Title: Quantizing non-linear phase spaces, causal diamonds and the Casimir matching principle

Speakers: Rodrigo Andrade E Silva

Series: Quantum Gravity

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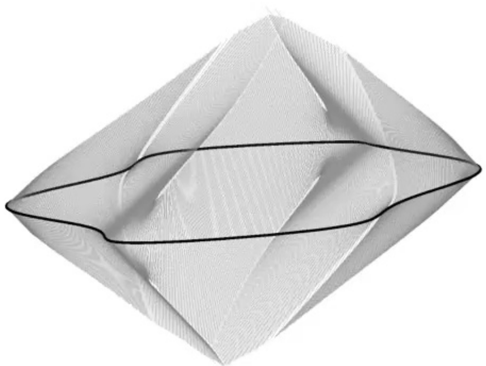
Abstract: In the quest to understand the fundamental structure of spacetime (and subsystems) in quantum gravity, it may be worth exploring the ultimate consequences of non-perturbative canonical quantization, carefully taking into account the constraints and gauge invariance of general relativity. As the reduced phase space (or even the pre-phase space) of gravity lacks a natural linear structure, a generalization of the standard method of quantization (based on global conjugate coordinates) is required. One such generalization is Isham's method based on transitive groups of symplectomorphisms, which we test in some simple examples. In particular, considering a particle that lives on a sphere, in the presence of a magnetic monopole flux, we algebraically recover Dirac's charge quantization condition from a "Casimir matching principle", which we propose as an important tool in selecting natural representations. Finally, we develop the non-perturbative reduced phase space quantization of causal diamonds in (2+1)-dimensional gravity. By solving the constraints in a constant-mean-curvature time gauge and removing all the spatial gauge redundancy, we find that the phase space is the cotangent bundle of $Diff^+(S^1)/PSL(2, \mathbb{R})$. Applying Isham's quantization we find that the Hilbert space of the associated quantum theory carries a (projective) irreducible unitary representation of the BMS_3 group. From the Casimir matching principle, we show that the states are realized as wavefunctions on the configuration space with internal indices in unitary irreps of $SL(2, \mathbb{R})$. A surprising result is that the twist of the diamond boundary loop is quantized in terms of the ratio of the Planck length to the boundary length.

Papers on quantization of causal diamonds:
https://arxiv.org/abs/2308.11741
https://arxiv.org/abs/2310.03100

Zoom link https://pitp.zoom.us/j/98636934234?pwd=THBCcFpYWHYzWmY0YjViY3Q1a3VYdz09

Quantizing non-linear phase spaces, causal diamonds and the Casimir matching principle

Rodrigo Andrade e Silva



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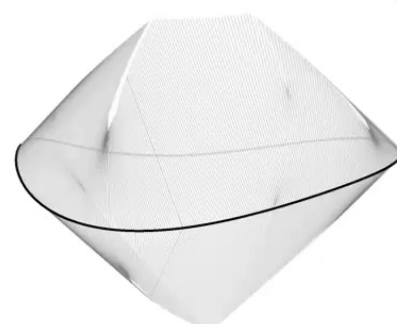


[arXiv:2308.11741, arXiv:2310.03100]

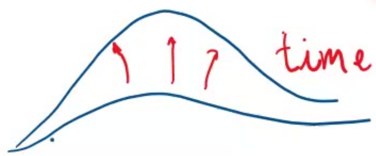
Peculiarities of quantum gravity

- Absence of local observables

$$[\mathcal{O}, C] \approx 0$$

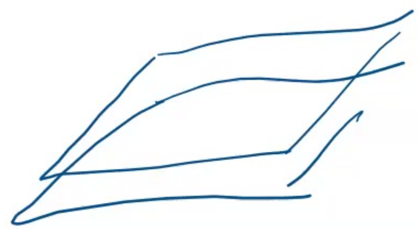
$$\Sigma \xrightarrow{\Psi} \Psi(\Sigma)$$

- Problem of time



- Quantum spacetime?

$$[h, \pi] \neq 0$$



- Perturbatively non-renormalizable in d > 3

$$[G] < 0$$

The canonical perspective

In the lack of experimental guidance, we should push the current theoretical framework to its limits… Lessons to be learned? Where exactly things break?

Canonical quantization is a magnificent tool, so it may be worth exploring it as carefully as possible.

→ The quantum theory "retains" the algebraic structure of classical observables

Consider the full structure of the phase space (constraints, gauge, topology)

No a priori assumptions about the Hilbert space

"Path integral is derived from the quantization"

Dirac's canonical quantization

If $\mathcal{P} = \mathbb{R}^{2n}$, take ***global*** coordinates x^i and p_i, satisfying $\{x^i, p_j\} = \delta^i_j$

Assume that they are represented self-adjointly, satisfying $[\hat{x}^i, \hat{p}_j] = i\delta^i_j$

Stone-von Neumann: *unique* unitary irrep given by wavefunctions $\Psi \in L^2(\mathbb{R}^n; d^n x)$,

$$\hat{x}^i \Psi(x) = x^i \Psi(x)$$

$$\hat{p}_j \Psi(x) = -i \frac{\partial}{\partial x^j} \Psi(x)$$

This quantization is *only* compatible with linear phase spaces: spectrum of x and p is whole $\mathbb{R}$

The uniqueness is very special and should not be taken for granted!

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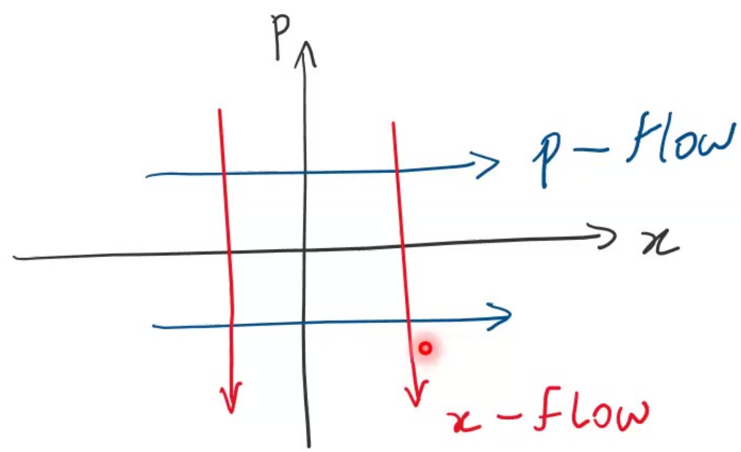
Contents

- Isham's group-theoretic quantization scheme
- The Casimir matching principle: a lesson from quantizing $\mathcal{P} = T^*S^2$
- Quantizing causal diamonds in (2+1)-dimensional gravity
- Refining the representations by Casimir matching
- Twist quantization

Isham's quantization scheme

For a linear phase space $\mathcal{P} = \mathbb{R}^{2n}$, with global Cartesian coordinates x^i and p_i, note that the p function generates translations in x, while the x function translates in $-p$



→ Think of the group of phase space translations, $G = \mathbb{R}^{2n}$, as the *foundation* of the quantization.

