Title: Quantum Analysis of the Bianchi IX model: Exploring Chaos

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Series: Quantum Gravity

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Abstract: According to the Belinski-Khalatnikov-Lifshitz conjecture, the Bianchi IX spacetime describes the evolution of each spatial point close to a generic spacelike singularity. However, near the singularity, quantum effects are expected to be relevant. Therefore, in this work a quantum analysis of the model is performed, mainly focusing on its chaotic nature. Considering some minimal approximations, it is possible to encode all the information of the quantum degrees of freedom in certain canonical variables, expanding thus the classical phase space. In this way, we can apply the usual methods of dynamical systems for studying chaos. In particular, two techniques are considered. On the one hand, an analytical study is carried out, which provides an isomorphism between the quantum dynamics of Bianchi IX and the geodesic flow on a Riemannian manifold. On the other hand, by means of numerical simulations, the fractal dimension of the boundary between points with different outcome in the space of initial data is studied. The main conclusion is that, although the quantum system is chaotic, the quantum effects considerably reduce this behavior as compared to its classical counterpart.

Zoom link https://pitp.zoom.us/j/91559466008?pwd=UzQvTGRkR3VQWm9MWDlaaVAyNi9EQT09

Quantum Analysis of the Bianchi IX model: Exploring Chaos

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Based on: arXiv:2307.00063, arXiv:2307.13040

Perimeter Institute for Theoretical Physics, December 7, 2023

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Introduction

Spacelike singularity

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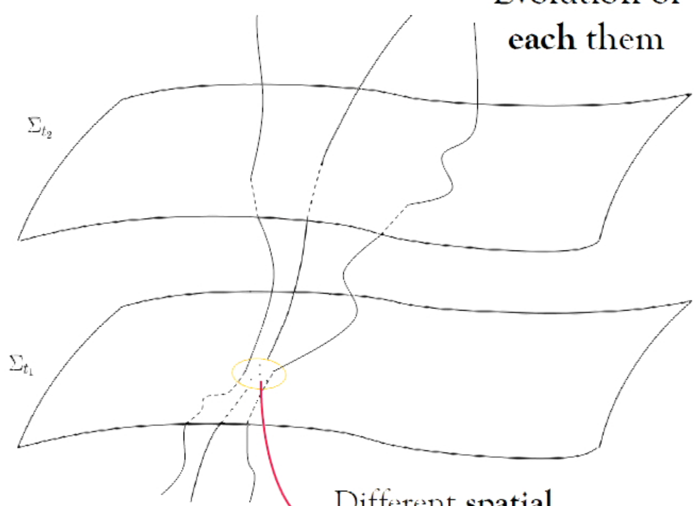
BKL conjecture

(1970)

Supported by many numerical studies



Evolution of **each them** *identical to the one given by the*

(vacuum Bianchi IX)

Mixmaster model

- Dominated by pure gravity
- Oscillatory
- Chaotic

Different **spatial points** decouple

Introduction

Spacelike singularity

Quantum effects

Main goal of my research

?

Mixmaster model

- Dominated by pure gravity
- Oscillatory
- Chaotic

Classical model: Mixmaster

Metric

$$ds^2 = -N^2 dt^2 + \sum_{i=1}^{3} a_i(t)^2 \sigma_i^2$$

Lapse function

Scale factors

1-forms on the 3-sphere

In terms of the *Euler angles* (θ, ϕ, ψ)

$$\sigma_1 := \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi,$$
$$\sigma_2 := \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi,$$
$$\sigma_3 := -d\psi - \cos\theta\, d\phi$$

- Spatially **homogenous** and **anisotropic**
- Solution of the **vacuum** Einstein field equations

} vacuum **Bianchi IX** spacetime

- Spacelike singularity at $t = 0$, where $a_i \to 0$
- Describe the evolution of **each** spatial point close to a generic spacelike singularity (**BKL conjecture**)

Classical model: Mixmaster

- Follow the ADM formalism

Misner variables

$$\beta_+ := -\frac{1}{2}\ln\left[\frac{a_3}{(a_1 a_2 a_3)^{1/3}}\right], \quad \beta_- := \frac{1}{2\sqrt{3}}\ln\left(\frac{a_1}{a_2}\right), \quad \alpha = \frac{1}{3}\ln(a_1 a_2 a_3)$$

Shape parameters

Spatial volume

Singularity $\alpha \to -\infty$

Hamiltonian constraint

$$C = \frac{1}{2}e^{-3\alpha}(-p_\alpha^2 + p_+^2 + p_-^2) + e^{\alpha}V(\beta_+, \beta_-) = 0$$

Potential

- p_α, p_+, p_- conjugate momenta: $\{\alpha, p_\alpha\} = \{\beta_+, p_+\} = \{\beta_-, p_-\}$=1

Classical model: Mixmaster

The dynamics?

- Freedom to choose any time parametrization (General Relativity)
- Choose an internal variable as the time variable
  - α is a monotonic function: choose it, gauge $\alpha = t$

Hamiltonian constraint

$$C = \frac{1}{2} e^{-3\alpha}(-p_\alpha^2 + p_+^2 + p_-^2) + e^\alpha V(\beta_+, \beta_-) = 0$$

Solve it for p_α

$$H := -p_\alpha = [p_+^2 + p_-^2 + 2e^{4\alpha} V(\beta_+, \beta_-)]^{1/2}$$

Physical Hamiltonian

Classical model: Mixmaster

Physical Hamiltonian

$$H := -p_\alpha = [p_+^2 + p_-^2 + 2e^{4\alpha} V(\beta_+, \beta_-)]^{1/2}$$

Equations of motion

$$\frac{d\beta_\pm}{d\alpha} = \{\beta_\pm, H\} = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H}, \qquad \frac{dp_\pm}{d\alpha} = \{p_\pm, H\} = -\frac{\partial H}{\partial \beta_\pm} = -\frac{e^{4\alpha}}{H}\frac{\partial V}{\partial \beta_\pm}$$

Qualitative dynamics close to the singularity ($\alpha \to -\infty$) ?

