spaces

We can now develop a general geometric setting for our combinatorial partition functions for $d = 1, 2, 3, 4$

We are interested in the instanton moduli space

$$\mathcal{M}_n^d := \frac{\{\text{solutions } A \text{ of } \Sigma \wedge F = *F \text{ with } \mathrm{ch}_d = n\}}{\text{gauge transformations}}$$

on $M_d = \mathbb{C}^d$ for rank $\mathrm{ch}_0 = 1$

Noncommutative instantons: Complex structure J corresponds to Poisson bivector $\theta = r\,\omega^{-1}$, $r > 0$. Berezin–Toeplitz quantization:

$$[z_i, \bar{z}_j] = \mathrm{i}\, r\, \delta_{ij} \quad , \quad [z_i, z_j] = 0 = [\bar{z}_i, \bar{z}_j]$$

represented on irreducible Fock module $\mathcal{H} = \mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]|\vec{0}\,\rangle$

Using Segal–Bargmann representation, instanton equations become algebraic equations for operators $Z_i \in \mathrm{End}(\mathcal{H})$:

$$[Z_i, Z_j] = 0 \quad , \quad \sum_{i=1}^{d} [Z_i, Z_i^\dagger] = d\, r\, \mathbb{1}_{\mathcal{H}}$$

Instanton Moduli Spaces

Moduli space can be described using a generalized ADHM parameterization as a moduli space defined by matrix polynomial equations:

$$\mathcal{M}_n^d \simeq \frac{\left\{ (B_i)_{i=1,\dots,d} \subset \mathrm{End}(\mathbb{C}^n)\ ,\ I \in \mathbb{C}^n \ \middle|\ \begin{array}{rcl} [B_i, B_j] & = & 0 \\ \sum_{i=1}^{d} [B_i, B_i^\dagger] + I \otimes I^\dagger & = & r\,\mathbb{1}_n \end{array} \right\}}{U(n)}$$

where $r > 0$ and $g \cdot (B_i, I) = (g\, B_i\, g^{-1}, g\, I)$ for $g \in U(n)$

Geometrically, this is the Hilbert scheme of n points in $\mathbb{C}^d$:

$$\mathcal{M}_n^d \simeq \mathrm{Hilb}^n(\mathbb{C}^d) := \{\text{ideals } \mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d] \mid \dim \mathbb{C}[z_1, \dots, z_d]/\mathcal{I} = n\}$$

Solutions Z_i parametrized by partial isometries

$$S : \mathcal{H} \longrightarrow \mathcal{H}_{\mathcal{I}} = \mathcal{I}[\bar{z}_1, \dots, \bar{z}_n]|\vec{0}\,\rangle$$

Instanton Moduli Spaces

This is **not smooth** in general: it has generic dimension $d \cdot n$ with an obstruction vector space over each point

Describe the virtual tangent bundle using the instanton deformation complex:

$$T^{\mathrm{vir}}\mathcal{M}_n^d \;=\; \left[T\mathcal{M}_n^d \xrightarrow{\;\mathcal{D}\;} \mathrm{Ob}\,\mathcal{M}_n^d\right]$$

where $\mathcal{D} \;=\;$ linearization of instanton equations followed by projector onto subspace orthogonal to $U(n)$-orbit ('normal bundle')

This defines the virtual dimension:

$$\mathrm{vdim}\,\mathcal{M}_n^d \;:=\; \mathrm{rk}\,T\mathcal{M}_n^d - \mathrm{rk}\,\mathrm{Ob}\,\mathcal{M}_n^d \;=\; \dim\ker\mathcal{D} - \dim\mathrm{coker}\,\mathcal{D}$$

and Euler class: $e(T^{\mathrm{vir}}\mathcal{M}_n^d) \;:=\; \dfrac{e(T\mathcal{M}_n^d)}{e(\mathrm{Ob}\,\mathcal{M}_n^d)}$

Let $\mathcal{V}_n^d \longrightarrow \mathcal{M}_n^d$ be a vector bundle of rank $\mathrm{vdim}\,\mathcal{M}_n^d$