Isham's quantization scheme

[Isham 84, 89]

"Find a transitive group G of symplectic symmetries of the phase space, and then construct the quantum theory based on unitary irreducible (projective) representations of G."

Each element ξ_i of the algebra is associated with a Hamiltonian charge H_i.

The Poisson algebra of these charges is homomorphic* to the algebra of G.

Transitiveness implies that this set of charges is complete (i.e., any observable can be locally expressed in terms of them).

Quantization proceeds by finding unitary irreducible representations of this algebra

$$\{H_i, H_j\} = c_{ij}^k H_k \quad \longrightarrow \quad \frac{1}{i\hbar}\left[\hat{H}_i, \hat{H}_j\right] = c_{ij}^k \hat{H}_k$$

Affine quantization

In the case of gravity, the (pre-)phase space is $\mathcal{P} = T^*\big(Riem(\Sigma)\big)$.

$\rightarrow$ $h - h'$ may not belong to $Riem(\Sigma)$

The group of translations, $(Sym_n\mathbb{R} \times Sym_n\mathbb{R})^\Sigma$, does not act naturally on $\mathcal{P}$

$$\cancel{\{h_{ab}(x), \pi^{cd}(x')\} = \delta^c_{(a}\delta^d_{b)}\,\delta(x,x')}$$

But the affine group, $(\kappa, \Lambda) \in (Sym_n\mathbb{R} \rtimes GL_n\mathbb{R})^\Sigma$, does:

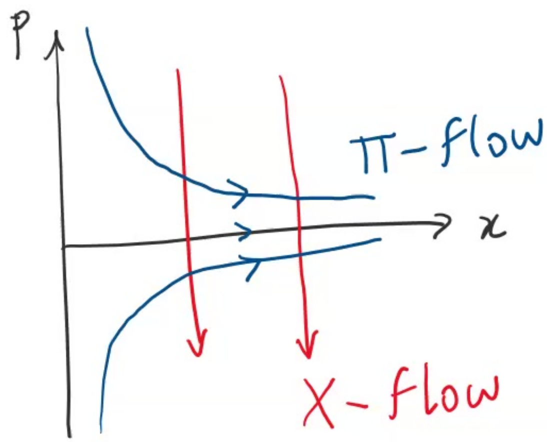
$$(\kappa, \Lambda)(h, \pi) = (\Lambda h, \pi\Lambda^{-1} - \kappa)$$

[Klauder, Isham, Pilati]

The half-line

Consider "1+1 gravity", with reduced phase space $\mathcal{P} = T^*\mathbb{R}^+ = \mathbb{R}^+ \times \mathbb{R}$

The affine group, $(k, \lambda) \in \mathbb{R} \rtimes \mathbb{R}^+$, acts as $(x, p) \mapsto (\lambda x, \lambda^{-1} p - k)$



The *canonical* charges are:

$$X \coloneqq x$$
$$\Pi \coloneqq xp$$

with Poisson algebra: $\{X, \Pi\} = X$

In the quantum theory: $\frac{1}{i\hbar}[\hat{X}, \widehat{\Pi}] = \hat{X}$

Unitary irrep on Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^+; dx/x)$

[Klauder, Isham, Pilati]

A particle on the sphere

[RAS & Jacobson (arXiv:2011.04888)]

Now let us consider a particle living on a 2-sphere, in the presence of a magnetic monopole flux $\int B = 4\pi g$.

S^2 admits no global coordinates. Moreover, $H \sim (p - eA)^2$ is not globally-defined.

We can deal with the second issue by modifying the symplectic structure

$$\omega = d\theta = dp_i \wedge dq^i \rightarrow d(p_i + eA_i) \wedge dq^i = dp_i \wedge dq^i + ed\left(A_i dq^i\right) = d\theta + eB$$

but we still need to worry with the first issue.

A particle on the sphere

How to construct a transitive group of symplectomorphisms on $\mathcal{P} = T^*S^2$?

It is natural to start with $K = SO(3)$ acting on $S^2 \sim SO(3)/SO(2)$.

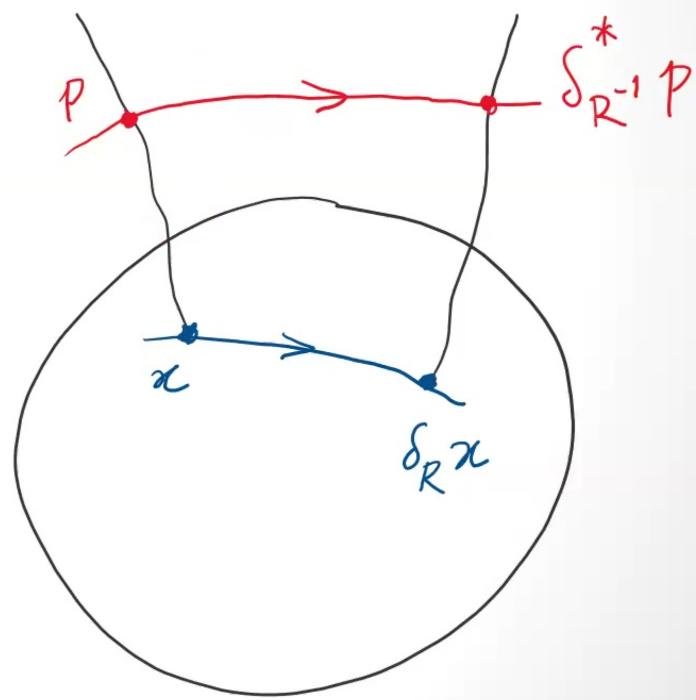
$$\delta_R(x) = Rx$$

Any action on the configuration space lifts to an action on the cotangent bundle

$$\tilde{\delta}_R(p) \coloneqq \delta^*_{R^{-1}}p$$

This action is symplectic, but not transitive.

