Title: Quantum error-correcting codes from Abelian anyon theories

Speakers: Tyler Ellison

Series: Quantum Matter

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Abstract: To perform reliable quantum computations in the midst of noise from the environment, it is imperative to use a quantum error-correcting code -- i.e., a scheme for redundantly encoding information so that errors may be detected and corrected as they occur. One of the most promising classes of quantum error-correcting codes are those based on topological phases of matter, such as the celebrated toric code. Although there is a rich classification of topological phases of matter, the toric code has by far received the most attention as a practical quantum error-correcting code, due to its simple representation within the stabilizer formalism.

In this talk, I will discuss three works in which we extend the stabilizer formalism to topological orders beyond that of the toric code. This includes the construction of two-dimensional stabilizer codes characterized by Abelian topological orders with gapped boundaries, three-dimensional stabilizer codes that host arbitrary two-dimensional Abelian topological orders on their surface, and two-dimensional subsystem codes also characterized by arbitrary Abelian topological orders. This work thus opens the door to encoding and processing quantum information using the exotic properties exhibited by the wide range of Abelian topological phases of matter.

Zoom link https://pitp.zoom.us/j/94640905425?pwd=aDd0Qnl1TUU0QytaNWJJLzEyZlQrQT09

Yale

Quantum error-correcting codes from Abelian anyon theories

Tyler Ellison


Yu-An Chen


Arpit Dua

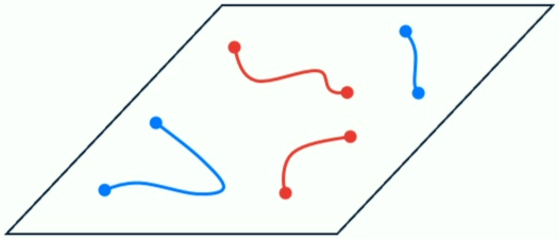

Wilbur Shirley


Nat Tantivasadakarn


Dom Williamson

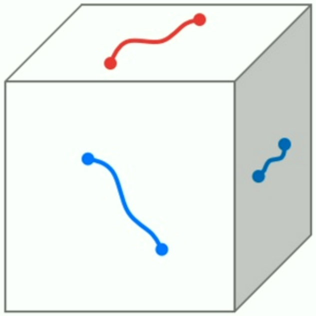
Main takeaways

2D topological Pauli stabilizer codes



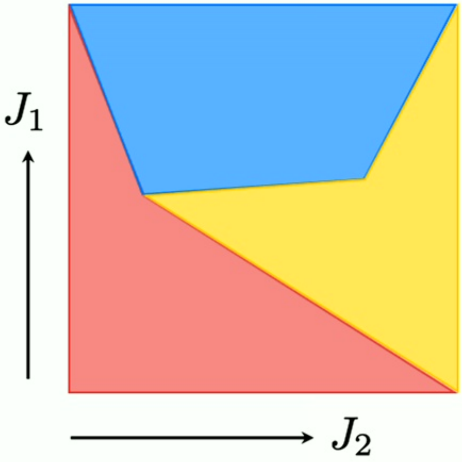
More exotic anyon theories than the toric code

3D topological Pauli stabilizer codes



Hosts anyon theory on the boundary

2D topological Pauli subsystem codes

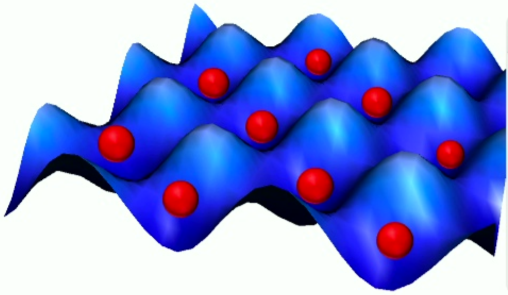


Outline

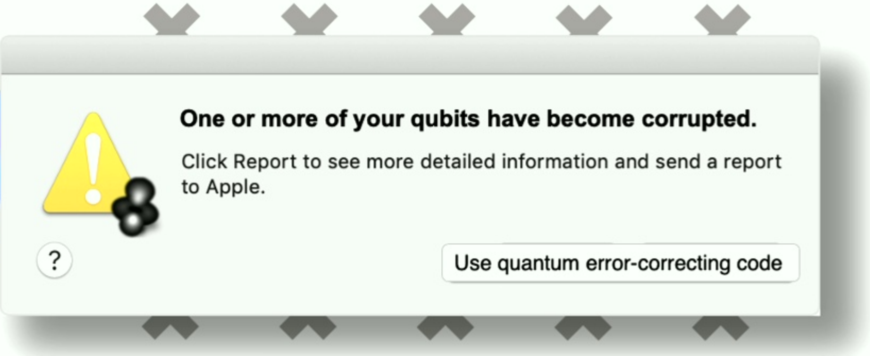
- Motivation and background
  - $\mathbb{Z}_4$ toric code
  - Abelian anyon theories
- 2D topological Pauli stabilizer codes
- 3D topological Pauli stabilizer codes
- 2D topological Pauli subsystem codes
- Future directions

Motivation and background

Motivation



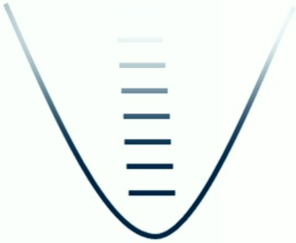
Neutral atom array



One or more of your qubits have become corrupted.

Click Report to see more detailed information and send a report to Apple.

Superconducting qubits



Bosonic qubit

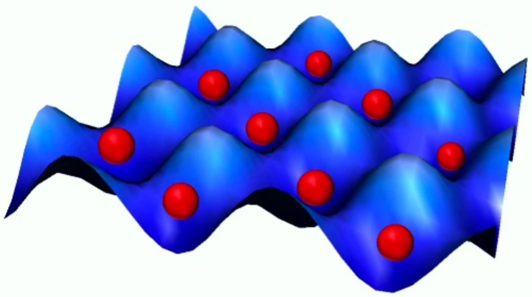
Quantum computers have the potential to:

- Solve quantum chemistry problems → advancement in drug design, carbon fixation
- Simulate complex condensed matter systems

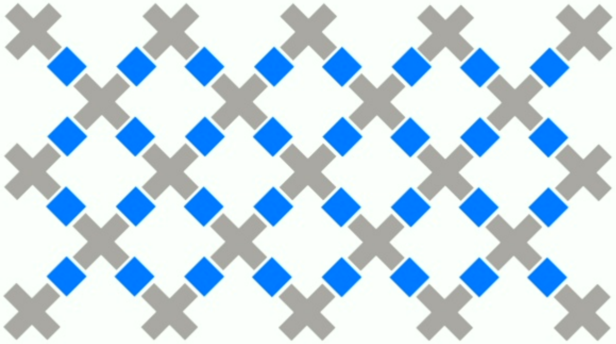
Problem: errors are a serious obstruction to reliable computations

- Small components
- Quantum noise

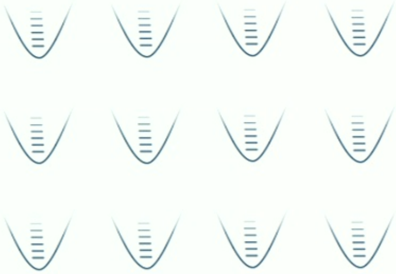
Motivation



Neutral atom array



Superconducting qubits



Bosonic qubits

Solution: quantum error-correcting codes!