Generalized Instanton Partition Functions

Geometric partition functions compute intersection theory of $\mathcal{M}_n^d$:

$$Z_d(q) \;=\; \sum_{n=0}^{\infty} q^n \int_{[\mathcal{M}_n^d]^{\rm vir}} e(\mathcal{V}_n^d)$$

$\mathcal{M}_n^d$ is acted upon by the torus group $T \;=\; U(1)^d \subset H$:

$$\vec{t}\cdot(B_i, I) \;=\; (t_i\, B_i\,,\, I) \qquad , \qquad t_i \;=\; \mathrm{e}^{\,\mathrm{i}\,\epsilon_i} \in U(1)$$

Fixed points $\left(\mathcal{M}_n^d\right)^T$ are isolated and given by monomial ideals $\mathcal{I} \in \mathrm{Hilb}^n(\mathbb{C}^d)^T$, which are in correspondence with d-dimensional Young diagrams for $d \geqslant 2$

Virtual localization theorem:

$$Z_d(q;\vec{\epsilon}) \;=\; \sum_{n=0}^{\infty} q^n \sum_{f\in(\mathcal{M}_n^d)^T} \frac{e_T(\mathrm{Ob}_f\,\mathcal{M}_n^d)}{e_T(T_f\mathcal{M}_n^d)}\; e_T(\mathcal{V}_n^d)_f$$

Can be computed as fluctuation determinants in instanton background

$d = 1$: **Vortices**

$\mathcal{M}_n^1$ is smooth of dimension n: $\mathrm{Hilb}^n(\mathbb{C}) \simeq \mathrm{Sym}^n(\mathbb{C}) := \mathbb{C}^n/S_n \simeq \mathbb{C}^n$

No obstructions, so compute Euler characteristic with $\mathcal{V}_n^1 = T\mathcal{M}_n^1$

T-fixed points: $t\,B = g(t)\,B\,g(t)^{-1}$, $g(t)\,I = I$

There is a **unique** fixed point satisfying $[B, B^\dagger] + I \otimes I^\dagger = r\,\mathbb{1}_n$

Euler classes cancel in localization formula:

$$Z_1(q) = \sum_{n=0}^{\infty} q^n = \mathcal{G}_1(q)$$

$\mathcal{G}_1(q)$ is the generating function for Euler characteristics of vortex moduli spaces

$d = 2$: **Instantons**

$\mathcal{M}_n^2$ is smooth of dimension $2n$

No obstructions, so compute Euler characteristic with $\mathcal{V}_n^2 \; = \; T\mathcal{M}_n^2$

T-fixed points: Monomial ideals $\mathcal{I}_{\boldsymbol{\lambda}} \subset \mathbb{C}[z_1, z_2]$ of codimension n correspond to partitions $\boldsymbol{\lambda}$ with $|\boldsymbol{\lambda}| \; = \; n$ ($\mathcal{I}_{\boldsymbol{\lambda}}$ generated by $z_1^{i-1}\, z_2^{\lambda_i}$)

Euler classes cancel in localization formula:

$$Z_2(q) \; = \; \sum_{n=0}^{\infty} q^n \sum_{|\boldsymbol{\lambda}|=n} 1 \; = \; \mathcal{G}_2(q)$$

$\mathcal{G}_2(q)$ is the generating function for Euler characteristics of instanton moduli spaces (Hilbert schemes of points on $\mathbb{C}^2$)

$d = 3$: $SU(3)$-Instantons

$\mathcal{M}_n^3$ is a scheme with branches of varying dimension

vdim $\mathcal{M}_n^3 = 0$, so compute 'volume' $\int_{[\mathcal{M}_n^3]^{\mathrm{vir}}} 1$

T-fixed points: Monomial ideals $\mathcal{I}_{\boldsymbol{\pi}} \subset \mathbb{C}[z_1, z_2, z_3]$ of codimension n correspond to plane partitions $\boldsymbol{\pi}$ with $|\boldsymbol{\pi}| = n$