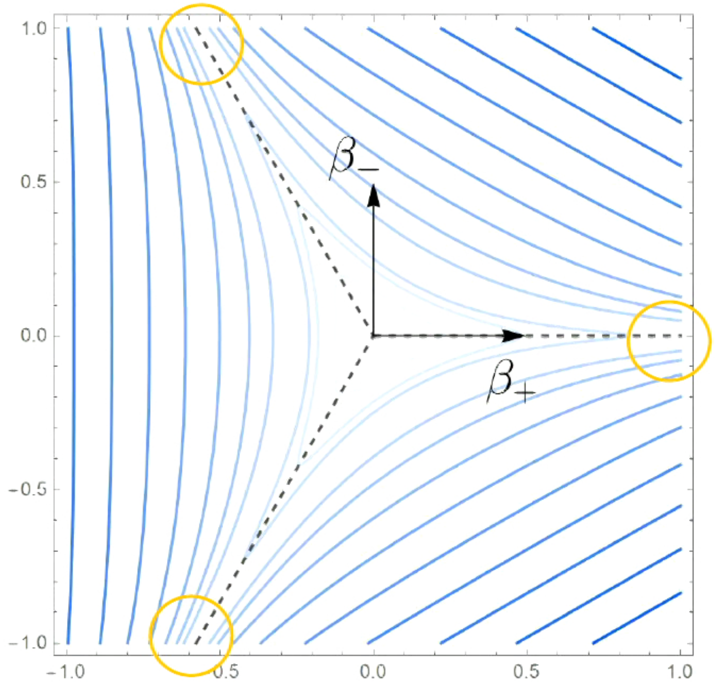
$\downarrow$

Analyze the potential $V(\beta_+, \beta_-)$

Classical model: Mixmaster

$$V(\beta_+,\beta_-) := \frac{1}{6}\left[e^{-8\beta_+} + 2e^{4\beta_+}\left(\cosh 4\sqrt{3}\beta_- - 1\right) - 4e^{-2\beta_+}\cosh 2\sqrt{3}\beta_-\right]$$

Equipotential lines



- $V(\beta_+,\beta_-)$ is **negligible:**

$$H = [p_+^2 + p_-^2 + 2e^{4\alpha}V(\beta_+,\beta_-)]^{1/2} \approx (p_+^2 + p_-^2)^{1/2}$$

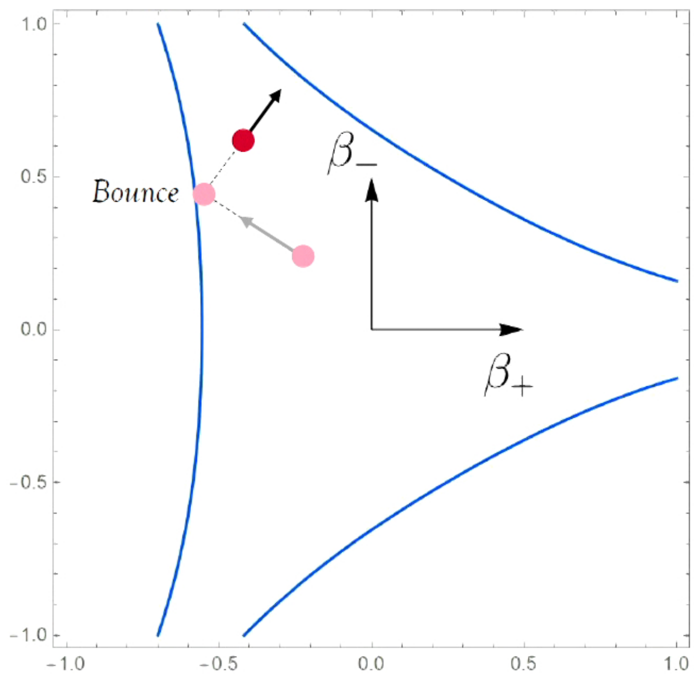
Free dynamics

- $V(\beta_+,\beta_-)$ is **not** negligible:

Infinite potential walls

Along the 3 symmetry semiaxes, no potential wall is encountered: **3 exits**

Classical model: Mixmaster



Analogy

The system is a point particle with coordinates (β_+, β_-) moving in a **potential well** with **3 exits**