- Use many qubits to encode a small number of "logical" qubits
- Ex. Toric code

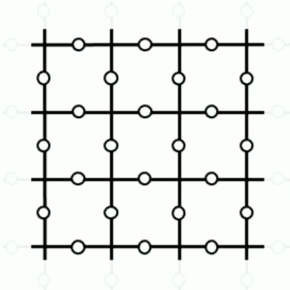
This work: Search for new quantum error correcting codes to

- Improve error thresholds
- Reduce computational overheads
- Better suited for current hardware

Background: $\mathbb{Z}_4$ toric code and Abelian anyon theories

Background: $\mathbb{Z}_4$ toric code

Hilbert space:

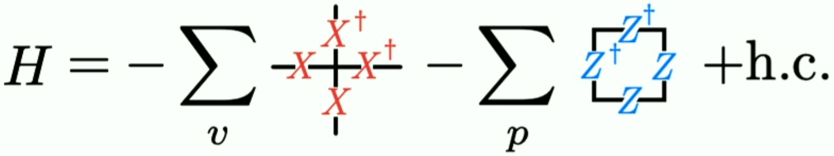


$\circ$ = 4-dimensional qudit

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$X^4 = 1\,, \quad Z^4 = 1\,, \quad ZX = i\,XZ$$

Hamiltonian:



(Deconfined $\mathbb{Z}_4$ gauge theory with matter)

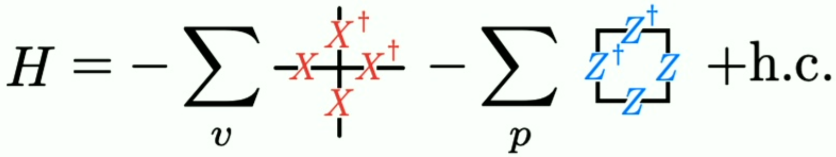
- Topological Pauli stabilizer code
- 16-dimensional ground state subspace on a torus
- Locally indistinguishable ground states

A.Y. Kitaev 1997; J.B. Kogut 1979

Background: $\mathbb{Z}_4$ toric code

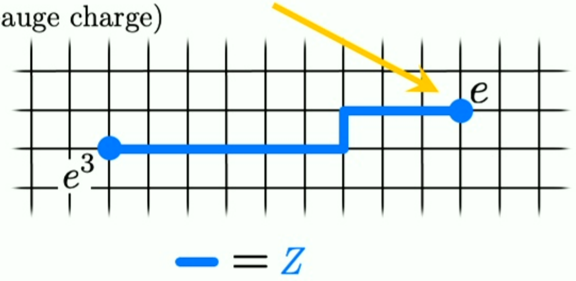
<u>Anyons:</u> local excitations that cannot be created by local operators

(Stabilizer codes)



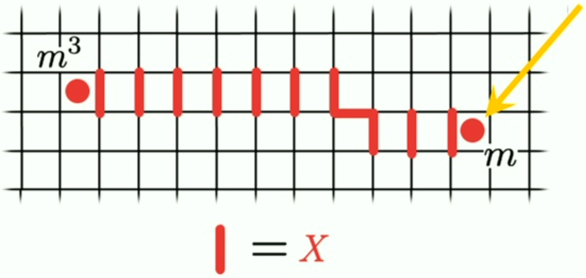
Violates vertex stabilizer

(Gauge charge)



Violates plaquette stabilizer

(Gauge flux)



A.Y. Kitaev 1997

Background: $\mathbb{Z}_4$ toric code

Anyon types:

	1	e	e^2	e^3
1	1	e	e^2	e^3
m	m	em	e^2m	e^3m
m^2	m^2	em^2	e^2m^2	e^3m^2
m^3	m^3	em^3	e^2m^3	e^3m^3

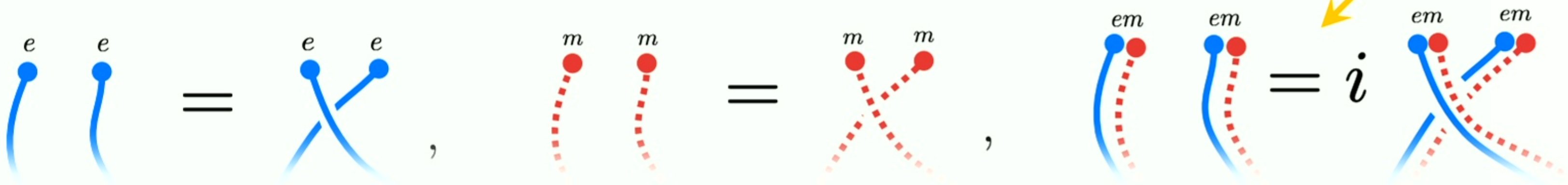
Local excitations created by local operators

Fusion rules: $\mathbb{Z}_4 \times \mathbb{Z}_4$ group generated by e and m

Examples: $e \times m = em, \quad em \times em = e^2m^2, \quad e^2m^2 \times e^2m^2 = 1$

Exchange statistics: Exchange statistics of $e^p m^q$ is i^{pq}

Semion!



A.Y. Kitaev 1997

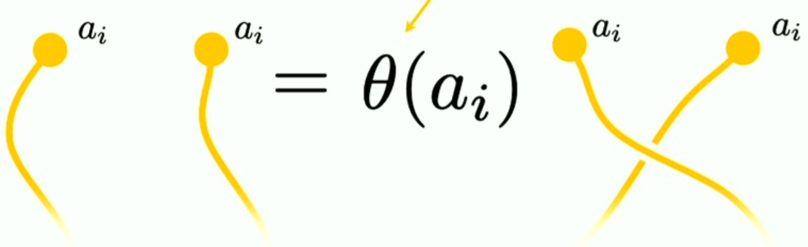
Background: Abelian anyon theories

Anyon types: $\{1, a_1, a_2, \ldots, a_n\}$

Fusion rules: $a_i \times a_j = a_k, \quad a_i \times 1 = a_i$

Exchange statistics:

(Satisfies certain consistency conditions)

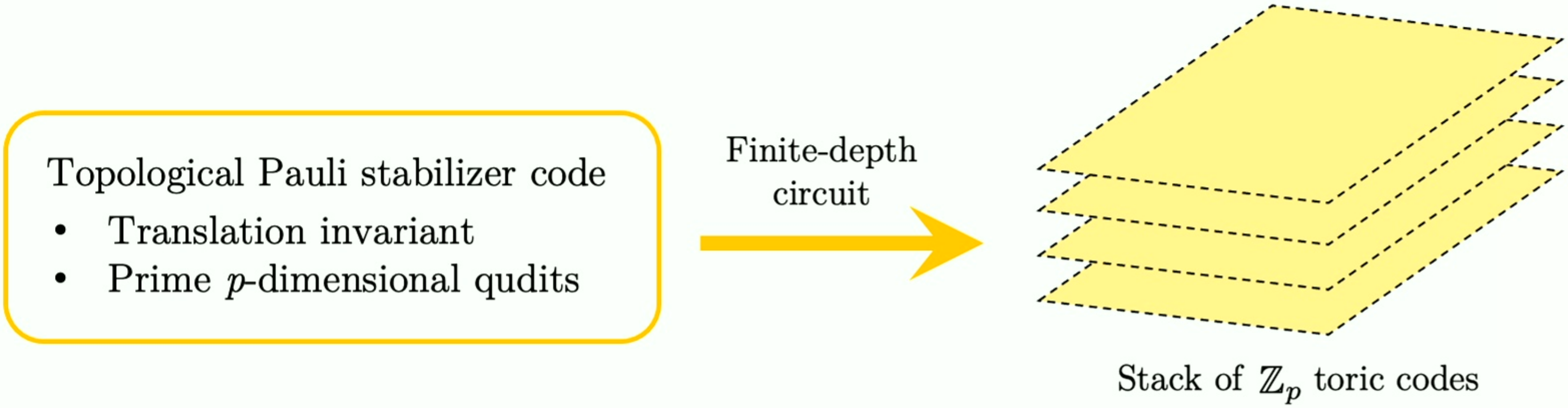


$$= \theta(a_i)$$

(Braiding relations are determined by the exchange statistics)

Classification of 2D topological Pauli stabilizer codes

- What Abelian anyon theories are captured by 2D topological Pauli stabilizer codes?



Topological Pauli stabilizer code

- Translation invariant
- Prime p-dimensional qudits

Finite-depth circuit

Stack of $\mathbb{Z}_p$ toric codes

Are more exotic anyon theories captured by topological Pauli stabilizer codes on composite-dimensional qudits?

