Title: Spark algebras and quantum groups

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Series: Mathematical Physics

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Abstract: I will discuss an explicit way to construct Hopf algebras and quasi-triangular Hopf algebras (their Drinfeld doubles) within 3d TQFT, using extended operators on boundary conditions -- dubbed `spark' algebras. The representation categories of these algebras capture boundary and bulk line operators. Overall, the construction realizes Tannakian duality geometrically; in perturbative TQFT's, it is closely connected to the holographic Koszul duality of Costello and Paquette. I'll illustrate the construction for Dijkgraaf-Witten theory (a.k.a. gauge theory with finite gauge group), and then sketch an application to the B-type topological twist of 3d N=4 gauge theories, which initially motivated these investigations. (Work in progress with T. Creutzig and W. Niu.)

Zoom link https://pitp.zoom.us/j/93746215441?pwd=YjdhaDNFeko3VDVKQW5ZV1MzL1cvUT09

Spark Algebras & Quantum Groups

Perimeter Institute, 7 Dec. 2023

Tudor Dimofte, U. of Edinburgh

w/ T. Creutzig, W. Niu

Intro

Physics: in (say) a topological QFT

local operators : algebras vector space + operations

a lot of data, especially if not semisimple
- line operators : categories reps of some algebra?
- surface operators : 2-categories reps of some monoidal category?
- ⋮ ⋮

How to organize this?

Optimistic answer: via (higher) rep thy.

E.g. in a G gauge thy, expect line operators $\overset{\text{something like}}{\sim}$ $\mathrm{Rep}(G) = U(\mathfrak{g})\text{-mod}^{\text{integrable}}$

4d BF thy $\mathcal{C} = \mathrm{Rep}\, G$

4d $\mathcal{N}=4$ SYM, B twist $\mathcal{C} = D^b\mathrm{Coh}^{\mathrm{Ad}(G)}(\mathfrak{g}_c) \supset \mathrm{Rep}(G)$

3d G_k CS $\mathcal{C} \simeq U_q(\mathfrak{g})\text{-mod}^{s.s.} \sim \mathrm{Rep}(G)$ as $k \to \infty$

Where do these algebras representing categories come from?

In math: Tannakian duality

Given a functor $\mathcal{F}: \mathcal{C} \to \mathrm{Vect}$ respecting all extra structures on $\mathcal{C}$, e.g. $\otimes$, rigidity, ...

"fiber functor"

let A = Symmetries of $\mathcal{F}$ endowed w/ extra dual structure e.g. Δ, S, ...

Then $\mathcal{C} \simeq A\text{-mod}$.

* w/ appropriate assumptions

In physics: how to get a fiber functor? What are its symmetries?

Roughly, $\mathcal{F} \sim$ a trivializing boundary condition or vacuum at ∞

$A \sim$ (possibly extended) operators/excitations on $\mathcal{F}$.

Some manifestations of this

there have been many!

- late 90's → 2000's 2d A/B models & categories of branes represented via quiver algebras

  entwined w/ notion of Koszul duality for A_∞-cats

  cf. Gaiotto-Moore-Witten '15
  TD @ StringMath '17
  Butson-Rapcak '23
  Gaiotto-Khan '23

- 2010's 4d holomorphic-topological twists & Yangians
  ($\mathcal{C}$ = line operators)

  cf. Costello '13
  Costello-Yamazaki-Witten '17
  ...

  ⇝ Koszul duality & holography Costello-Paquette '20
  ≃ Tannakian duality for perturbative QFT w/ vacuum @ ∞

- 2009 [Witten] fiber functor for Satake category via transverse boundaries in 4d SYM...

Today: general fiber functors from pairs of transverse boundary conditions.

For 3d TQFT ⇝ Hopf algebras & Drinfeld double construction.

Motivation: braided ⊗ cats in twisted 3d $\mathcal{N}=4$ gauge thy's

comparing w/ (complicated) VOA answers

Inspiration: papers/thesis of Nanna Aamand $U_q(\mathfrak{g})$ in perturbative 3d CS

computed via transverse b.c.