We need to add some sort of "vertical'' translations



A particle on the sphere

Any 1-form field w on $Q = S^2$ defines a momentum translation

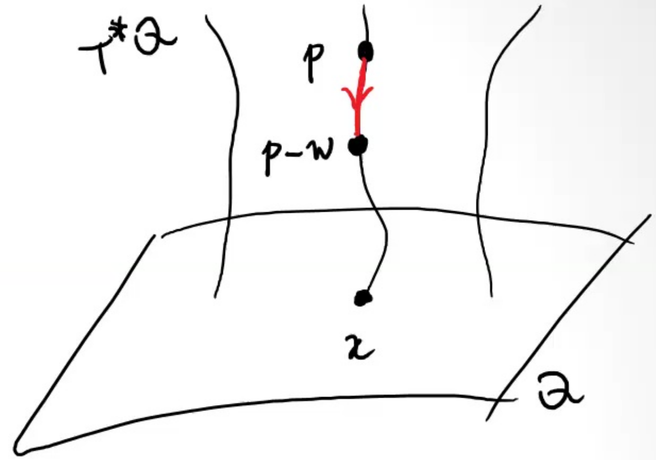
$$f_w(p) \coloneqq p - w$$

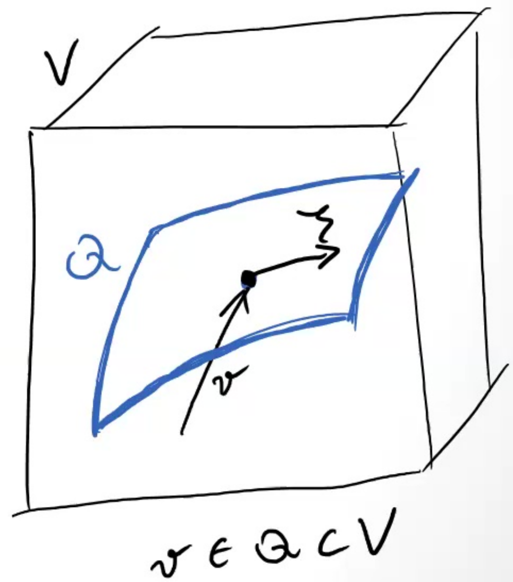
which is a symplectomorphism if $dw = 0$.

But we want a "*minimal*" set of translations:

"Find a representation of K on a vector space V such that at least one orbit in V is homeomorphic to Q"

Any $w \in V^*$ can be restricted to $Q \subset V$ to define an exact 1-form field on Q.





A particle on the sphere

The "canonical group" is thus

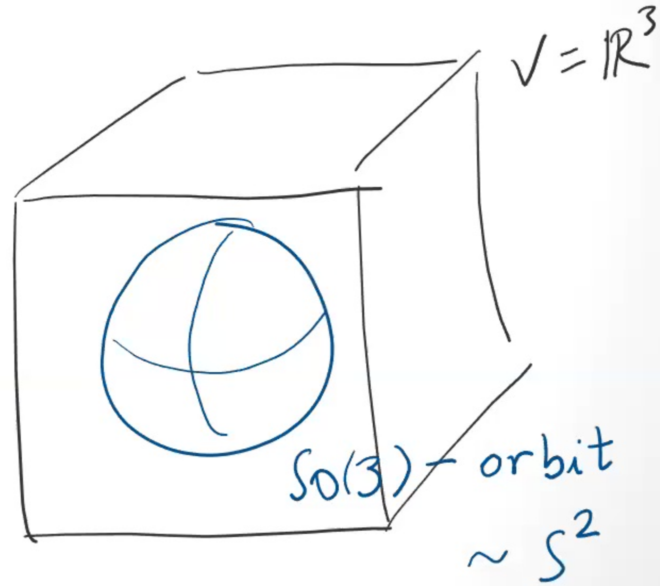
$$G = V^* \rtimes K$$

momentum translations   configuration translations

In the case of the sphere,
with $K = SO(3)$, we can take $V = \mathbb{R}^3$

$$G = (\mathbb{R}^3)^* \rtimes SO(3) = E(3)$$

i.e., the Euclidean group in 3d.



A particle on the sphere

$$G = (\mathbb{R}^3)^* \rtimes SO(3)$$

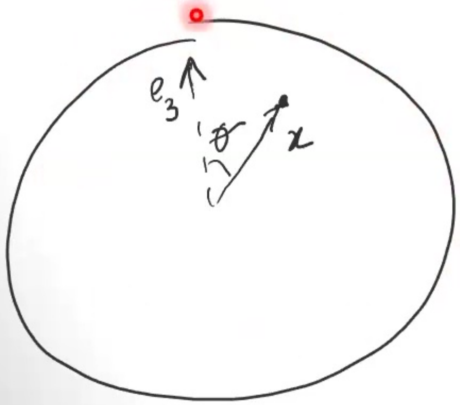
momentum translations

$$Q_w = w(x)$$

configuration translations

$$P_n = p(X_n) - eg\, n \cdot x$$

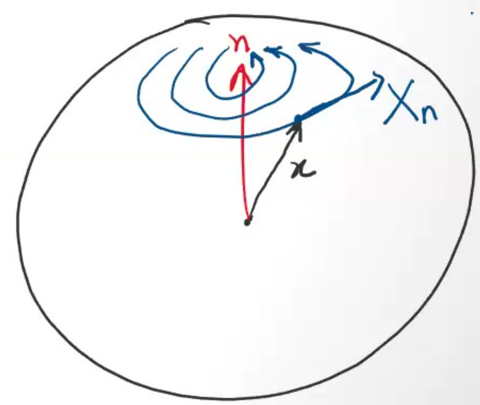
S^2 embedded into $V = \mathbb{R}^3$ as unit sphere