H. Bombin, G. Duclos-Cianci, D. Poulin 2011; H. Bombin 2011; J. Haah 2018

2D topological Pauli stabilizer codes

PRX Quantum **3**, 010353

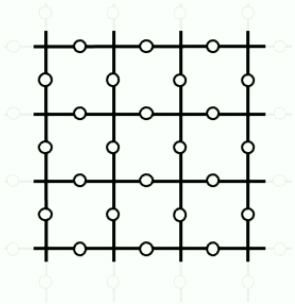
Double semion stabilizer code

Hilbert space:

$Z^4 = 1\,,$

$X^4 = 1\,,$

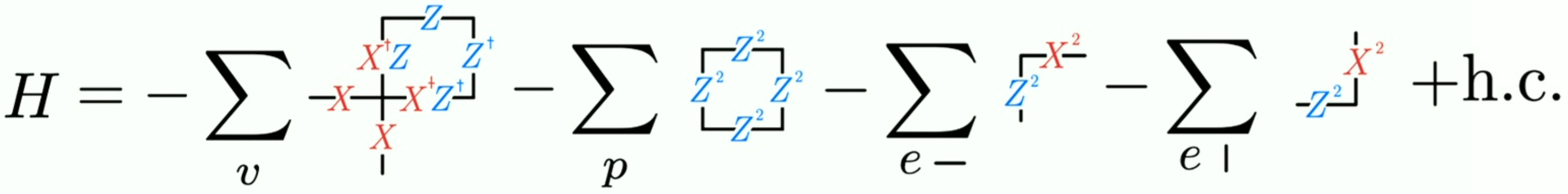
$ZX = i\,XZ$



$\circ$ = 4-dimensional qudit

- Topological Pauli stabilizer code
- 4-dimensional ground state subspace on a torus
- Locally indistinguishable ground states

Hamiltonian:



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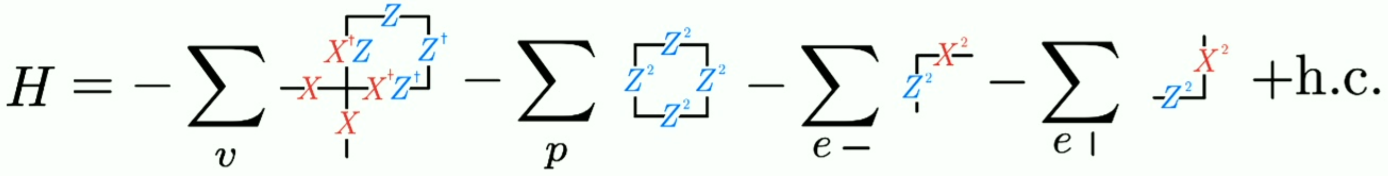
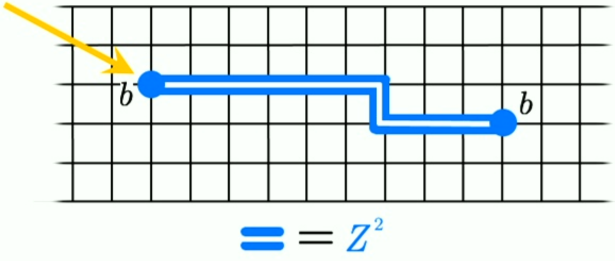
Double semion stabilizer code excitations

Anyons: local excitations that cannot be created by local operators

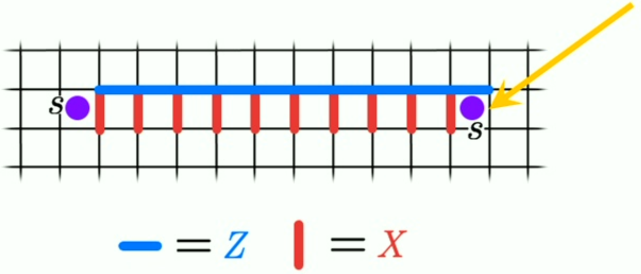
(Stabilizer codes)



Violates vertex term



Violates vertex terms and plaquette terms



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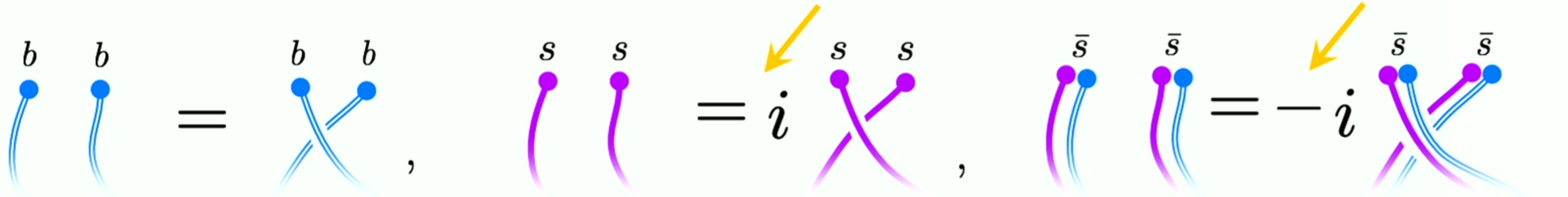
Double semion stabilizer code anyon theory

Anyon types: $\{1, b, s, \bar{s}\}$

Fusion rules: $\mathbb{Z}_2 \times \mathbb{Z}_2$ group generated by b and s

$$b \times b = 1 \ , \qquad s \times s = 1 \ , \qquad \bar{s} \times \bar{s} = 1 \ , \qquad b \times s = \bar{s} \ , \ \ldots$$

Exchange statistics:



Semion!

Anti-semion!

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Double semion stabilizer code anyon theory

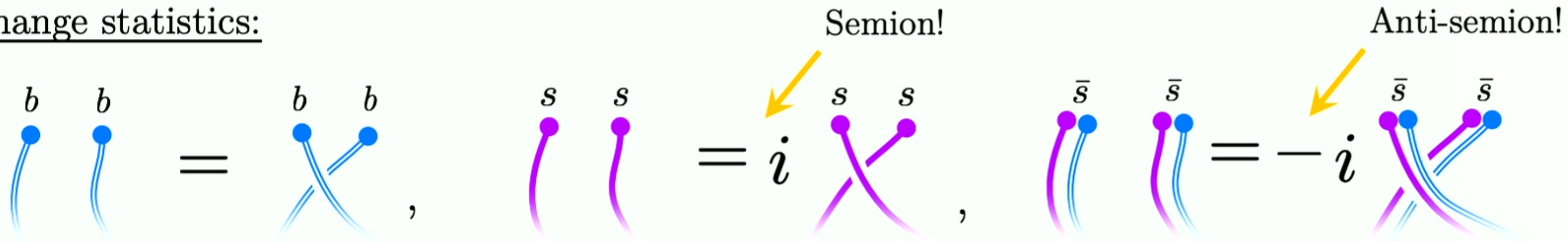
Anyon types: $\{1, b, s, \bar{s}\}$

Distinct from toric code or a stack of toric codes!

Fusion rules: $\mathbb{Z}_2 \times \mathbb{Z}_2$ group generated by b and s

$$b \times b = 1 \ , \quad s \times s = 1 \ , \quad \bar{s} \times \bar{s} = 1 \ , \quad b \times s = \bar{s} \ , \ \ldots$$

Exchange statistics:



Semion!

Anti-semion!