Perspiration: Johnson-Freyd & Reutter WIP on higher braiding

overlaps w/ constructions today

$\boxed{\text{Setup}}$ (in 3d)

Examples: $\mathcal{T}$ = DW(G) G finite
$\mathcal{N}$ = Neumann $\mathcal{D}$ = Dirichlet

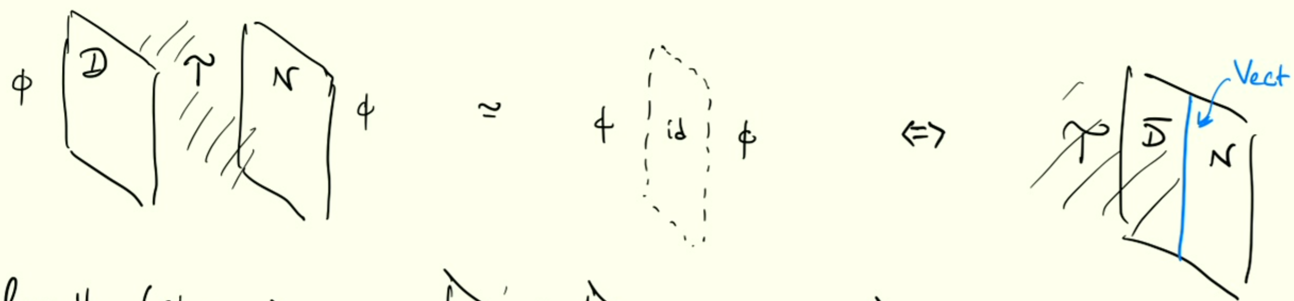
$\mathcal{T}$ = 3d BF thy (similar)

$\mathcal{T}$ = 3d hypermultiplet (X,Y), B-twist
$\mathcal{N} = \{Y|_\partial = 0\}$ $\mathcal{D} = \{X|_\partial = 0\}$

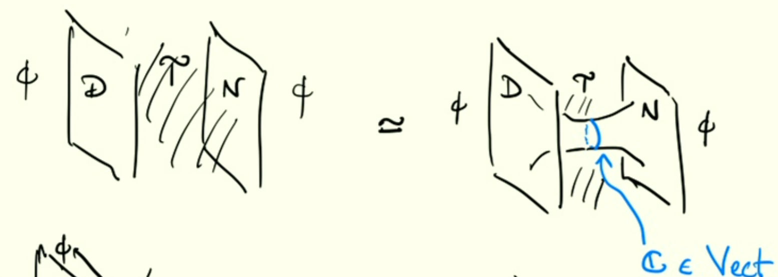
$\mathcal{T}$: 3d TQFT, oriented

$(\mathcal{D}, \mathcal{N})$: a pair of transverse, "complete", oriented b.c.

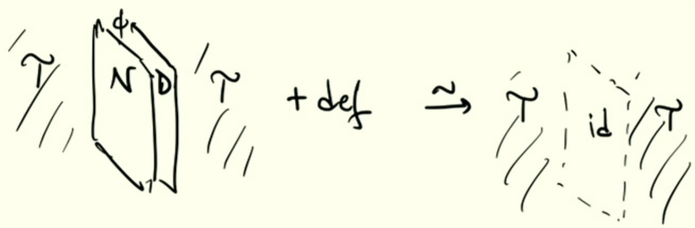
transverse:



locally (stronger)



complete: $\exists$ a deformation (a.k.a. condensation, or coupling)



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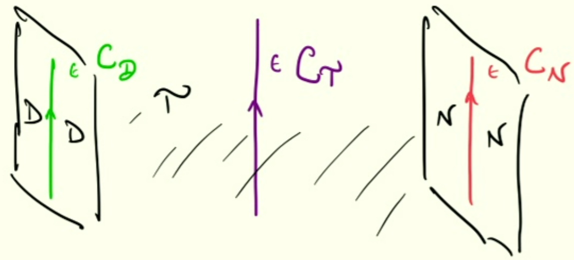
Setup (in 3d) $\mathcal{T}$: 3d TQFT, oriented

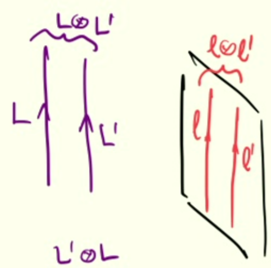
$(\mathcal{D}, \mathcal{N})$: a pair of transverse, "complete", oriented b.c.

Introduce three categories



Recall: categories w/ morphisms defined as

all monoidal w/

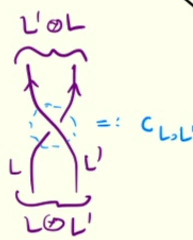


(etc)

Expect: $\mathcal{C}_\mathcal{T} = Z(\mathcal{C}_\mathcal{D}) = Z(\mathcal{C}_\mathcal{N})$

Drinfeld centers.