If $w = e^3$,

$$Q_{e^3} = e^3(x)$$

$$= x^3 = \cos\theta$$



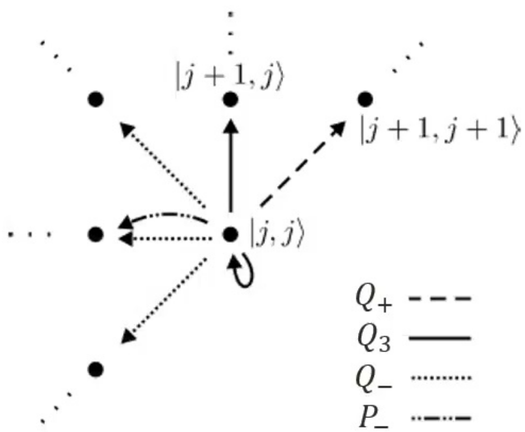
A particle on the sphere

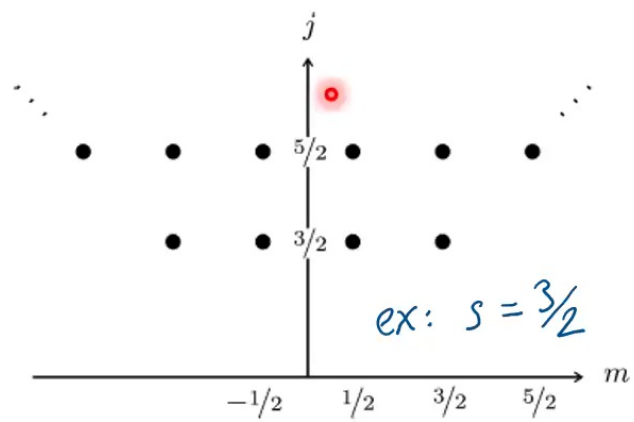
There are two independent Casimir operators

$$\hat{Q}_i\hat{Q}_i = \rho^2$$
$$\hat{P}_i\hat{Q}_i = s$$

As the algebra is invariant under $\hat{Q}_i \to \lambda\hat{Q}_i$, we can choose $\rho = 1$.

We find irreps classified in terms of s:





States $|j, m\rangle$

$$-j \leq m \leq j$$

$$j \geq j_0 = |s|$$

A particle on the sphere

In a basis, the classical algebra reads

$$\{P_i, P_j\} = \varepsilon_{ijk} P_k$$
$$\{P_i, Q_j\} = \varepsilon_{ijk} Q_k$$
$$\{Q_i, Q_j\} = 0$$

so quantization proceeds by finding a unitary irrep of the corresponding algebra

$$\left[\hat{P}_i, \hat{P}_j\right] = i\hbar\, \varepsilon_{ijk} \hat{P}_k$$
$$\left[\hat{P}_i, \hat{Q}_j\right] = i\hbar\, \varepsilon_{ijk} \hat{Q}_k$$
$$\left[\hat{Q}_i, \hat{Q}_j\right] = 0$$

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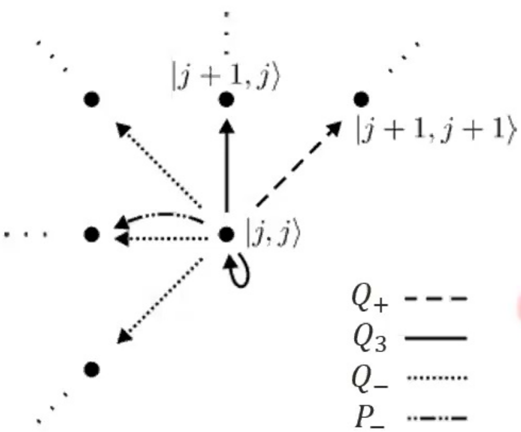

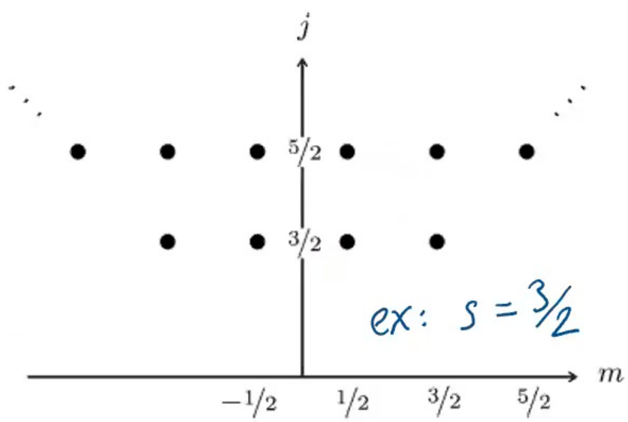

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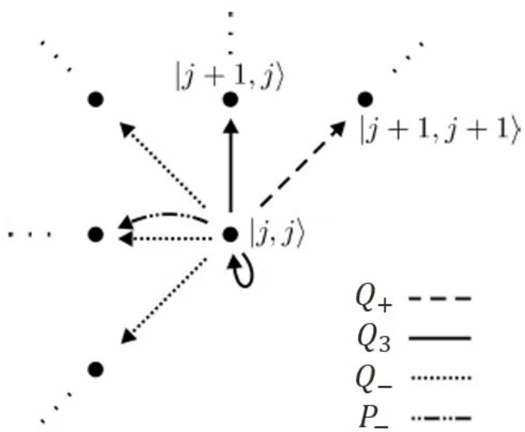
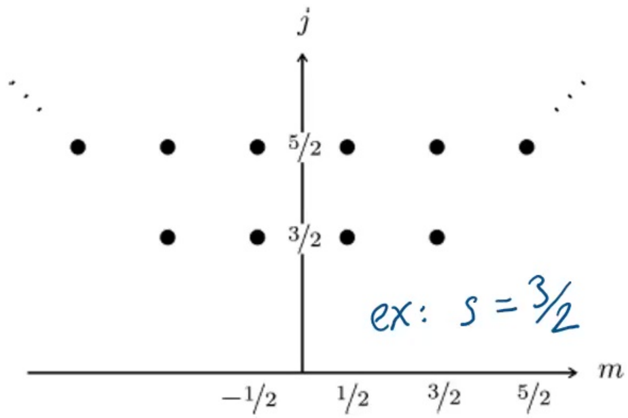




States $|j, m\rangle$

$$-j \leq m \leq j$$

$$j \geq j_0 = |s|$$

$s \in \frac{1}{2}\mathbb{Z}$

Casimir matching principle

How to select the *"correct"* representation?

Note that classical the two Casimirs are

$$Q_i Q_i = x^2 = 1$$
$$P_i Q_i = -eg\, x^2 = -eg$$

Both classically and quantum mechanically, Casimirs take the same value for all states.

We propose that the value of the classical and quantum Casimirs should match.