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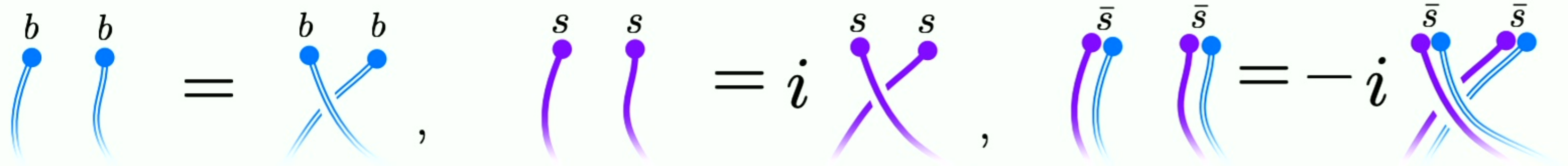
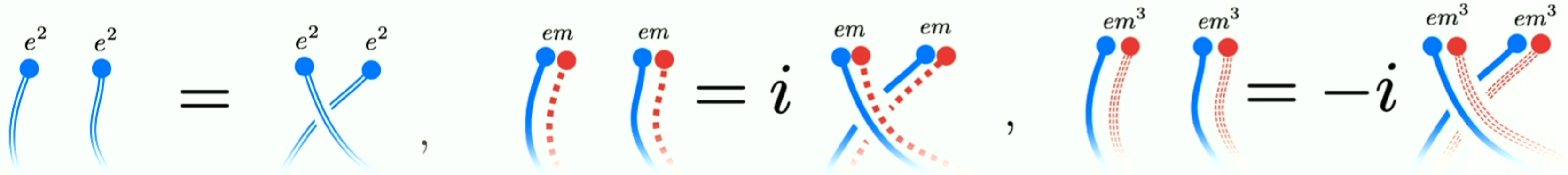
Construction of the double semion stabilizer code

Double semion exchange statistics:



$\mathbb{Z}_4$ toric code exchange statistics:



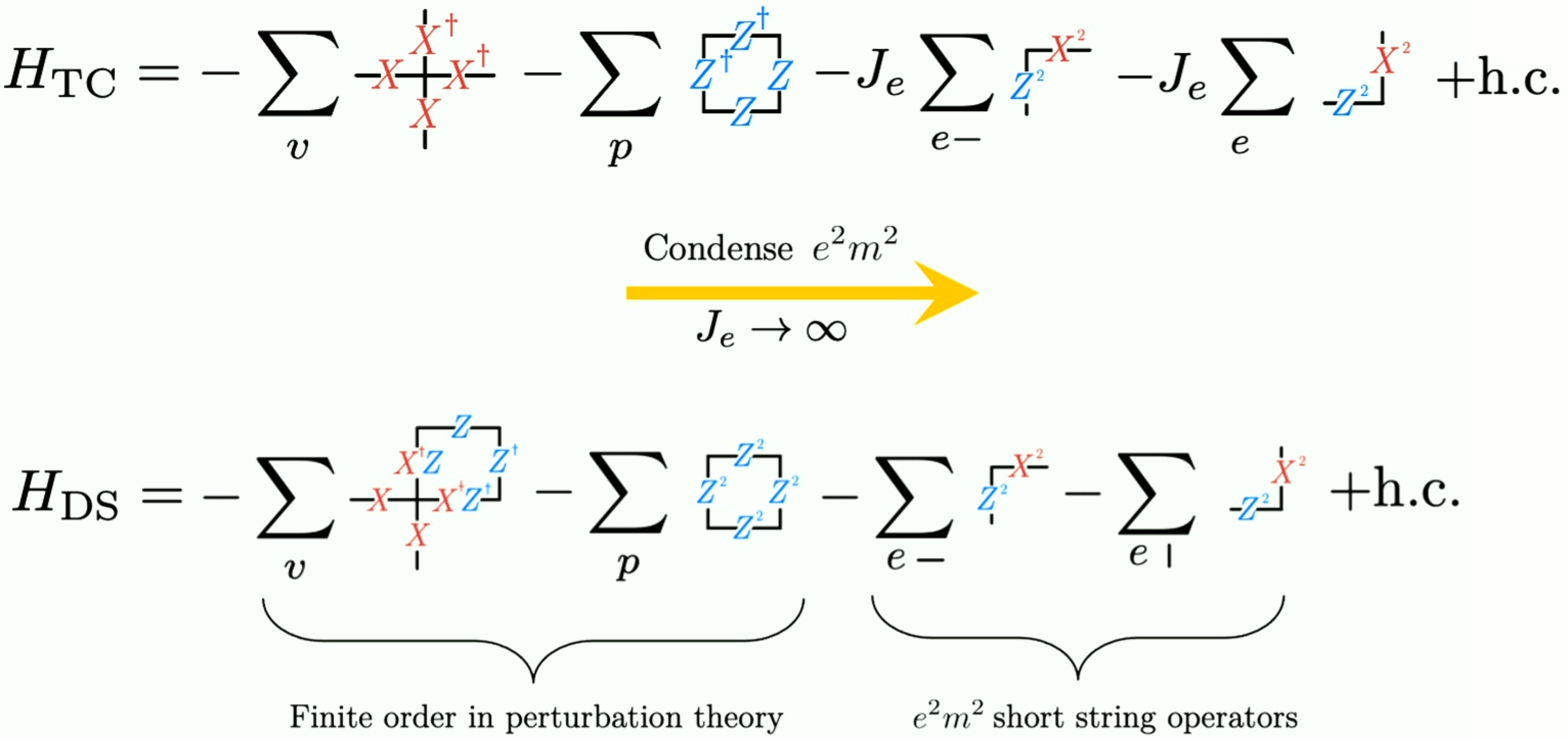
Fusion rules do not match

$s \times s = 1$ $\qquad$ $em \times em = e^2m^2$

$\bar{s} \times \bar{s} = 1$ $\qquad$ $em^3 \times em^3 = e^2m^2$

$\Longrightarrow$ Condense e^2m^2

Construction of double semion stabilizer code



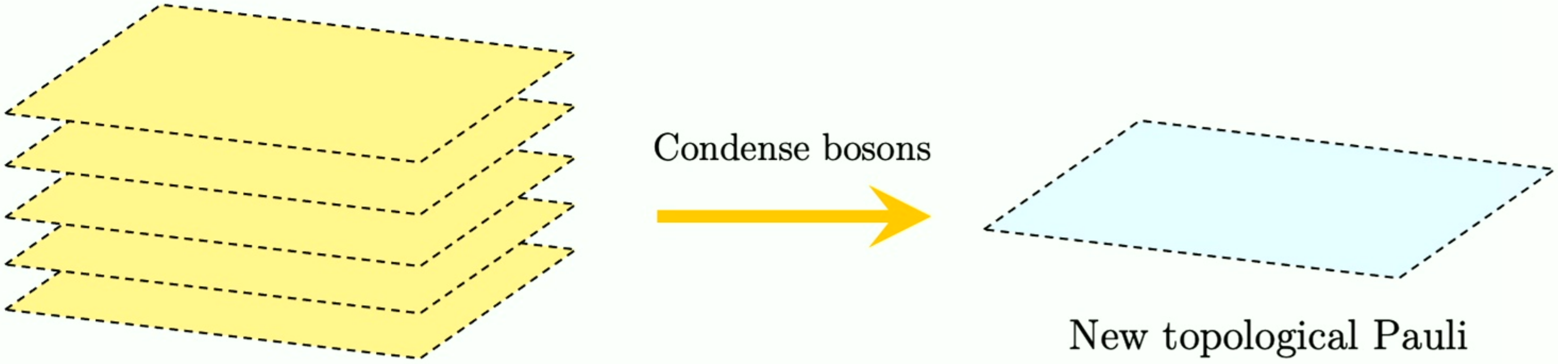
Condense e^2m^2

$J_e \to \infty$

Finite order in perturbation theory

e^2m^2 short string operators

General construction



Stack of toric codes

New topological Pauli stabilizer code

Exhausts all Abelian anyon theories that admit a gapped boundary!