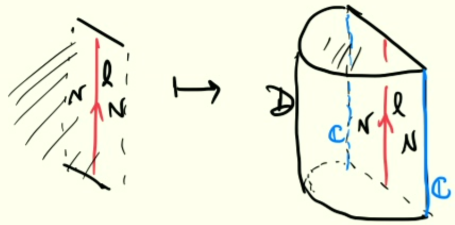
and $\mathcal{C}_\mathcal{T}$ is braided

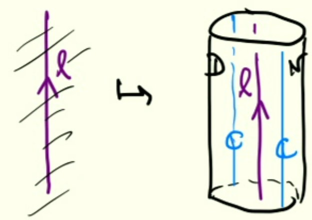


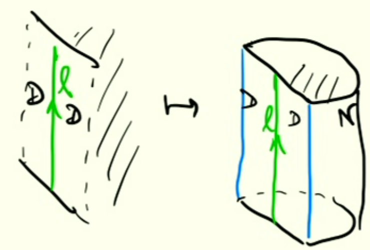
we can define three fiber functors...

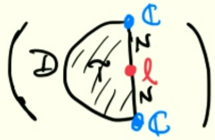
Fiber functors

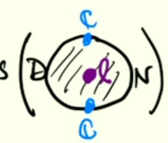
$\mathcal{F}_N : \mathcal{C}_N \to \mathrm{Vect}$ $\qquad$ $\mathcal{F}_T : \mathcal{C}_T \to \mathrm{Vect}$ $\qquad$ $\mathcal{F}_D : \mathcal{C}_D \to \mathrm{Vect}$

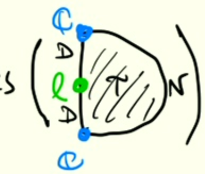




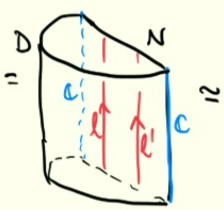
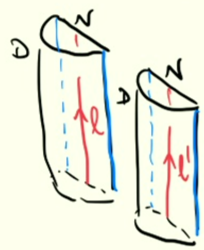


$\ell \mapsto$ States()

$\ell \mapsto$ States()

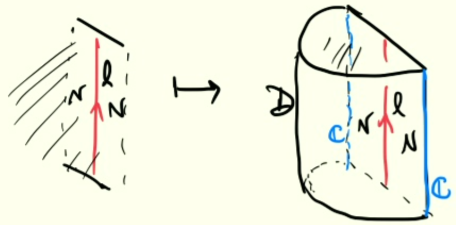
$\ell \mapsto$ States()

all <u>monoidal</u>, due to transversality:

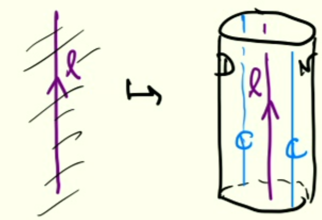
$\mathcal{F}_N(\ell \otimes \ell') =$  $\cong$  $= \mathcal{F}_N(\ell) \otimes \mathcal{F}_N(\ell')$ $\qquad$ (etc.)

Fiber functors

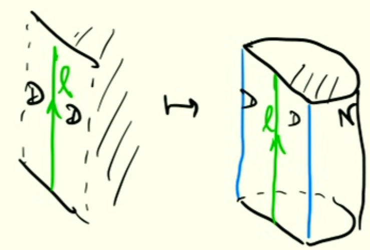
$F_N : C_N \to \mathrm{Vect}$



$F_T : C_T \to \mathrm{Vect}$



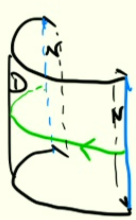
$F_D : C_D \to \mathrm{Vect}$



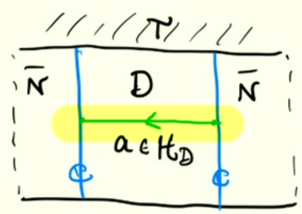
Tannakian philosophy: find symmetries of the functors.