~ similar to the billiard picture

$$H = [p_+^2 + p_-^2 + 2e^{4\alpha} V(\beta_+, \beta_-)]^{1/2}$$

Potential walls move away as $\alpha \to -\infty$

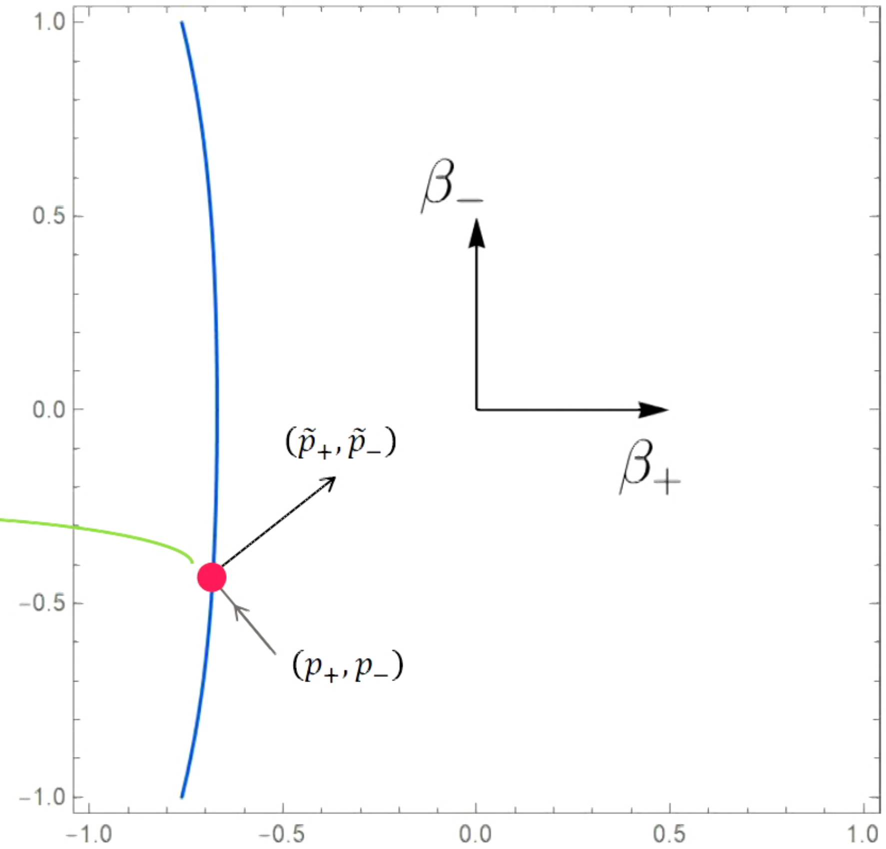
Classical model: Mixmaster

Bounces

Specific **transition laws**

$$\tilde{p}_- = p_-$$

$$\tilde{p}_+ = \frac{1}{3}\left[4(p_+^2 + p_-^2)^{1/2} - 5p_+\right]$$

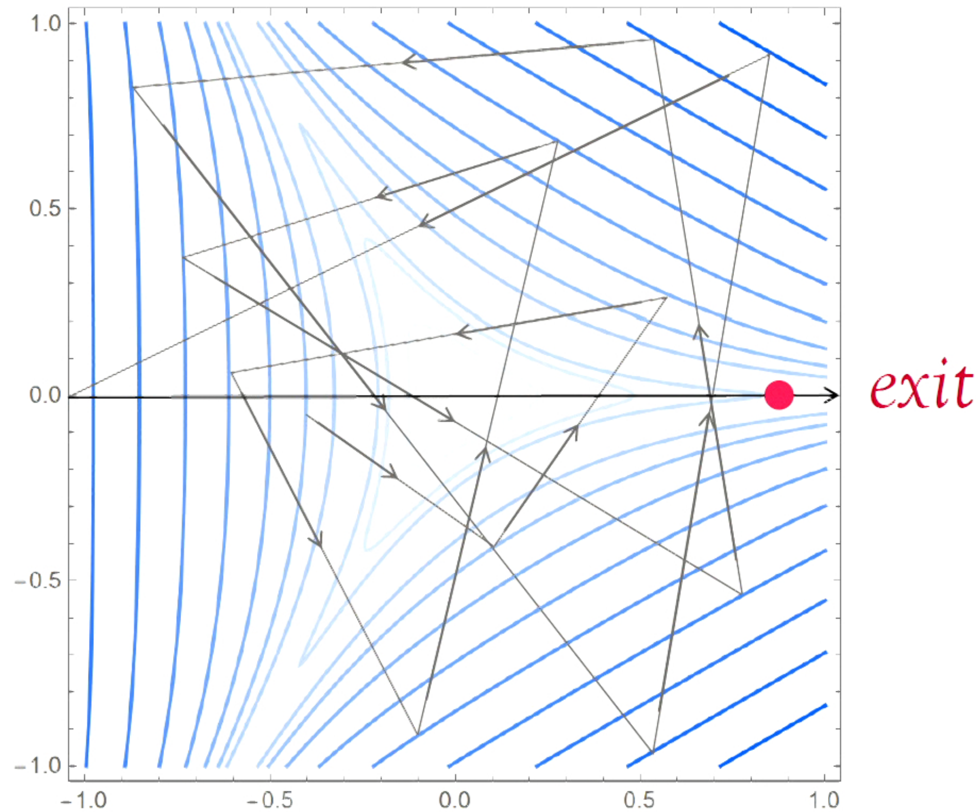


Classical model: Mixmaster

Constantly bouncing against the potential walls **(billiard)**

Final state
$\alpha \to +\infty$

(most points) **"escape"** through one of the 3 exits



Classical model: Mixmaster

Chaotic motion

Slight difference in the initial conditions: **not escape** or escape from **another exit**

How to prove this?

- Nature of general relativity: **observer independent methods** → ~~*Usual dynamical techniques*~~