For the sphere, we get

$$-eg = s \in \frac{1}{2}\mathbb{Z}$$

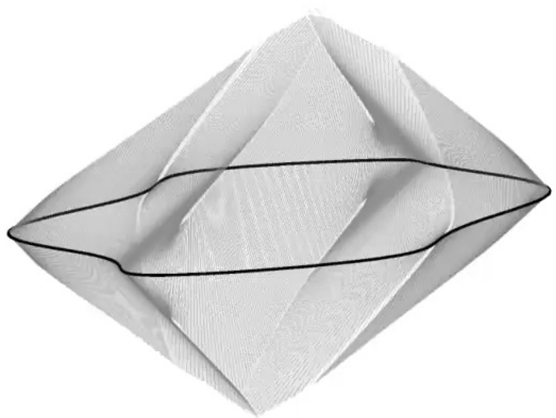
which is the famous Dirac charge discretization condition!

Causal diamonds: motivations

[RAS (arXiv:2308.11741, arXiv:2310.03100), RAS & Jacobson 23]

We wish to consider two things:

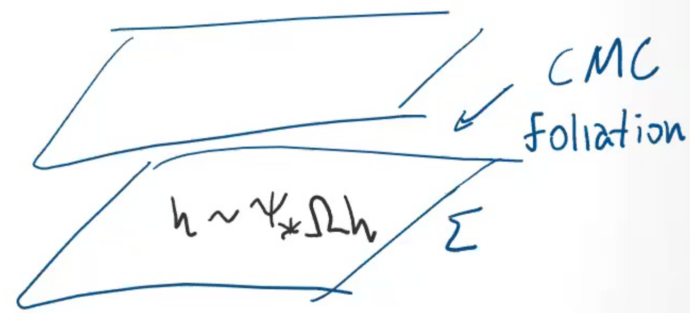
1. Describe quantum gravity in finite regions



Quantize causal diamonds?

2. Do it non-perturbatively

Moncrief's program



[Moncrief 89, Fischer, Carlip ..., Witten 22]

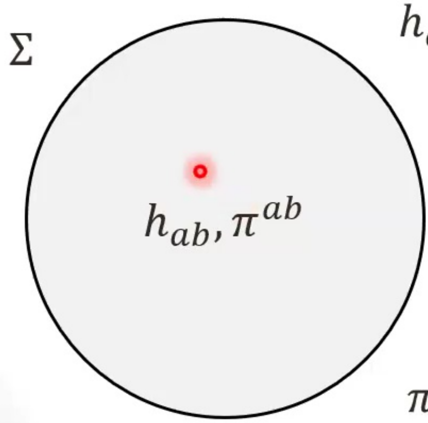
Causal diamonds: the system

[RAS (arXiv:2308.11741, arXiv:2310.03100), RAS & Jacobson 23]

2+1 dimensional Einstein-Hilbert gravity with $\Lambda \leq 0$

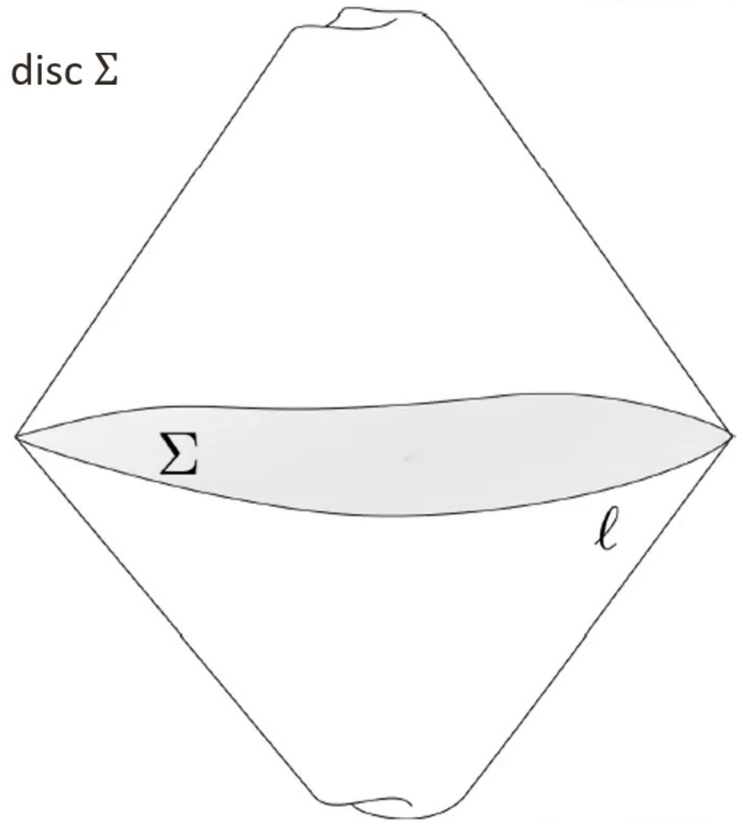
Spacetime is domain of dependence of a topological disc Σ

Dirichlet condition for the induced boundary metric (boundary length ℓ)



$$h_{ab}\Big|_{\partial D} = \gamma$$

$$\pi^{ab} = \sqrt{\det h}\left(K^{ab} - K h^{ab}\right)$$



CMC reduction

[RAS Part I: Classical reduction]

1. Show that any diamond can be foliated by

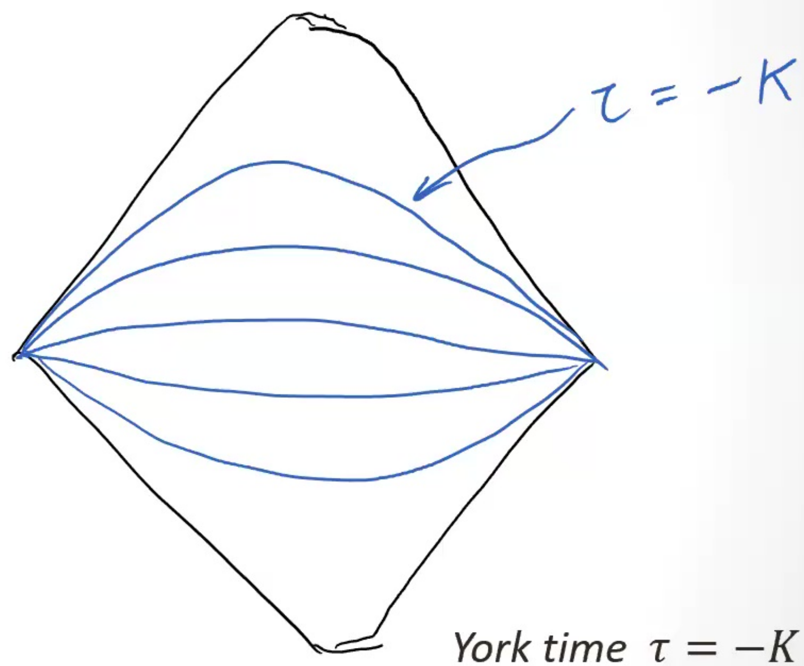
CMCs = Constant-mean-curvature surfaces

$$K = h_{ab}K^{ab} = \text{constant}$$

where K^{ab} is the extrinsic curvature.