Define H_D := operators supported in an interval on D, between N's
= "spark algebra on D"

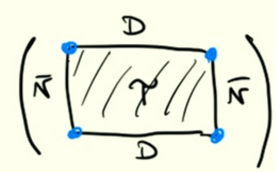
$H_D \ni a$



or



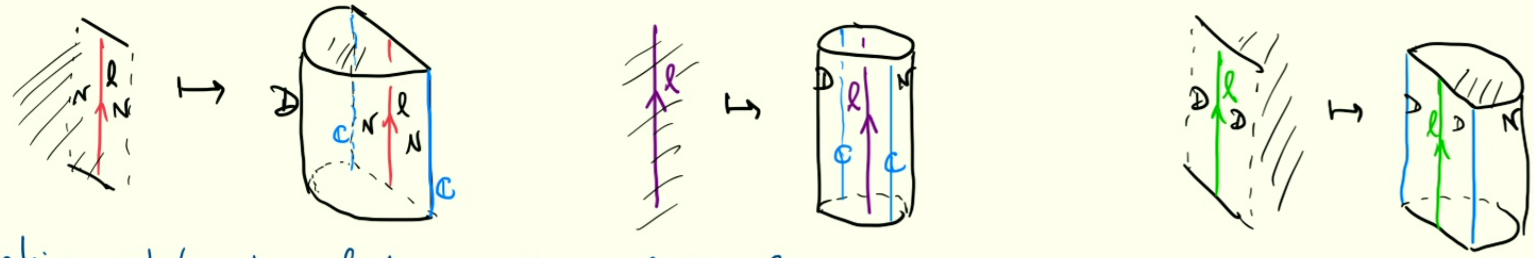
or $H_D :=$ States



obtain H_D as interval factorization / Hochschild homology of C_D...

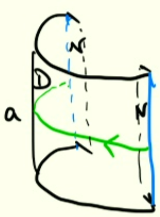
Fiber functors

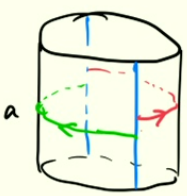
$\mathcal{F}_N : \mathcal{C}_N \to \mathrm{Vect}$ $\qquad$ $\mathcal{F}_T : \mathcal{C}_T \to \mathrm{Vect}$ $\qquad$ $\mathcal{F}_D : \mathcal{C}_D \to \mathrm{Vect}$



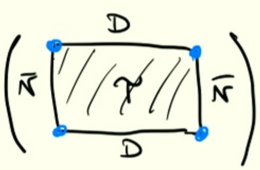
Tannakian philosophy: find symmetries of the functors.

Sparks on $\mathcal{D}$ $\qquad$ Sparks on S^1 $\qquad$ Sparks on $\mathcal{N}$

$\mathcal{H}_D \ni a$ 

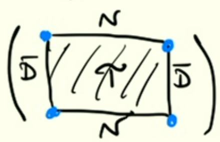
$U \ni$ 

$\alpha \in \mathcal{H}_N$

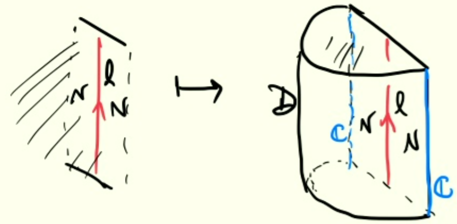
$\mathcal{H}_D := \mathrm{States}\Big($  $\Big)$

$U := \mathrm{States}\Big($  $\Big)$

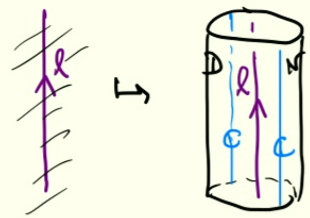
$\simeq \mathcal{H}_D \otimes \mathcal{H}_N$

$\mathcal{H}_N = \mathrm{States}\Big($  $\Big)$

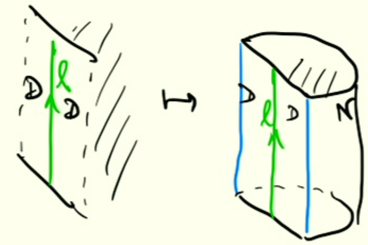
$\tilde{F}_N : C_N \to \mathrm{Vect}$ $\quad$ $\tilde{F}_T : C_T \to \mathrm{Vect}$ $\quad$ $\tilde{F}_D : C_D \to \mathrm{Vect}$