At each fixed point: $e_T(\mathrm{Ob}_{\boldsymbol{\pi}}\mathcal{M}_n^3) = (-1)^{|\boldsymbol{\pi}|}\, e_T(T_{\boldsymbol{\pi}}\mathcal{M}_n^3)$

Euler classes cancel up to a sign in localization formula:

$$Z_3(q) = \sum_{n=0}^{\infty} q^n \sum_{|\boldsymbol{\pi}|=n} (-1)^{|\boldsymbol{\pi}|} = \mathcal{G}_3(-q)$$

$\mathcal{G}_3(-q)$ is the generating function for Donaldson–Thomas invariants of $\mathbb{C}^3$ (virtual numbers of ideal sheaves on $\mathbb{C}^3$)

$d = 4$: $SU(4)$-Instantons

$\mathcal{M}_n^4$ has virtual dimension $\operatorname{vdim} \mathcal{M}_n^4 = n$

Take the tautological bundle:

$$\mathcal{V}_n^4 = \{(B_i, I) \mid \text{ADHM eqns}\} \times_{U(n)} \mathbb{C}^n$$

This carries an extra $U(1)$-action with equivariant parameter m

T-fixed points: Solid partitions $\boldsymbol{\sigma}$ with $|\boldsymbol{\sigma}| = n$

Instanton partition function:

$$Z_4(q; \vec{\epsilon}, m) = \sum_{n=0}^{\infty} q^n \sum_{|\boldsymbol{\sigma}|=n} \frac{e_T(\mathrm{Ob}_{\boldsymbol{\sigma}} \mathcal{M}_n^4)\; e_{T\times U(1)}(\mathcal{V}_n^4)_{\boldsymbol{\sigma}}}{e_T(T_{\boldsymbol{\sigma}} \mathcal{M}_n^4)}$$

Signs: $\mathrm{Ob}\, \mathcal{M}_n^4$ is a real vector bundle, so its determinant requires a choice of orientation, which amounts to choosing a sign $(-1)^{o(\boldsymbol{\sigma})}$ at each fixed point $\boldsymbol{\sigma}$ to compute the Euler class

$d = 4$: $SU(4)$-Instantons

Computing the Euler classes results in a complicated combinatorial series with non-trivial measure:

$$Z_4(q;\vec{\epsilon},m) \;=\; \sum_{n=0}^{\infty} q^n \sum_{|\boldsymbol{\sigma}|=n} (-1)^{o(\boldsymbol{\sigma})} \prod_{\vec{p},\vec{p}\,'\in\boldsymbol{\sigma}} \frac{\vec{p}\cdot\vec{\epsilon}-m}{\vec{p}\cdot\vec{\epsilon}}\, R\big((\vec{p}-\vec{p}\,')\cdot\vec{\epsilon}\,|\vec{\epsilon}\big)$$

where $R(x|\vec{\epsilon}) \;:=\; \dfrac{x\,(x-\epsilon_1-\epsilon_2)\,(x-\epsilon_1-\epsilon_3)\,(x-\epsilon_2-\epsilon_3)}{(x-\epsilon_1)\,(x-\epsilon_2)\,(x-\epsilon_3)\,(x-\epsilon_4)}$

A closed form was conjectured by Cao–Kool and Nekrasov–Piazzalunga, and proven rigorously by Kool–Rennemo in Donaldson–Thomas theory on $\mathbb{C}^4$

Theorem: There exists choices of orientation $o(\boldsymbol{\sigma})$ such that

$$Z_4(q;\vec{\epsilon},m) \;=\; \mathcal{G}_3(-q)^{-\frac{m\,(\epsilon_1+\epsilon_2)\,(\epsilon_1+\epsilon_3)\,(\epsilon_2+\epsilon_3)}{\epsilon_1\,\epsilon_2\,\epsilon_3\,\epsilon_4}}$$

The uniform measure on solid partitions (with weight given simply by $q^{|\boldsymbol{\sigma}|}$) is not natural — they look 3-dimensional when suitably counted