→ $\Lambda \leq 0$ used here.

(More generally, we could include “attractive” matter.)



York time $\tau = -K$

[York 72]

CMC reduction

[RAS Part I: Classical reduction]

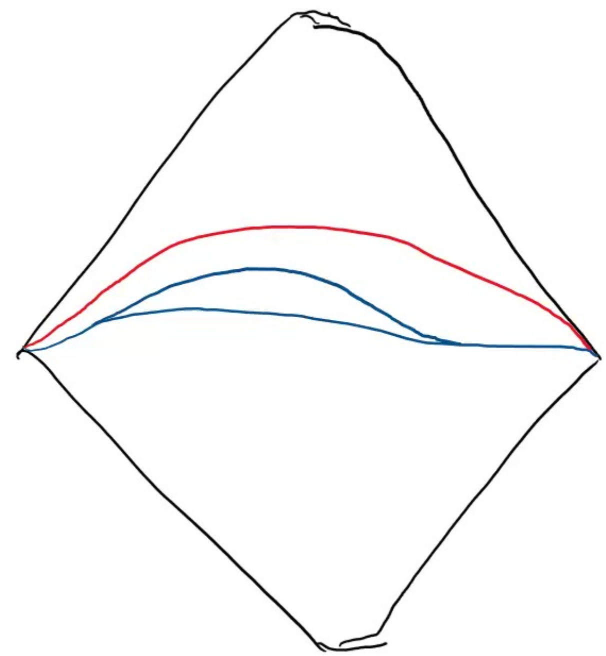
2. Show that the gauge is attainable

When there are corners/boundaries, deformations near the corner may not be gauge

→ Fixing the induced metric on the corner, $h_{ab}|_\partial = \gamma$, enters here.

(More generally, we could extend the phase space by edge modes.)

[Donnelly, Freidel...]



CMC reduction

[RAS Part I: Classical reduction]

3. Show that the Lichnerowicz equation has *one and only one* solution

$$\nabla^2\phi - R_{(h)} + e^{-\phi}\sigma^{ab}\sigma_{ab} - e^{\phi}\chi = 0$$

[Lichnerowicz 44, Moncrief 89]

where $\chi = -2\Lambda + \frac{1}{2}\tau^2$.

This provides a characterization of the *constraint surface* in the pre-phase space

→ $\chi \geq 0$ is used here.

CMC reduction

[RAS Part I: Classical reduction]

4. Quotient by spatial diffeomorphisms, obtaining the *reduced phase space*

$$\tilde{\mathcal{P}} = T^*[ConfGeo(\Sigma)]$$

$$[h_{ab} \sim \Psi_* e^{\lambda} h_{ab}]$$

with $\Psi|_{\partial\Sigma} = I$
and $\lambda|_{\partial\Sigma} = 0$

→ A choice of symplectic form is made here

$$\omega = \int \delta\pi^{ab} \wedge \delta h_{ab}$$

The reduced phase space

[RAS Part I: Classical reduction]

We find that the configuration space is

$$\mathcal{Q} \coloneqq ConfGeo(\Sigma) = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$$

so the reduced phase space is

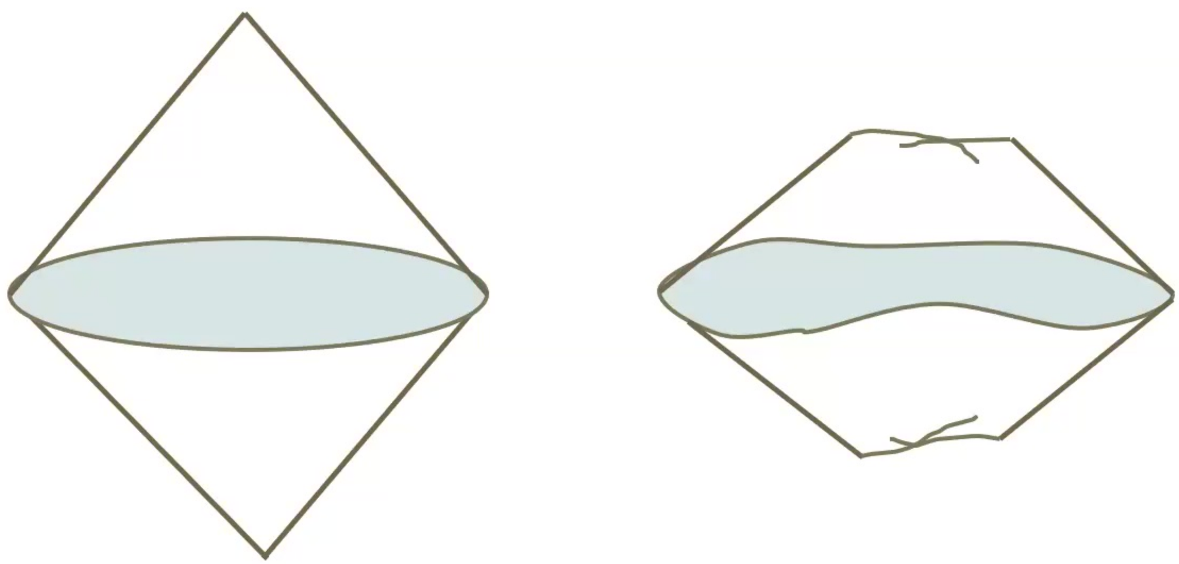
$$\tilde{\mathcal{P}} = T^*\big(\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})\big)$$

where symplectic form is the one associated with the cotangent bundle structure.