Sparks on D

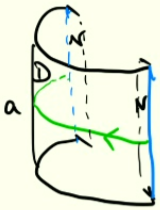


Sparks on S^1

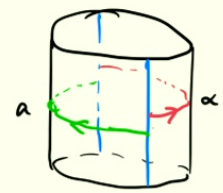


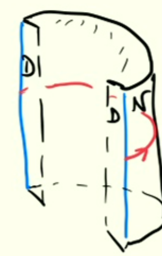
Sparks on N

$H_D \ni a$



$U \ni$





$\alpha \in H_N$

Claims:

[cf Reutter '17 at PI]

- H_D & H_N are <u>Hopf algebras</u> w/ a nondegenerate Hopf pairing

  $h : H_D \otimes H_N \to \mathbb{C}$ $\qquad$ $h(a^i, \alpha_j) :=$  $= h^i_j$

3-ball

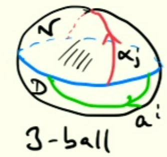
  $h(ab, \alpha) = \sum h(a, \alpha^{(1)})\, h(b, \alpha^{(2)})$
  w/ $\Delta\alpha = \sum \alpha^{(1)} \otimes \alpha^{(2)}$

- U is (intrinsically) a quasitriangular Hopf algebra and $U \simeq \mathrm{Double}(H_D) = \mathrm{Double}(H_N)$

  $$R = \sum_{i,j} (h^{-1})^j{}_i\, \alpha_j \otimes a^i$$

- $C_N \simeq H_D\text{-mod}$ $\qquad$ $C_T \simeq U\text{-mod}$ $\qquad$ $C_D \simeq H_N\text{-mod}$

$\boxed{\text{Claims:}}$ • $\mathcal{H}_D$ & $\mathcal{H}_N$ are <u>Hopf algebras</u> [cf Reutter '17 at PI] w/ a nondegenerate Hopf pairing

$h : \mathcal{H}_D \otimes \mathcal{H}_N \to \mathbb{C} \qquad h(a^i, \alpha_j) :=$  $= h^i_j$

• U is (intrinsically) a quasitriangular Hopf algebra and $U \simeq \mathrm{Double}(\mathcal{H}_D) \simeq \mathrm{Double}(\mathcal{H}_N)$

$$R = \sum_{i,j} (h^{-1})^j{}_i \, \alpha_j \otimes a^i$$

• $\mathcal{C}_N \simeq \mathcal{H}_D\text{-mod} \qquad \mathcal{C}_T \simeq U\text{-mod} \qquad \mathcal{C}_D \simeq \mathcal{H}_N\text{-mod}$

<u>Perturbatively:</u> In a dg setting (e.g. BV formalism),

[Costello-Paquette] $\Rightarrow$

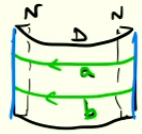
$$\mathcal{H}_D \simeq (\text{Local ops on } N)^! \leftarrow \text{Koszul dual}$$
$$\mathcal{H}_N \simeq (\text{Local ops on } D)^!$$
$$U \simeq (\text{Bulk local ops})^!$$

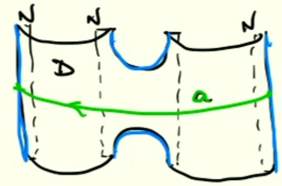
Claim • $h : \mathcal{H}_D \otimes \mathcal{H}_N \to \mathbb{C}$ is given by the bulk shifted-Poisson (E_3) bracket

$$\Rightarrow R = (E_3\text{-bracket})^{-1}$$

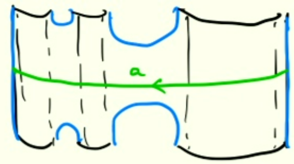
Proofs by picture

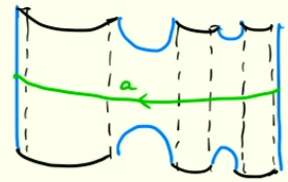
1) $\mathcal{H}_D$ is Hopf [Reutter '17 @ PI]

$ab :=$ 

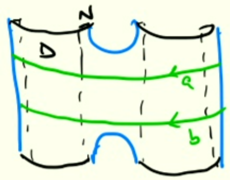
$\Delta(a) :=$ 

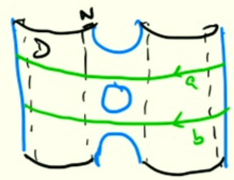
$S(a) = \dots$

check $\Delta_1\Delta(a) = \Delta_2\Delta(a)$ 

$\simeq$ 

$=: \Delta\Delta(a)$

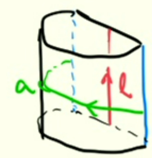
$\Delta(ab) =$ 

transverse $\simeq$ 

$= \Delta(a)\Delta(b)$

+ antipode relations

$\Rightarrow$ $\otimes$ functor $C_N \to \mathcal{H}_D$-mod is automatic

$\mathcal{H}_D$ acts on all $F_N(\ell)$ 

and $a|_{F_N(\ell\otimes\ell')} = \Delta(a)|_{F_N(\ell)\otimes F_N(\ell')}$ (etc.)