Modular: For every a_i there exists a_j, such that $\theta(a_i a_j) \neq \theta(a_i)\theta(a_j)$

Lagrangian subgroup: Subgroup of bosons $\mathcal{L}$, such that, for every $a \notin \mathcal{L}$, there exists $b \in \mathcal{L}$ with $\theta(ab) \neq \theta(a)\theta(b)$

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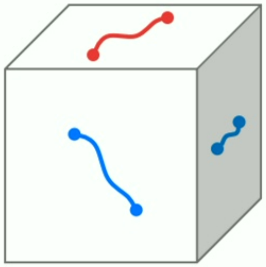
Beyond 2D stabilizer codes

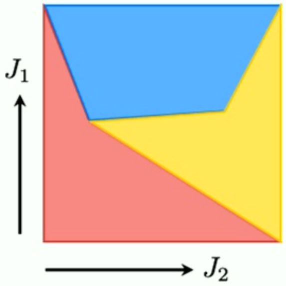
Can Abelian anyon theories without gapped boundaries be captured by codes beyond 2D stabilizer codes?

3D topological Pauli stabilizer codes



Anyon theories without gapped boundaries!

2D topological Pauli subsystem codes



Non-modular anyon theories!

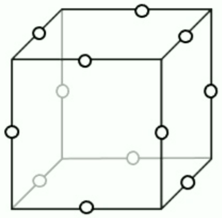
3D topological Pauli stabilizer codes

PRX Quantum **3**, 030326

Chiral semion boundary code

Hilbert space:

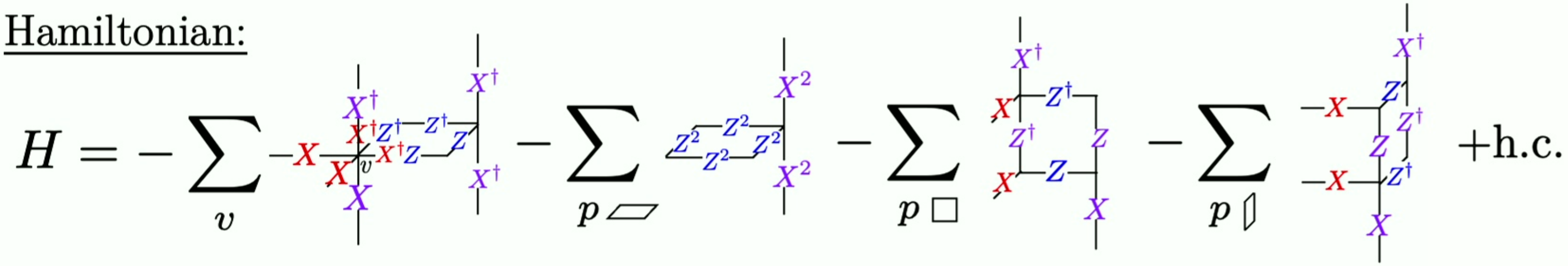
$$Z^4 = 1\,,$$
$$X^4 = 1\,,$$
$$ZX = i\,XZ$$



$\circ$ = 4-dimensional qudit

- Topological Pauli stabilizer code
- 2-dimensional ground state subspace on a solid torus
- Locally indistinguishable ground states

Hamiltonian:



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Chiral semion boundary code

Hilbert space:

$$Z^4 = 1\,,$$
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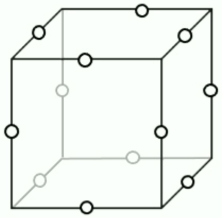

○ = 4-dimensional qudit

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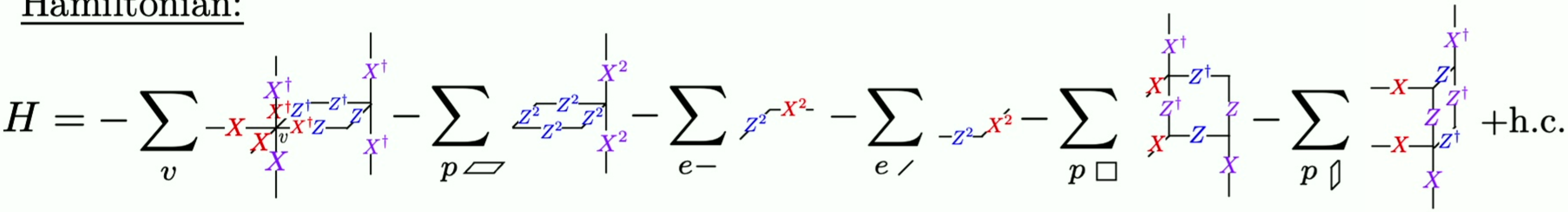

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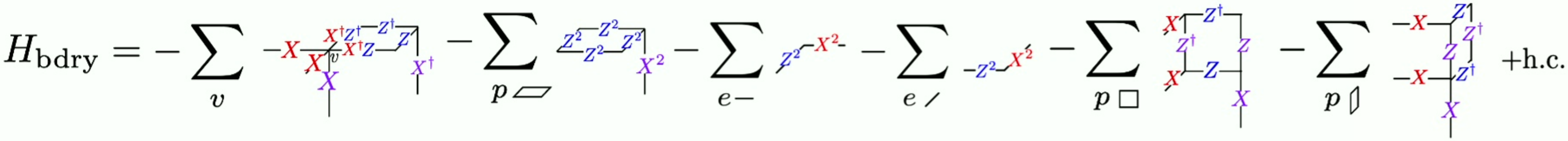
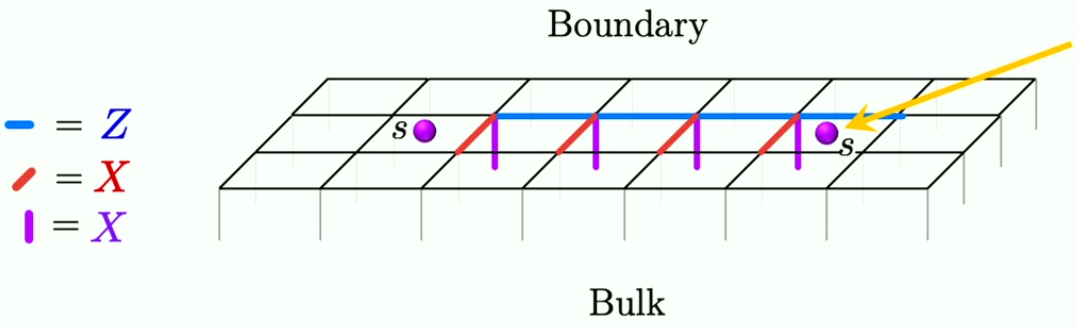
Chiral semion boundary code

<u>Anyons:</u> local excitations that cannot be created by local operators

(Stabilizer codes)





Violates vertex terms and plaquette terms

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Chiral semion anyon theory

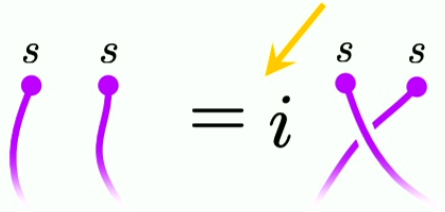
Anyon types: $\{1, s\}$

$s =$ 

Fusion rules: $\mathbb{Z}_2$ group generated by s

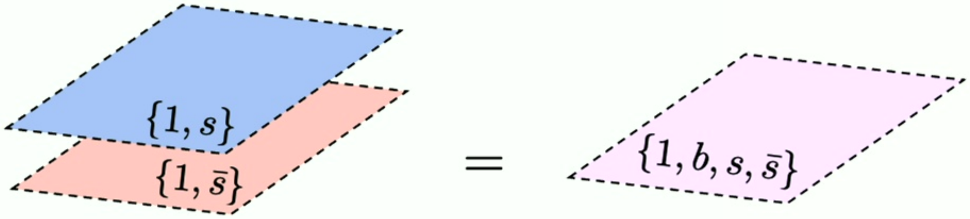
$$s \times s = 1$$

Exchange statistics:



Semion!

Chiral semion theory does not admit gapped boundaries!