1. **Fractal methods:**

N. J. Cornish and J. J. Levin, *The Mixmaster universe: A Chaotic Farey Tale*, Phys. Rev. D **55**, 7489 (1997)

2. **Lyapunov exponent:**

G. P. Imponente and G. Montani, *On the Covariance of the Mixmaster Chaoticity*, Phys. Rev. D **63**, 103501 (2001)

Classical model: Mixmaster

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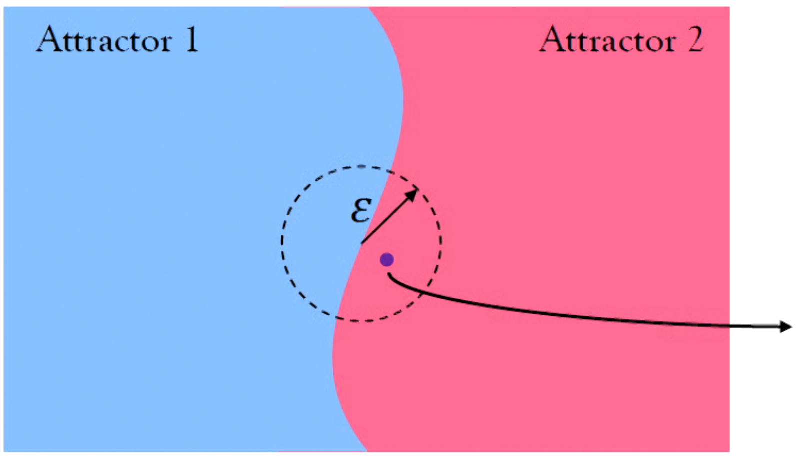
N. J. Cornish and J. J. Levin, *The Mixmaster universe: A Chaotic Farey Tale*, Phys. Rev. D **55**, 7489 (1997)

Main ideas:

- Systems with more than one **attractor (repeller), exit** or **final outcome**
  - **Divide the space of initial conditions** accordingly, in terms of the attractor/exit they end up (basins of attraction)



$*\ \varepsilon \equiv$ *radius of the uncertainty in the initial conditions*

$*\ f(\varepsilon) \equiv$ *fraction of the space of initial conditions with uncertain outcomes for each* ε

Uncertain ouctome: Attractor 1 or Attractor 2?

Classical model: Mixmaster

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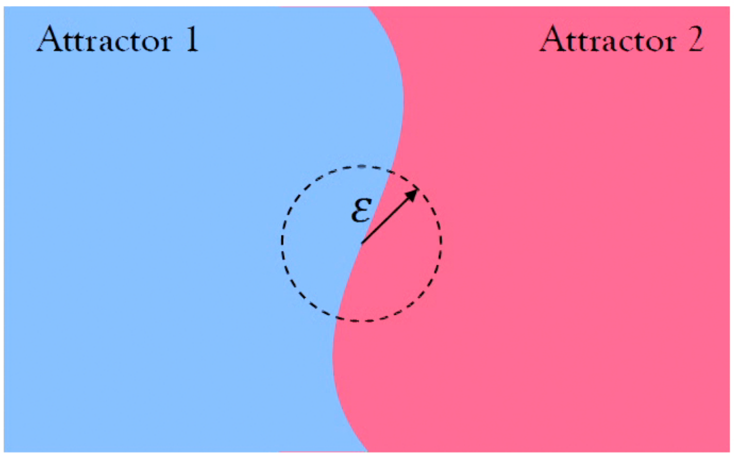

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$$f(\varepsilon) \propto \varepsilon$$

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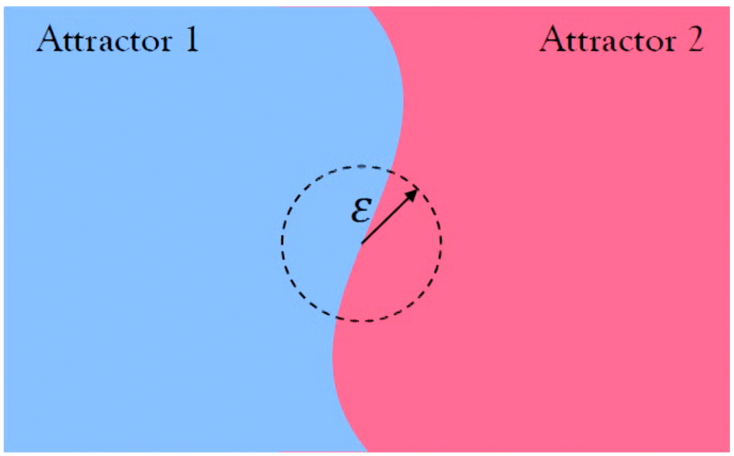

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- **Chaotic:** *amplification* of the initial error

$$f(\varepsilon) \propto \varepsilon^{\delta} \quad (0 \leq \delta < 1)$$

Classical model: Mixmaster

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(Uncertainty exponent) $\delta = D - D_b \begin{cases} = 1, & \text{smooth boundary} \\ & \end{cases}$

Dimension of the space of initial conditions

Dimension of the boundary

- **Not chaotic:** *no* amplification of the initial error

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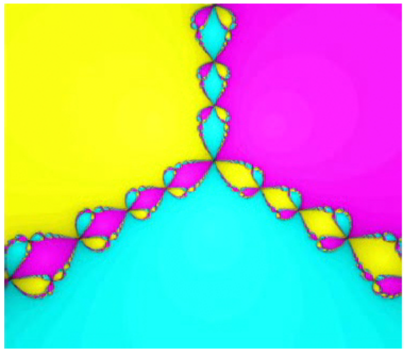
(Uncertainty exponent) $\delta = \underbrace{D}_{\text{Dimension of the space of initial conditions}} - \underbrace{D_b}_{\text{Dimension of the boundary}} \begin{cases} = 1, & \text{smooth boundary} \\ < 1, & \text{fractal boundary } (D_b \text{ not an integer}) \end{cases}$



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Classical model: Mixmaster

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Dimension of the space of initial conditions (D)

Dimension of the boundary (D_b)

Compute δ → Quantify the level of chaos !