Physical interpretation of the states

In 2+1 there are no local degrees of freedom

Classical states correspond to shapes of causal diamonds in AdS_3 (or $Mink_3$)



Group-theoretic quantization

[RAS Part II: Quantization]

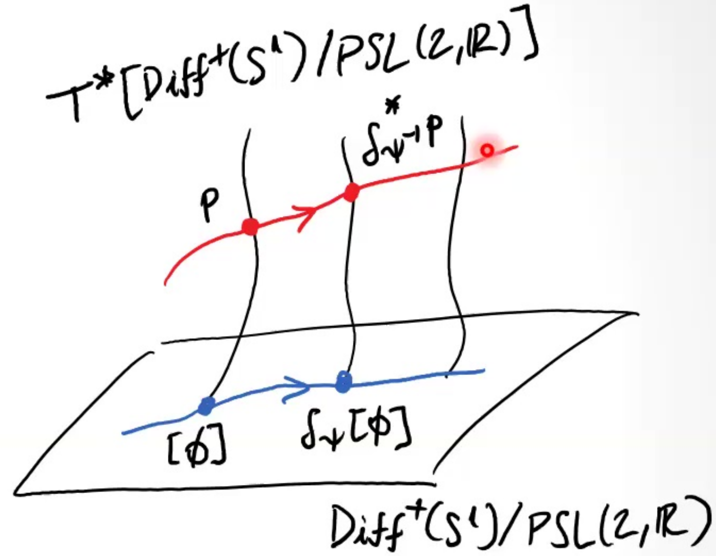
The configuration space $Q = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$ is a homogeneous space for $\mathrm{Diff}^+(S^1)$

$$\delta_\psi[\phi] \coloneqq \psi[\phi] \coloneqq [\psi \circ \phi]$$

so it is natural to start with $K = \mathrm{Diff}^+(S^1)$.

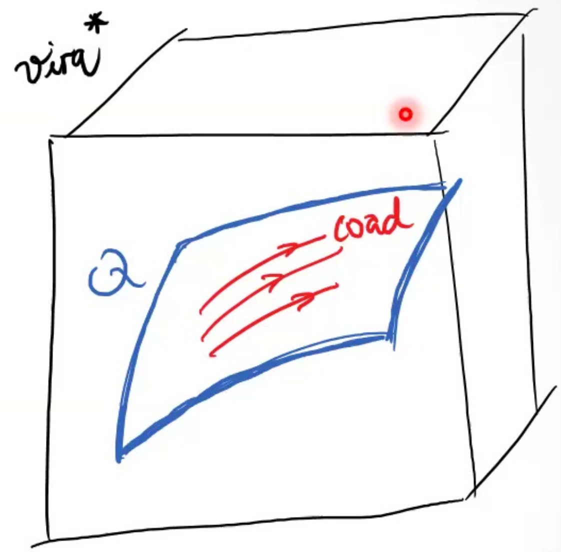
The next step is to construct the vertical translations

But we could *not* find any representation of $\mathrm{Diff}^+(S^1)$ with an orbit isomorphic to Q.



Canonical group for the diamond

The *coadjoint representation* of the *Virasoro group* Vira (extension of $\mathrm{Diff}^+(S^1)$ by $\mathbb{R}$) on its dual Lie algebra $\mathfrak{vira}^* \sim \mathfrak{diff}^* \oplus_S \mathbb{R}$, does have an orbit isomorphic to $\mathcal{Q}$. *[Witten, Segal]*



Thus, taking $K = \mathrm{Vira}$ and $V = \mathfrak{vira}^*$, we get a transitive group of symplectomorphisms of $\tilde{\mathcal{P}} = T^*[\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})]$

$$G = (\mathfrak{vira}^*)^* \rtimes \mathrm{Vira}$$

momentum translations   configuration translations

Canonical algebra for the diamond

Elements of $\mathfrak{vira}$ are characterized by $\hat{\hat{\xi}} = \xi(\theta)\partial_\theta + x\hat{c}$. In a Fourier basis:

Configuration transl.	$L_n = \left(0; e^{in\theta}\partial_\theta\right)$	$R = (0; \hat{c})$	*central*
Momentum transl.	$K_n = \left(e^{in\theta}\partial_\theta; 0\right)$	$T = (\hat{c}; 0)$	*elements*

which gives

$$[L_n, L_m] = i(n-m)L_{n+m} - 4\pi i n^3 \delta_{n+m,0} R$$

$$[K_n, L_m] = i(n-m)K_{n+m} - 4\pi i n^3 \delta_{n+m,0} T$$

$$[K_n, K_m] = 0$$

Canonical charges

We can evaluate the canonical charges generated on the phase space. In this Fourier basis,

$$L_n \mapsto P_n \qquad\qquad K_n \mapsto Q_n$$

with central charges $R \mapsto 0$ and $T \mapsto 1$.

Their Poisson algebra is

$$\{P_n, P_m\} = i(n-m)P_{n+m}$$

$$\{Q_n, P_m\} = i(n-m)Q_{n+m} - 4\pi i n^3 \delta_{n+m,0}$$

$$\{Q_n, Q_m\} = 0$$

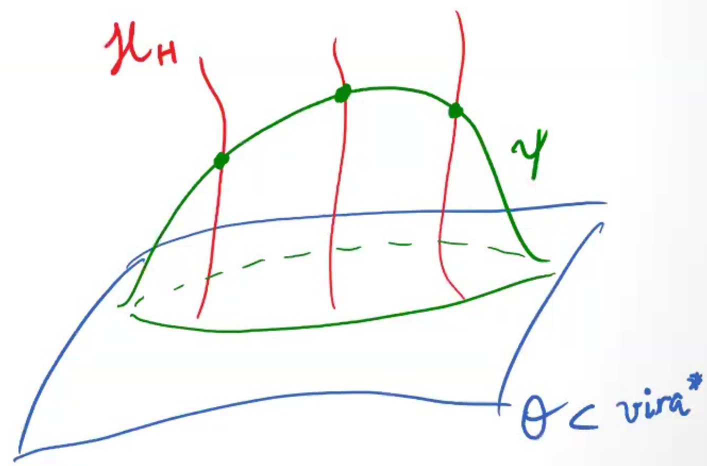
This corresponds to the BMS_3 algebra (symmetries of *2+1* asymptotically flat spacetimes at null infinity). *[Barnich & Compere 07; Oblak 17]*

Quantum diamonds

The quantum theory is based on some unitary irreducible (projective) representation of the canonical group $G = (\mathfrak{vira}^*)^* \rtimes \mathrm{Vira}$

Since G is a semi-direct product of the form $abelian \rtimes group$, we “can” apply Mackey’s theory of induced representations. *[Oblak 16]*

A representation is given by “wavefunctions” Ψ that are sections of a vector bundle, whose base space is some coadjoint orbit ϑ of Vira and fibers $\mathcal{H}_H$ carry a (projective) unitary irrep of the corresponding little group H ($\vartheta \sim \mathrm{Vira/H}$).