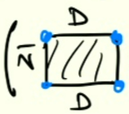
Equivalence of cats requires "<u>completeness</u>" of b.c..

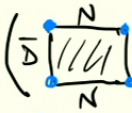
Proofs by picture

2) Hopf pairing $h(a,\alpha) =$ 

3-ball

is just the pairing of

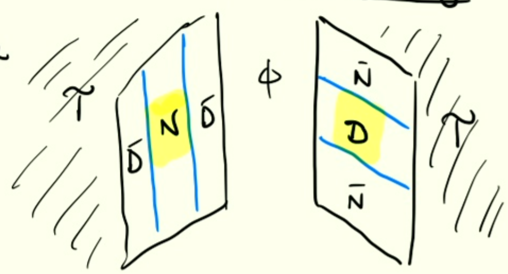
$\mathcal{H}_D = \mathrm{States}\Big($ 

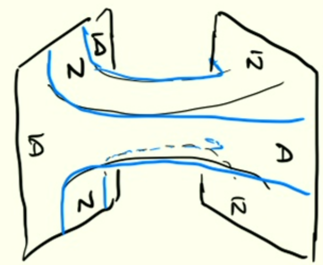
$\Big)$ & $\mathcal{H}_N = \mathrm{States}\Big($ 

$\Big)$

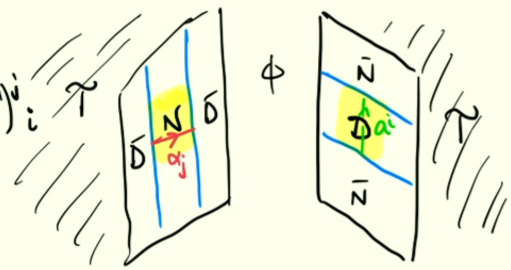
dual, if finite dimensional

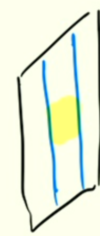
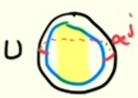
Key property: it inverts the <u>box gluing</u>

I.e. to glue 

$\rightsquigarrow$ 

$\simeq$

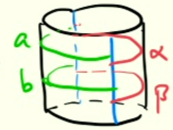
deform by $\sum_{i,j} (h^{-1})^{ij}$ 

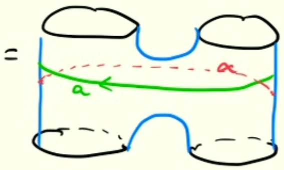
Proof:  $\cup$ 

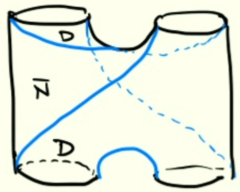
$\simeq$ 

$\forall \alpha^i$

Proofs by picture

3) The S^1 algebra is quasitriangular Hopf $\quad (a\alpha)(b\beta) =$ 

$\Delta(a\alpha) :=$ 

$R :=$ 

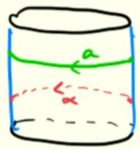
~ twisted $\Delta(1)$

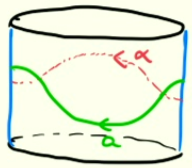
$\Rightarrow \quad R\,\Delta\,R^{-1} = \Delta^{op}$

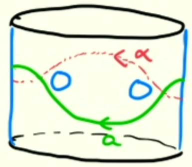
Check that it's the double:

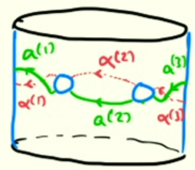
$\Delta\Delta(a) = \sum a^{(1)} \otimes a^{(2)} \otimes a^{(3)}$

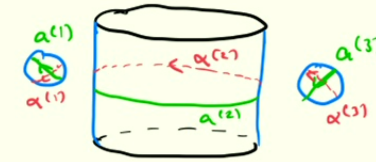
$\Delta\Delta(\alpha) = \sum \alpha^{(1)} \otimes \alpha^{(2)} \otimes \alpha^{(3)}$