- **Not chaotic:** *no* amplification of the initial error

$$f(\varepsilon) \propto \varepsilon \qquad \textbf{Smooth boundary}$$

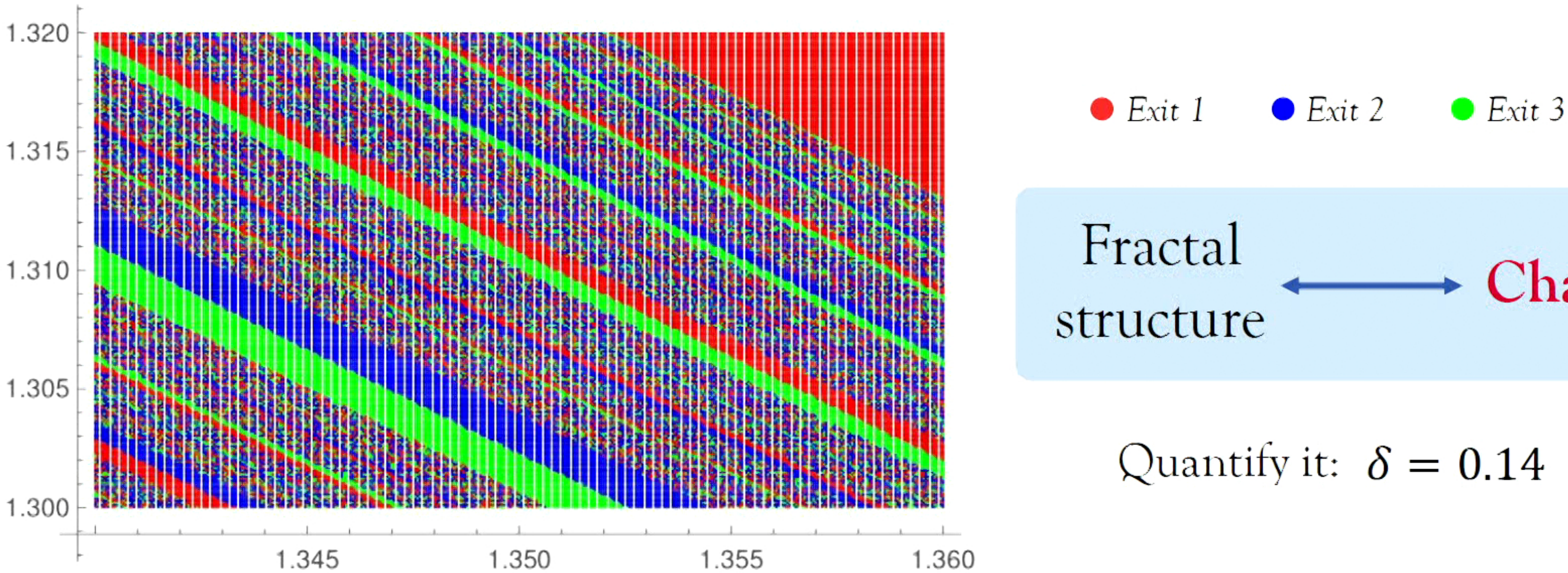
- **Chaotic:** *amplification* of the initial error

$$f(\varepsilon) \propto \varepsilon^{\delta} \quad (0 \leq \delta < 1) \qquad \textbf{Fractal boundary}$$

Classical model: Mixmaster

1. **Fractal methods:** N. J. Cornish and J. J. Levin, *The Mixmaster universe: A Chaotic Farey Tale*, Phys. Rev. D **55**, 7489 (1997)

Classify **each point** of the space of **initial data** in terms of its **final state (exit)**



Fractal structure ⟷ Chaos

Quantify it: $\delta = 0.14$

Classical model: Mixmaster

2. Lyapunov exponent:

G. P. Imponente and G. Montani, *On the Covariance of the Mixmaster Chaoticity*, Phys. Rev. D **63**, 103501 (2001)

Main ideas:

- The **value** of the **Lyapunov exponent** is *not* **invariant**: depends on the time parametrization
  - The **sign** of the Lyapunov exponent is **invariant:** $\lambda > 0 \rightarrow$ *chaotic*
- The Lyapunov exponent is an **accurate** way of representing chaos $\longleftrightarrow$ some **specific conditions** are **satisfied**

With the **usual variables** to describe the Mixmaster model they are *not* **satisfied**

Change of variables

Misner-Chitre variables:

$$\alpha = -e^{\Gamma}\xi, \qquad \beta_+ := e^{\Gamma}\sqrt{\xi^2 - 1}\cos\theta, \qquad \beta_- := e^{\Gamma}\sqrt{\xi^2 - 1}\sin\theta, \qquad (\theta \in (0,\pi), \xi \geq 1, \Gamma \in \mathbb{R})$$

Classical model: Mixmaster

2. Lyapunov exponent:

G. P. Imponente and G. Montani, *On the Covariance of the Mixmaster Chaoticity*, Phys. Rev. D **63**, 103501 (2001)

- **Each trajectory** on the phase space is **isomorphic** to a **geodesic** on a Riemannian manifold with metric:

$$ds^2 = E^2\left[\frac{d\xi^2}{\xi^2-1} + (\xi^2-1)d\theta^2\right]$$

✓ *Necessary conditions to compute the Lyapunov exponent*

Chaotic geodesic flow ⟷ The Mixmaster model is **chaotic**

- The geodesic deviation equation shows that **initially close** geodesics **exponentially diverge** → Lyapunov exponent $\lambda > 0$ ⟶ The system is **chaotic**

Complete picture?

Quantum effects?