Casimir matching

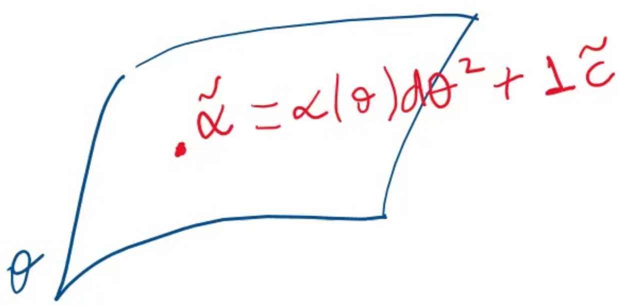
[Oblak 16]

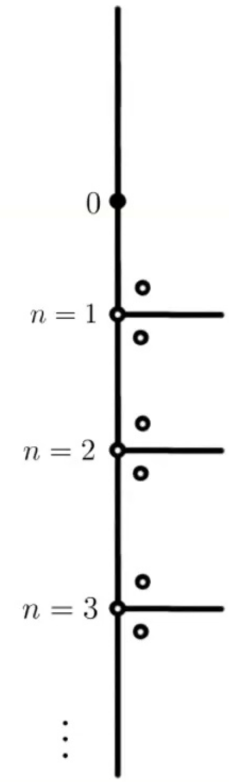
There are many (regular) coadjoint orbits of Virasoro.

Elements of $\mathfrak{vira}^*$ are characterized by $\tilde{\alpha} = \alpha(\theta)d\theta^2 + a\tilde{c}$

The simplest Casimirs are $\hat{T}$ and $\hat{R}$.

Classically $T = 1$ implies orbit $\tilde{\alpha} = \alpha(\theta)d\theta^2 + \tilde{c}$





and $R = 0$ implies that the central $\mathbb{R}$ "inside" H is represented trivially.

"Generalized" Casimirs: monodromy and winding

- $SL(2,\mathbb{R})$-conjugacy classes of monodromy matrix $\mathbf{M}$

Defined from Hill's operator: $\tilde{\alpha} \mapsto 4a\frac{\partial^2}{\partial\theta^2} + \alpha(\theta)$, acting on densities F of weight $-{}^1/_2$

Given a basis of two (normalized) solutions, $F_1(\theta)$ and $F_2(\theta)$, the monodromy matrix is

$$\mathbf{F}(\theta + 2\pi) = \mathbf{M}\,\mathbf{F}(\theta) \qquad \mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

The conjugacy class of $\mathbf{M}$ is *constant* along each coadjoint orbit.

$$\mathfrak{M} := \left[\mathscr{P}\exp\int_0^{2\pi} d\theta \begin{pmatrix} 0 & 1 \\ -\frac{\sum_{n\in\mathbb{Z}}\widehat{Q}_n e^{-in\theta}}{8\pi} & 0 \end{pmatrix}\right]$$

- Winding number w: complete "wrappings" of $F_1(\theta)/F_2(\theta)$ as θ goes around S^1

"Generalized" Casimirs: monodromy

- $SL(2,\mathbb{R})$-conjugacy classes of monodromy matrix $\mathbf{M}$

Classically, $$[\mathbf{M}_{\mathcal{Q}}] = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

n-fold cover of $PSL(2,\mathbb{R})$

Implies orbits $\vartheta = \mathrm{Diff}^+(S^1)/PSL^{(n)}(2,\mathbb{R})$, with $n \in 2\mathbb{N}-1$

- Winding number w

Classically, $$w_{\mathcal{Q}} = 1 \Rightarrow n = 1$$

Thus $\vartheta = \mathrm{Diff}^+(S^1)/PSL(2,\mathbb{R})$, so

$\mathcal{H}$ = {*Wavefunctions on $\mathcal{Q}$ with indices in a projective unitary irrep of* $\mathrm{SL}(2,\mathbb{R})$}

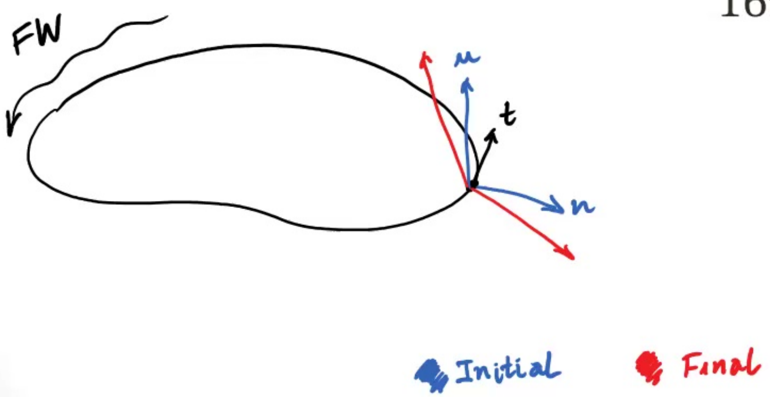
Spin/Twist

The charge P_0 can be interpreted as the ***spin*** of the diamond.

It corresponds to the $SO(2)$ subgroup of $\mathrm{Diff}^+(S^1) \subset \mathrm{Vira}$

We can also show that P_0 is proportional to the ***twist*** $\mathcal{T}$ of the diamond corner

$$P_0 = \frac{\ell}{16\pi^2 \ell_P}\mathcal{T}$$



The twist is related with the holonomy induced on the *normal bundle* of the loop:

"If we Fermi-Walker transport a frame along the loop, it will come back boosted by a (hyperbolic) angle equal to the twist"

Conclusion

- The consequences of a “careful” canonical quantization deserve more attention, especially in the context of gravity.
- $\mathbb{C}$-valued wavefunctions on $\mathcal{Q}$ should not be taken for granted. Additional principles are needed to select preferred representations (e.g., Casimir matching).
- We studied the quantization of causal diamonds in 2+1 gravity, implementing Moncrief’s program at the classical level and Isham’s method to quantize it.
- What is the nature of these quantum causal diamonds?
  Can the *twist quantization* be promoted to a general statement about loops in AdS_3?
  Generalizations (with matter, higher dimensions)…
  Construct the whole “quantum spacetime’’ by gluing causal diamonds?
  Thermodynamics/entanglement of causal diamonds?

Thank You!