- $a\alpha =$ 

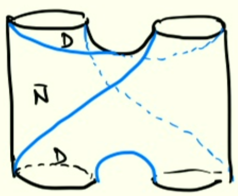
$=$ 

trans. $\simeq$ 

def Δ $=$ 

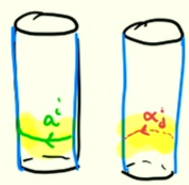
trans. $\simeq$ 

$= \sum h(S(a^{(1)}), \alpha^{(1)})\, \alpha^{(2)} a^{(2)}\, h(a^{(3)}, \alpha^{(3)})$

- $R =$ 

$=$ 

glue

$\simeq \sum (h^{-1})^j{}_i$ 

$= \sum (h^{-1})^j{}_i \; a^i \otimes \alpha_j$

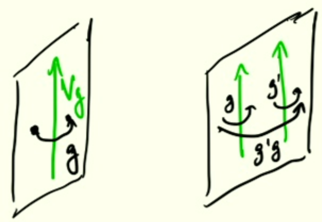
Examples

Dijkgraaf-Witten theory (finite G gauge thy) $\quad Z(M) \sim \sum_{G\text{-bundles } E\to M} \frac{1}{\mathrm{Aut}(E)}$

N : Neumann $\quad$ exists only if 3-cocycle (CS term) $=0$ $\qquad \mathcal{C}_N = \mathrm{Rep}(G)$ Wilson lines

D : Dirichlet $\quad$ trivializes G bundle $\qquad W_\rho \otimes W_{\rho'} \simeq W_{\rho\otimes\rho'}$

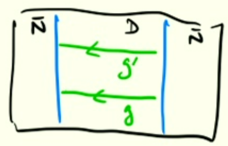
line ops on D are defects
that change the trivialization



$\mathcal{C}_D = \mathrm{Vect}_G$

$V_g \otimes V_{g'} \simeq V_{gg'}$

$\mathcal{H}_D = \mathbb{C}G$?

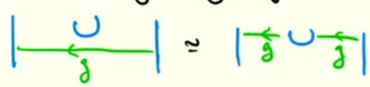


✓

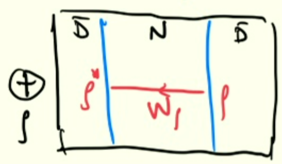
$\mathcal{H}_D\text{-mod} \simeq \mathcal{C}_N$

$\mathbb{C}G \qquad \mathrm{Rep}(G)$

easy: $\Delta g = g \otimes g$ $\qquad \leftrightarrow \quad \otimes$ in $\mathrm{Rep}(G)$



$\mathcal{H}_N = ?$



$= \bigoplus_\rho \rho^* \otimes \rho \simeq \mathbb{C}[G]$ functions on G

commutative product,

$\Delta \delta_g = \sum_{hk=g} \delta_h \otimes \delta_k$

$\mathcal{H}_N\text{-mod} \simeq \mathcal{C}_D$

$\mathbb{C}[G] \qquad \mathrm{Vect}_G$

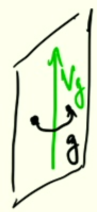
Examples

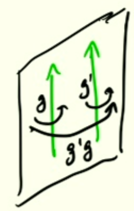
Dijkgraaf-Witten theory (finite G gauge thy) $Z(M) \sim \sum_{G\text{-bundles } E \to M} \frac{1}{\mathrm{Aut}(E)}$

$\mathcal{N}$ = Neumann exists only if 3-cocycle (CS term) $= 0$ $\quad \mathcal{C}_{\mathcal{N}} = \mathrm{Rep}(G)$ Wilson lines

$\mathcal{D}$ = Dirichlet trivializes G bundle $\quad W_\rho \otimes W_{\rho'} \simeq W_{\rho \otimes \rho'}$

line ops on $\mathcal{D}$ are defects that change the trivialization





$\mathcal{C}_{\mathcal{D}} = \mathrm{Vect}_G$

$V_g \otimes V_{g'} \simeq V_{gg'}$

$\mathcal{H}_{\mathcal{D}} = \mathbb{C}G = \mathbb{C}\langle g \rangle_{g \in G}$

$\mathcal{H}_{\mathcal{N}} = \mathbb{C}[G] = \mathbb{C}\langle \delta_g \rangle_{g \in G}$ $\quad \Rightarrow \quad U \simeq \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{N}} \quad g\,\delta_h = \delta_{ghg^{-1}}\, g \quad R = \sum_{g \in G} \delta_g \otimes g$