Quantum model: formalism

Canonical quantization: $\beta_\pm \to \hat{\beta}_\pm, \quad p_\pm \to \hat{p}_\pm, \quad [\hat{\beta}_m, \hat{p}_n] = i\hbar\delta_{mn} \quad (m, n \in \{+,-\})$

Instead of solving the **Schrödinger** equation $\hat{H}\psi(\beta_\pm, \alpha) = i\hbar\frac{\partial}{\partial\alpha}\psi(\beta_\pm, \alpha)$, we will study how the **quantum moments** of the wave function **evolve,** based on this equivalence:

Information of the quantum state

Wave function $\psi(\beta_\pm, \alpha)$

equivalent

$$\left\{\Delta(\beta_+^m p_+^n \beta_-^k p_-^l) := \left\langle\left(\hat{\beta}_+ - \beta_+\right)^m (\hat{p}_+ - p_+)^n \left(\hat{\beta}_- - \beta_-\right)^k (\hat{p}_- - p_-)^l\right\rangle\right\}_{m,n,k,l=0}^{+\infty}$$

Infinite set of quantum moments

$\beta_\pm := \langle\hat{\beta}_\pm\rangle, \quad p_\pm := \langle\hat{p}_\pm\rangle$

Quantum model: formalism

Motivation?

- Probability distribution $P(x)$ $\xleftrightarrow{\text{equivalent}}$ Moment generating function

$$M(s) := \int_E e^{sx}\, P(x)\, dx \quad \left(\langle x^n \rangle = \left. \frac{d^n M(s)}{ds^n} \right|_{s=0} \right)$$

Information of the quantum state

$$|\psi(x,t)|^2 \xleftrightarrow[\text{equivalent}]{} \left\{ \Delta(x^i p^j) := \left\langle (\hat{x} - x)^i (\hat{p} - p)^j \right\rangle_{\mathrm{Weyl}} \right\}_{i,j=0}^{+\infty}$$

$$x := \langle \hat{x} \rangle, \qquad p := \langle \hat{p} \rangle$$

Totally symmetric ordering

Dimensions

$[\Delta(x^i p^j)] = [\hbar]^{(i+j)/2}$

$i + j = order$

Central and completely symmetric quantum moments

e.g. $\mathrm{Var}(x) = \Delta(x^2)$

Quantum model: formalism

The **dynamics** of the **quantum moments?** Governed by the following Hamiltonian:

Hamiltonian operator

Taylor expansion around the expectation values $\beta_\pm := \langle\hat{\beta}_\pm\rangle,\ p_\pm := \langle\hat{p}_\pm\rangle$

$$H_Q := \langle\hat{H}(\hat{\beta}_+, \hat{p}_+, \hat{\beta}_-, \hat{p}_-, \alpha)\rangle = H + \sum_{m+n+k+l=2}^{+\infty} \frac{1}{m!\,n!\,k!\,l!} \frac{\partial^{m+n+k+l} H(\beta_+, p_+, \beta_-, p_-, \alpha)}{\partial\beta_+^m \partial p_+^n \partial\beta_-^k \partial p_-^l} \Delta(\beta_+^m p_+^n \beta_-^k p_-^l)$$

Classical Hamiltonian

Equations of motion

$$\{\langle\hat{f}\rangle, \langle\hat{g}\rangle\} = \frac{1}{i\hbar}\langle[\hat{f}, \hat{g}]\rangle$$

$$\frac{d\beta_\pm}{d\alpha} := \{\beta_\pm, H_Q\} = \frac{\partial H_Q}{\partial p_\pm}, \qquad \frac{dp_\pm}{d\alpha} := \{p_\pm, H_Q\} = -\frac{\partial H_Q}{\partial \beta_\pm}, \qquad \frac{d\Delta(\beta_+^m p_+^n \beta_-^k p_-^l)}{d\alpha} := \{\Delta(\beta_+^m p_+^n \beta_-^k p_-^l), H_Q\}$$

Completely **equivalent** to the evolution given by the **Schrödinger equation**

Quantum model: formalism

Remarks:

- It can be applied to **any** quantum system [Bojowald, Skirzewski (2006)]
- Known for a very long time: thanks to computers "rediscovered" in the last ~15 years
- Usually H_Q cannot be written in a closed-form (infinite series)
  - Particular exception: **polynomial** Hamiltonians
  - Practical purposes: apply a **cut-off**

$$H_Q := \langle \hat{H}(\hat{x}, \hat{p}, t) \rangle = H + \sum_{i+j=2}^{N} \frac{1}{i!\,j!} \frac{\partial^{i+j} H(x,p,t)}{\partial x^i \partial p^j} \Delta(x^i p^j)$$

~ Semiclassical study

Quantum model: formalism

Advantages:

- Deal directly with **measurable quantities** (expectation values) instead of with the wave function
- Clearer to **interpret:** work in the **phase space** directly
- Closer to the classical description: classical e.o.m. + quantum corrections
  - Immediate to obtain the **classical limit**
  - Very appropriate for **semiclassical** studies
- No need to define the **Hilbert space:** "assume" that there is an internal product
- Equations with **total derivatives** instead of partial (Schrödinger)

Quantum model: set-up

Our particular model: **Mixmaster**

H_Q cannot be written in a closed-form

$$H_Q := \langle \hat{H}(\hat{\beta}_+, \hat{p}_+, \hat{\beta}_-, \hat{p}_-, \alpha) \rangle = H + \sum_{i+j+k+l=2}^{+\infty} \frac{1}{i!\,j!\,k!\,l!} \frac{\partial^{i+j+k+l} H(\beta_+, p_+, \beta_-, p_-, \alpha)}{\partial\beta_+^i \partial p_+^j \partial\beta_-^k \partial p_-^l} \Delta(\beta_+^i p_+^j \beta_-^k p_-^l)$$

(classical Hamiltonian) $$H = [p_+^2 + p_-^2 + 2e^{4\alpha} V(\beta_+, \beta_-)]^{1/2}$$

How to deal with this?