Chiral semion × chiral antisemion

Double semion

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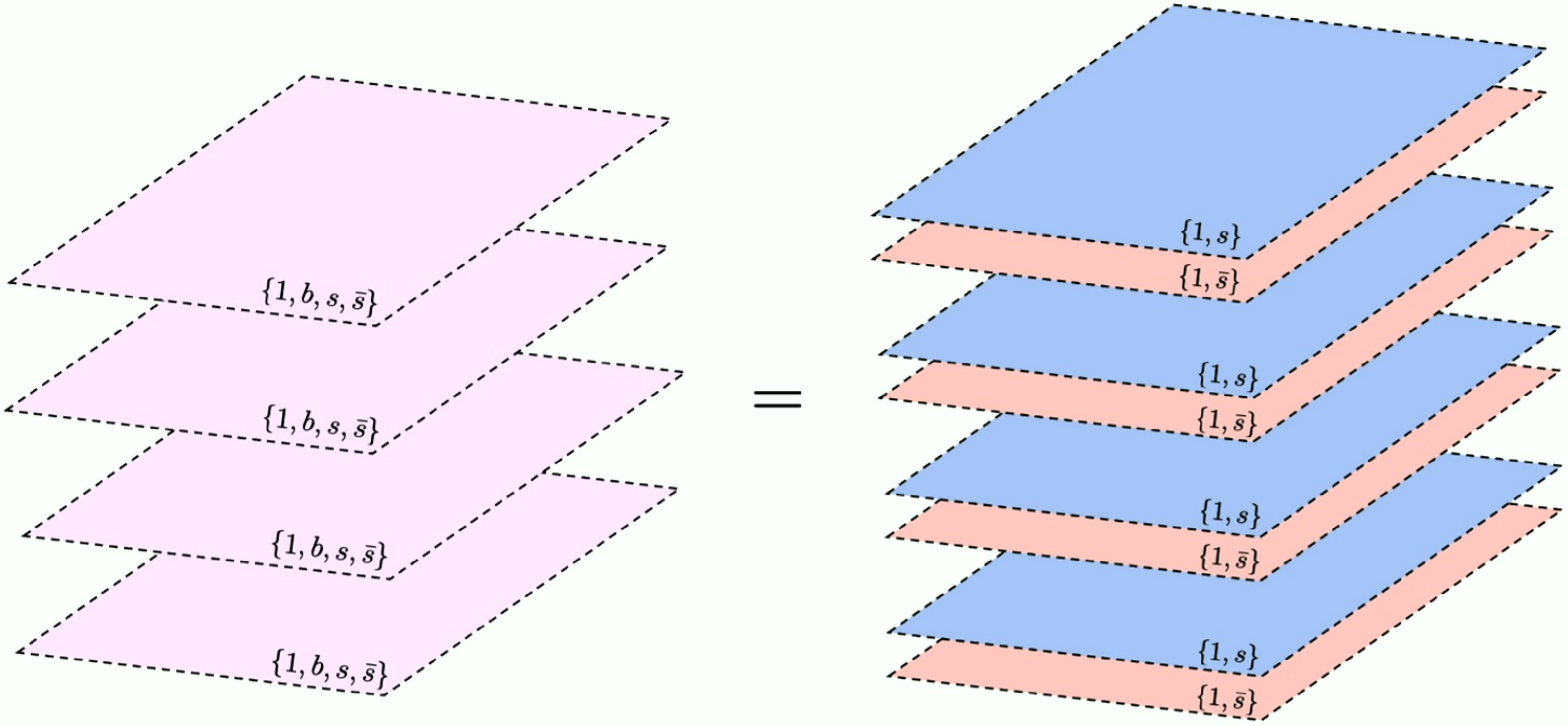
Construction of the chiral semion boundary code



Stack of double semion stabilizer codes

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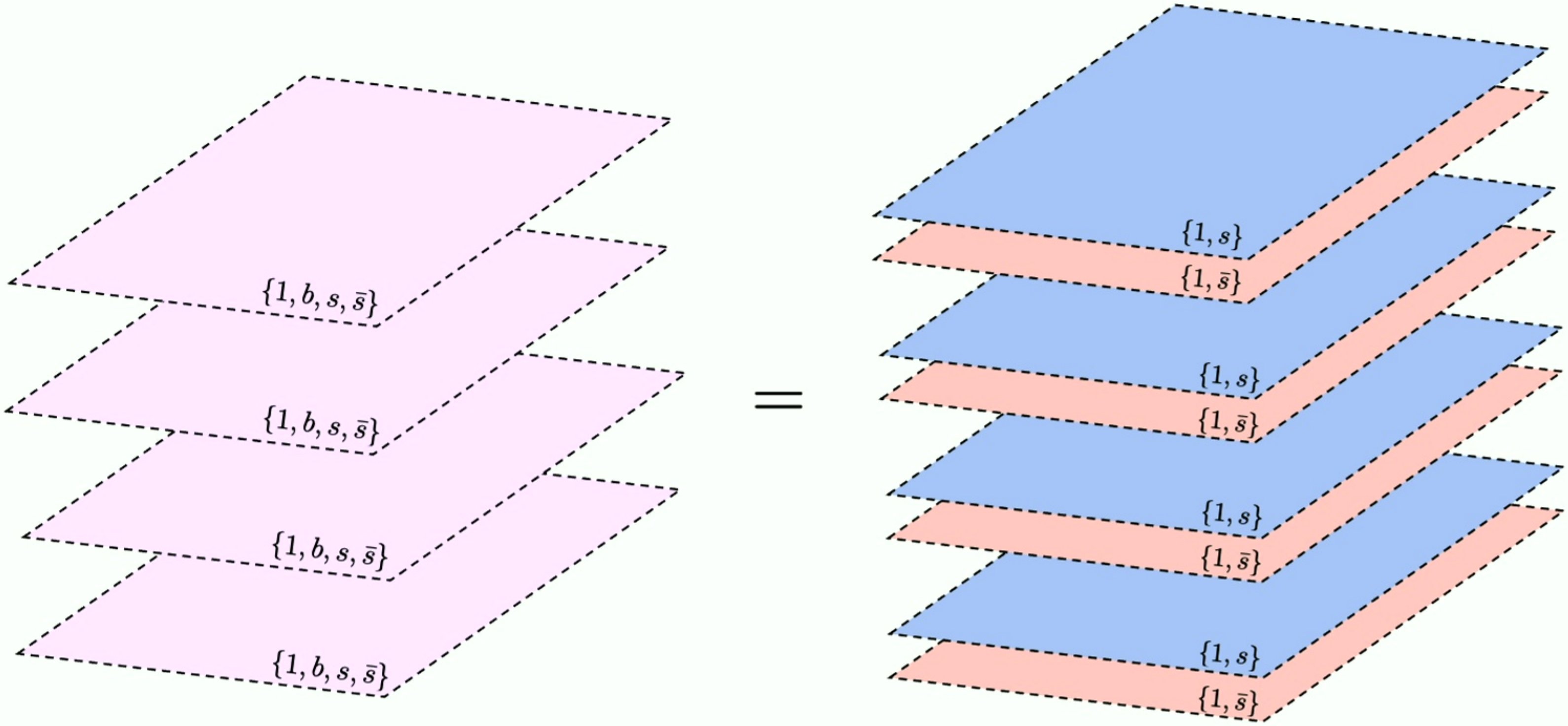
Construction of the chiral semion boundary code



$\bar{s}_i s_{i+1}$
is a boson!