$\boxed{\text{Note:}}$ Only partial info is required!

e.g. • $\mathcal{H}_{\mathcal{D}}$ as a Hopf algebra

OR • $\mathcal{H}_{\mathcal{D}}$ & $\mathcal{H}_{\mathcal{N}}$ as algebras (recover Δ from Hopf pairing)

$\Rightarrow$ full structure of $\mathcal{H}_{\mathcal{D}}, \mathcal{H}_{\mathcal{N}}, U$

B-twist of 3d $\mathcal{N}=4$ (G, V) gauge thy

G: compact; V: complex, unitary rep

hypermultiplets $(X, Y) \in V \oplus V^*$

17

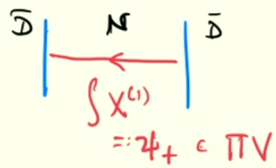
$\mathcal{N}$: Neumann on gauge field, $Y=0$
$\mathcal{D}$: Dirichlet on gauge field, $X=0$

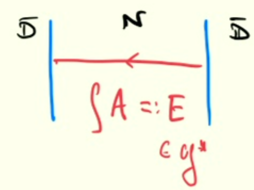
} unique 2d $\mathcal{N}=(2,2)$ completion, compatible w/ 2d B twist

[Kapustin-Rozansky-Saulina '08
Bullimore-TD-Gaiotto-Hilburn '16]

Perturbatively:

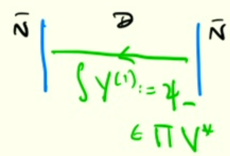
$\mathcal{H}_{\mathcal{N}}$ contains

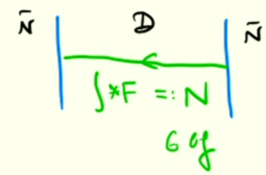




$\mathcal{H}_{\mathcal{N}} \simeq \mathrm{Sym}(\mathfrak{g}^*) \otimes \mathrm{Sym}\, \Pi V$
E ψ_+
commutative

$\mathcal{H}_{\mathcal{D}}$ contains





$\mathcal{H}_{\mathcal{D}} \simeq U(\mathfrak{g}) \# \mathrm{Sym}\, \Pi V^*$
N ψ_-
$[N, -]$ = action

Non-perturbatively: use [KRS] methodology to extract

categories
$\mathcal{C}_{\mathcal{D}} \simeq D^b\mathrm{Coh}(G_{\mathbb{C}} \times V^*)$
$\mathcal{C}_{\mathcal{N}} \simeq D^b\mathrm{Coh}(V/G)$

$\longleftrightarrow$

algebras (no Δ)
$\mathcal{H}_{\mathcal{N}} \simeq \mathbb{C}[G_{\mathbb{C}}] \otimes \mathrm{Sym}\, \Pi V$
$\mathcal{H}_{\mathcal{D}} \simeq$ "$\mathbb{C} G_{\mathbb{C}}$" $\# \mathrm{Sym}\, \Pi V^*$

18

B-twist of 3d $\mathcal{N}=4$ (G, V) gauge thy

compact — complex, unitary rep

hypermultiplets $(X, Y) \in V \oplus V^*$

$\mathcal{N}$: Neumann on gauge field, $Y=0$

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} unique 2d $\mathcal{N}=(2,2)$ completion, compatible w/ 2d B twist

[Kapustin - Rozansky - Saulina '08
Bullimore - TD - Garotto - Hilburn '16]

$$\mathcal{H}_{\mathcal{N}} \simeq \overset{e^E}{\mathbb{C}[G_{\mathbb{C}}]} \otimes \overset{\mathcal{4}_+}{\mathrm{Sym}\,\Pi V}$$

$$\mathcal{H}_{\mathcal{D}} \simeq \text{``}\mathbb{C}G_{\mathbb{C}}\text{''} \,\#\, \overset{\mathcal{4}_-}{\mathrm{Sym}\,\Pi V^*}$$

integrable $U(\mathfrak{g})$-modules

$$\Rightarrow \quad U \simeq \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{N}}$$

$$\text{w/} \quad R = \left(\text{standard } \mathbb{C}G \otimes \mathbb{C}[G]\right) \cdot e^{\mathcal{4}_- \otimes \mathcal{4}_+} .$$