- Apply a *cut-off*: D. Brizuela and S. F. Uria (2022)
- Consider *other approximations*: M. Bojowald, D. Brizuela, P. Calizaya Cabrera, and S. F. Uria (2023)

Quantum model: set-up

Consider *two assumptions:* (Semiclassical)

1. $H_Q := \langle \hat{H} \rangle \approx \sqrt{\langle \hat{H}^2 \rangle} \longrightarrow H_Q^2 \approx \langle \hat{H}^2(\hat{\beta}_+, \hat{p}_+, \hat{\beta}_-, \hat{p}_-, \alpha) \rangle = \langle \hat{p}_+^2 + \hat{p}_-^2 + 2e^{4\alpha} V(\hat{\beta}_+, \hat{\beta}_-) \rangle$

$$= p_+^2 + p_-^2 + \Delta(p_+^2) + \Delta(p_-^2) + 2e^{4\alpha} \sum_{i+j=2}^{+\infty} \frac{1}{i!\,j!} \frac{\partial^{i+j} V(\beta_+, \beta_-)}{\partial \beta_+^i \partial \beta_-^j} \Delta\left(\beta_+^i \beta_-^j\right)$$

2. **Gaussian-like** state all along the evolution:

$$\Delta(\beta_+^{2n} \beta_-^{2m}) = \frac{s_1^{2n}}{2^n} \frac{s_2^{2m}}{2^m} \frac{(2n)!}{n!} \frac{(2m)!}{m!}, \qquad \Delta(\beta_+^n \beta_-^m) = 0 \quad \text{otherwise} \quad (n, m \in \mathbb{N}, \quad s_1, s_2 \in \mathbb{R}^+)$$

$$\sum_{i+j=2}^{+\infty} \frac{1}{i!\,j!} \frac{\partial^{i+j} V(\beta_+, \beta_-)}{\partial \beta_+^i \partial \beta_-^j} \Delta\left(\beta_+^i \beta_-^j\right) = \underbrace{\frac{1}{6}\left[e^{-8\beta_+ + 32 s_1^2} + 2e^{4\beta_+ + 8 s_1^2}\left(e^{24 s_2^2} \cosh(4\sqrt{3}\beta_-) - 1\right) - 4e^{-2\beta_+ + 2s_1^2 + 6s_2^2} \cosh(2\sqrt{3}\beta_-)\right]}_{\textit{Quantum potential} \quad V_Q(\beta_+, \beta_-, s_1,\, s_2)}$$

Quantum model: set-up

$$H_Q \approx \left[p_+^2 + p_-^2 + \Delta(p_+^2) + \Delta(p_-^2) + 2e^{4\alpha}\, V_Q(\beta_+, \beta_-, s_1, s_2)\right]^{1/2}$$

Closed-form effective Hamiltonian

Some **variables** are *not* **canonically conjugate** !

↓

Transformation to **canonical variables** $\{s_i, p_{s_i}\} = \delta_{ij}\ (i,j = 1,2)$

Encode the quantum effects

$$\Delta(p_+^2) = p_{s_1}^2 + \frac{U_1}{s_1^2}, \quad \Delta(p_-^2) = p_{s_2}^2 + \frac{U_2}{s_2^2}, \quad \text{with } U_1, U_2 \text{ constants}$$

Thus

$$\boxed{H_Q \approx \left[p_+^2 + p_-^2 + p_{s_1}^2 + \frac{U_1}{s_1^2} + p_{s_2}^2 + \frac{U_2}{s_2^2} + 2e^{4\alpha}\, V_Q(\beta_+, \beta_-, s_1, s_2)\right]^{1/2}}$$

M. Bojowald, J. Phys. A: Math. Theor. 55, 504006 (2022)

Quantum model: set-up

Advantage of this approximation:

Work on an **extended** (finite) **phase space**

(2 classical + 2 quantum degrees of freedom)

↓

Use the **previous techniques** to study **chaos**

↓

Compare with the **classical results**

Is the quantum system more or less chaotic?

Quantum model: chaos

1. Lyapunov exponent: *Is it chaotic?*

Problem: in the present coordinate system $(\beta_+, \beta_-, s_1, s_2, p_+, p_-, p_{s_1}, p_{s_2})$ the Lyapunov exponent **cannot** be properly computed

⟶ **Extend** the *Misner-Chitre* canonical transformation: $(\Gamma, \xi, \theta) \to (\Gamma, \xi, \theta, \sigma, \phi)$

$$\alpha = -e^{\Gamma}\xi, \qquad \beta_+ := e^{\Gamma}\sqrt{\xi^2 - 1}\cos\theta, \qquad \beta_- := e^{\Gamma}\sqrt{\xi^2 - 1}\sin\theta\cos\sigma,$$

Classical Misner-Chitre transformation

$$s_1 = e^{\Gamma}\sqrt{\xi^2 - 1}\sin\theta\sin\sigma\, f_+(\phi, p_\phi), \qquad s_2 = e^{\Gamma}\sqrt{\xi^2 - 1}\sin\theta\sin\sigma\, f_-(\phi, p_\phi)$$

with $$f_\pm(\phi, p_\phi) := \frac{1}{\sqrt{2}|p_\phi|}\left[p_\phi^2 \pm \left(U_1 - U_2 - h(p_\phi)\sin 2\phi\right)\right]^{1/2},$$

$$h(p_\phi) := \sqrt{\left(p_\phi^2 - (U_1 + U_2)\right)^2 - 4U_1U_2}, \qquad \xi \geq 1, \qquad \Gamma \in \mathbb{R}, \qquad \theta, \sigma \in (0, \pi), \qquad \phi \in [0, 2\pi)$$