Stack of double semion stabilizer codes

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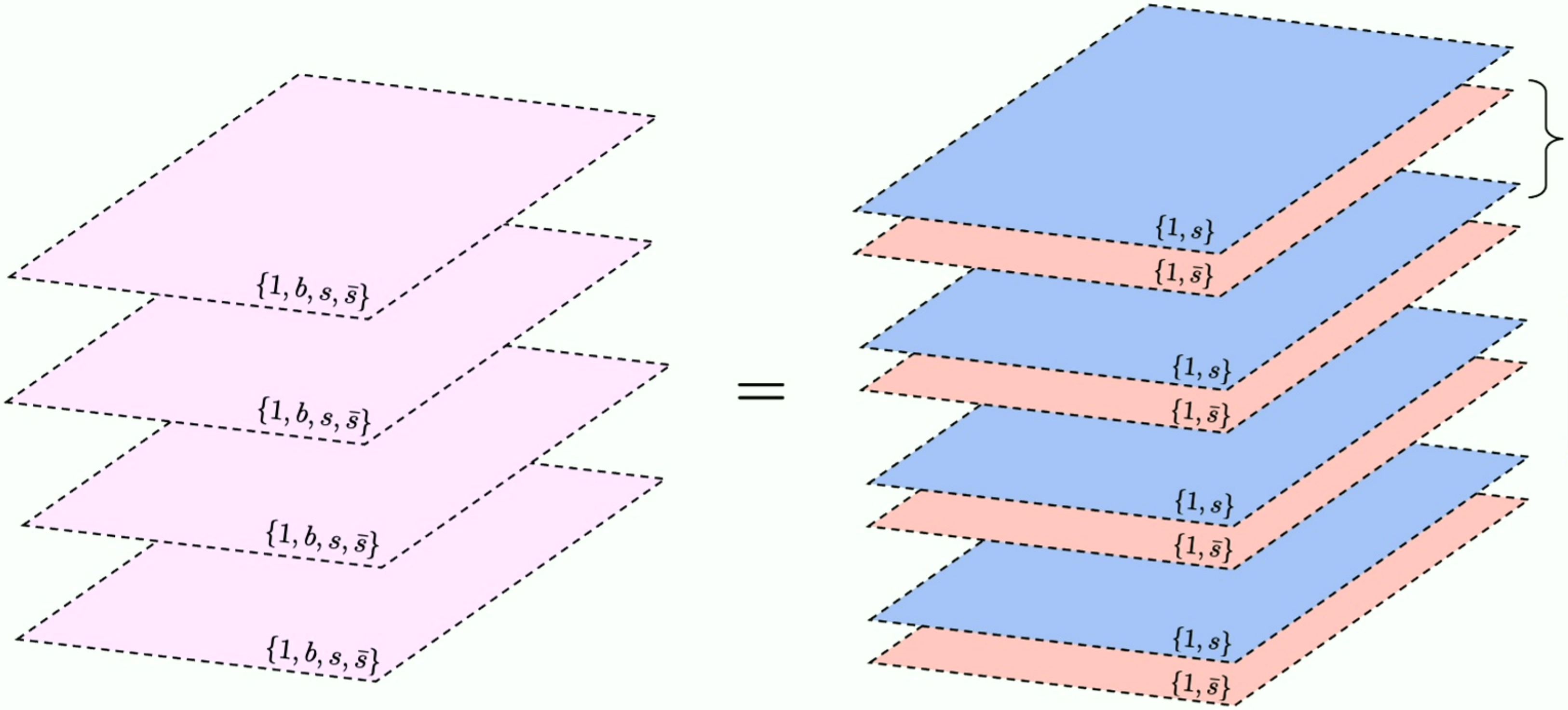
Construction of the chiral semion boundary code



Interlayer
condensation
of $\bar{s}_i s_{i+1}$

$\bar{s}_i s_{i+1}$
is a boson!

Stack of double semion stabilizer codes

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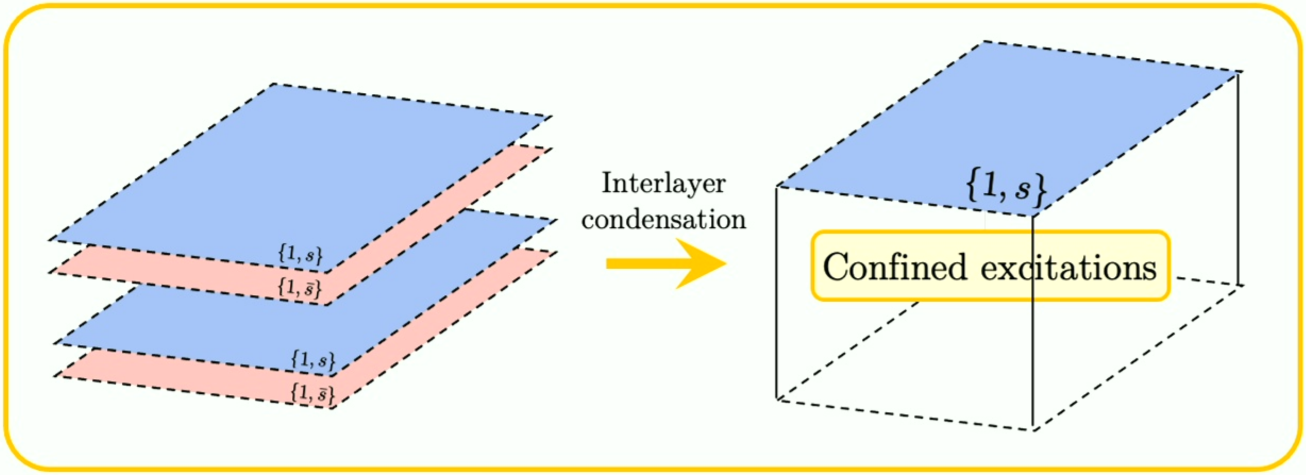
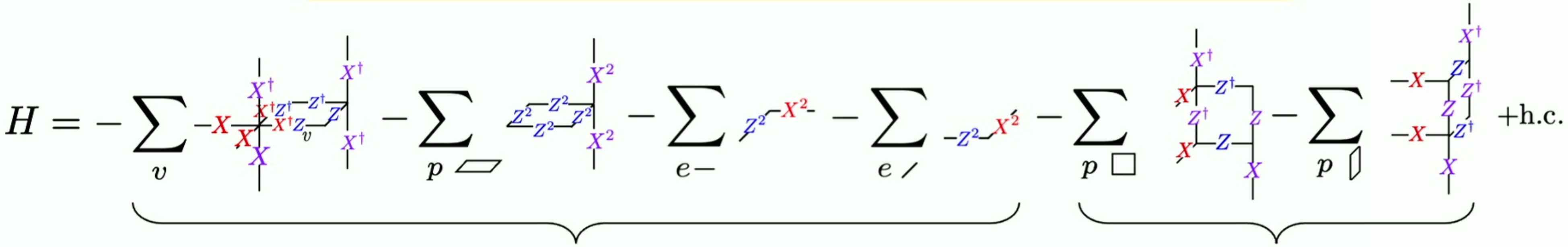
Construction of the chiral semion boundary code



Interlayer condensation

Confined excitations



Double semion stabilizer code layers

$\bar{s}_i s_{i+1}$ interlayer short string operators

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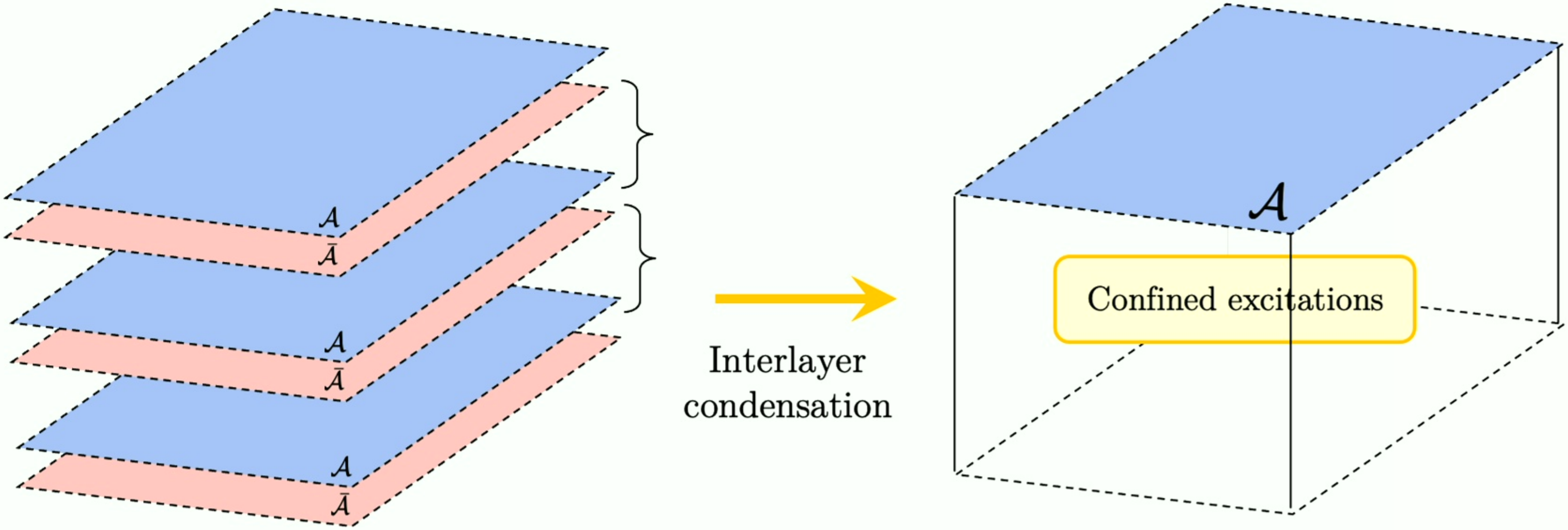
General construction of boundary codes