Quantum model: chaos

1. Lyapunov exponent:

- As in the classical case, **each trajectory** on the phase space is **isomorphic** to a **geodesic** on a Riemannian manifold with metric

$$ds^2 = E^2\left[\frac{d\xi^2}{\xi^2 - 1} + (\xi^2 - 1)d\theta^2 + (\xi^2 - 1)\sin^2\theta\, d\sigma^2 + (\xi^2 - 1)\sin^2\theta \sin^2\sigma\, d\phi^2\right]$$

✓ *Necessary conditions to compute the Lyapunov exponent*

Quantum model: chaos

1. Lyapunov exponent:

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Chaotic geodesic flow ⟷ *The quantum Mixmaster model is chaotic*

- The geodesic deviation equation shows that **initially close** geodesics **exponentially diverge** → Lyapunov exponent $\lambda > 0$ ⟷ *The quantum model is* **still chaotic**

Quantum model: chaos

2. Fractal methods: *More or less chaotic than the classical system?*

- **Quantum dynamics** follow the classical picture: **free motion** + **bounces**

  [D. Brizuela, S. F. Uria (2022)]

  - There are **3 exits** ~ classical exits

- Divide the **space of initial data** as classically: **exit** through which the system **escapes**

- For **different cross-sections** (initial values of the quantum moments) **measure** the

  uncertainty exponent ⟷ the **level of chaos**

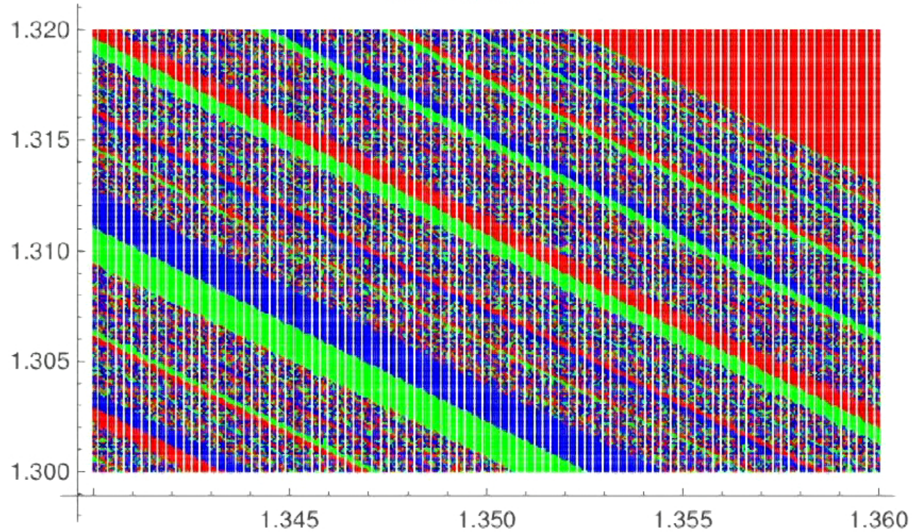
- **Compare** the results with the **classical** one

Quantum effects reduce the level of chaos!

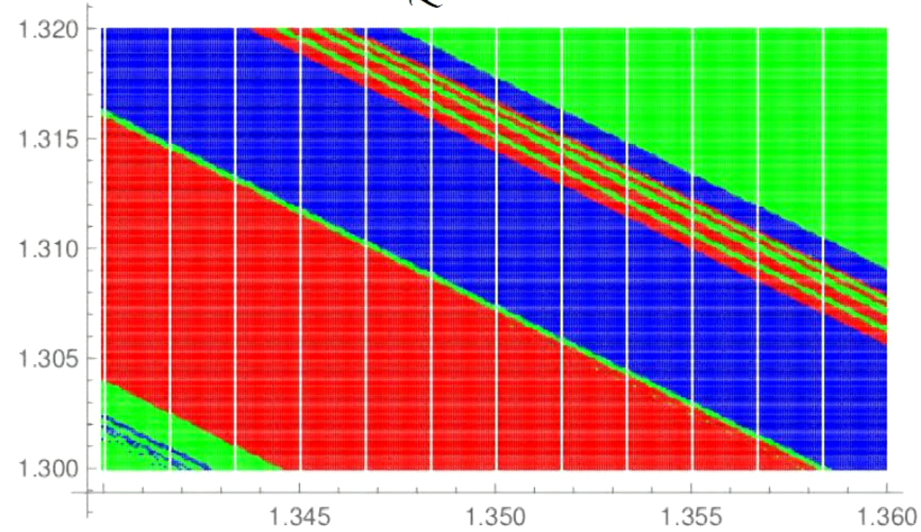
Quantum model: chaos

2. Fractal methods:

Classical



Quantum



Less fractal ⟷ Less chaotic

Quantum model: chaos

Interesting result: **implications?** (early universe…)

More approaches: **(Ongoing and future work)**

1. Apply iteratively the **discrete map** of the **quantum transition laws** obtained in [D. Brizuela, S. F. Uria (2022)]
2. Study **chaos far** from the **singularity**, where General Relativity prevails:
   - If the system is *still* chaotic, *no* **quantum theory** can fully **eliminate** the model's **chaotic nature**.
3. Consider a different quantization: **polymeric**

Conclusions

- Present the classical Mixmaster model: **chaotic** character
  - Numerically: **fractal methods**
  - Analytically: **Lyapunov exponent**
- Remark that **quantum effects** must be taken into account (close to the singularity)
- Build a **quantum** Mixmaster **model**
  - Perform a **decomposition** of the wave function into its **quantum moments**
  - Consider two minimal approximations:
    - **Semiclassical** regime
    - **General enough** for different theories of quantum gravity
    - Describe the system with **canonical variables** and **finite phase space**
- Study **quantum** chaos with the previous methods (numerical + analytical):

Quantum effects reduce the level of chaos!

Thanks for your attention