Stacks of 2D topological Pauli stabilizer codes

New 3D Pauli stabilizer code

Exhausts all <u>modular</u> Abelian anyon theories!

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2D topological Pauli subsystem codes

Quantum 7, 1137 (2023)

Subsystem codes

Gauge group:

May be non-Abelian!

$$\mathcal{G} \subset \text{Pauli group}$$

$$\Longrightarrow$$

Stabilizer group:

Abelian

$$\mathcal{S} = \text{center of } \mathcal{G}$$

(Up to roots of unity)

Stabilizer group defines the code space $\mathcal{H}_C$

$$\mathcal{H}_C \oplus \mathcal{H}_C^\perp$$

Gauge group induces factorization of code space

$$(\mathcal{H}_G \otimes \mathcal{H}_L) \oplus \mathcal{H}_C^\perp$$

Gauge subsystem

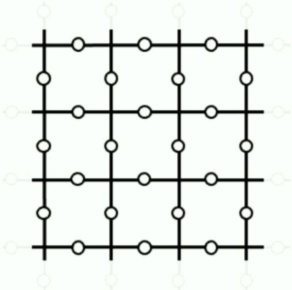
Logical subsystem

D.W. Kribs, R. Laflamme, D. Poulin, M. Lesosky 2005

$\mathbb{Z}_4^{(1)}$ subsystem code

Hilbert space:

$$X^4 = 1\,,$$
$$Z^4 = 1\,,$$
$$ZX = i\,XZ$$

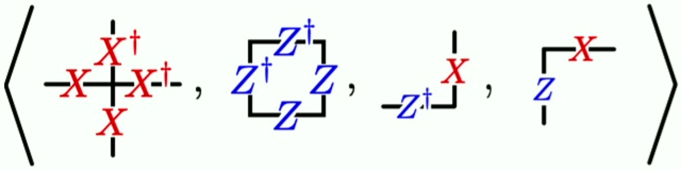


$\circ$ = 4-dimensional qudit

- Topological Pauli subsystem code
- 2-dimensional logical subsystem on a torus
- Robust against local errors

Gauge group:

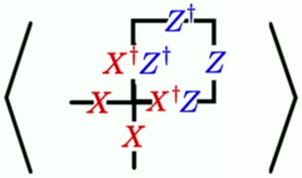
$\mathcal{G}_{\mathbb{Z}_4^{(1)}} =$



$\Longrightarrow$

Stabilizer group:

$\mathcal{S}_{\mathbb{Z}_4^{(1)}} =$



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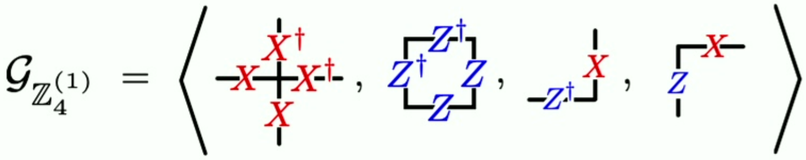
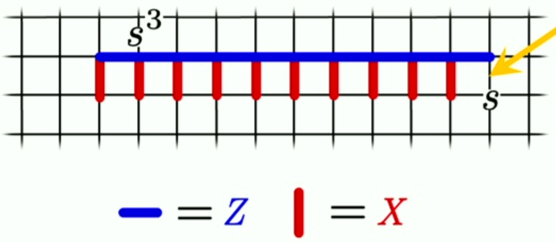
$\mathbb{Z}_4^{(1)}$ subsystem code

Anyons: Violations of gauge operators that cannot be created by local operators

(Subsystem codes)



Violates vertex gauge operators and plaquette gauge operators



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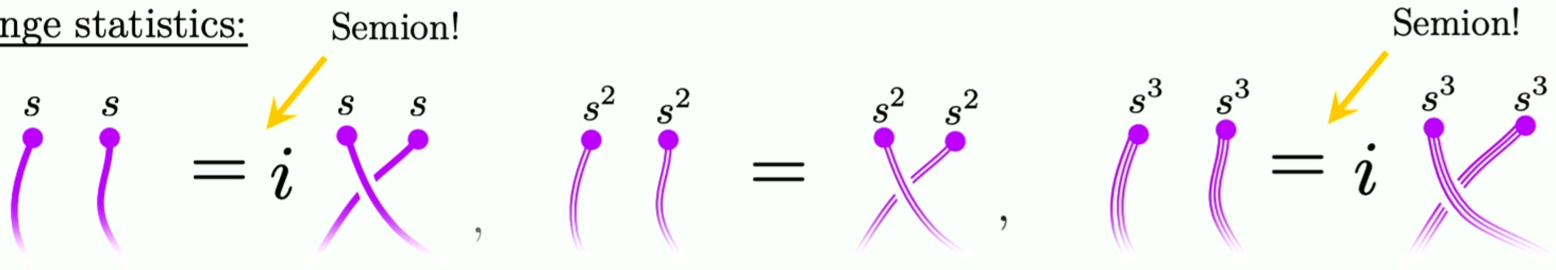
$\mathbb{Z}_4^{(1)}$ subsystem code anyons

Anyon types: $\{1, s, s^2, s^3\}$

Fusion rules: $\mathbb{Z}_4$ group generated by s

Non-modular anyon theory!

Exchange statistics:



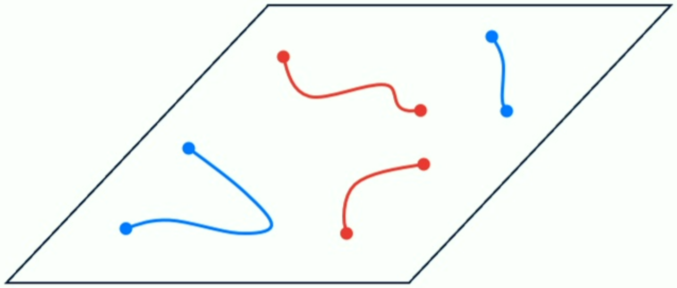
Subsystem codes

Stabilizer codes:

- Defines unique exactly-solvable Hamiltonian

$$H = -\sum_{S\in\mathcal{S}} S + \text{h.c.}$$

- Stabilizers are common conserved quantities

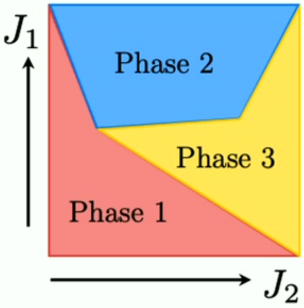


- Anyons are excitations of H

Subsystem codes:

- Defines Parameter space of Hamiltonians

$$H = -\sum_{G\in\mathcal{G}} J_G G + \text{h.c.}$$



Exhibits various phases and phase transitions

- Stabilizers are common conserved quantities
- (Informal) Anyons are excitations common to the gapped Hamiltonians

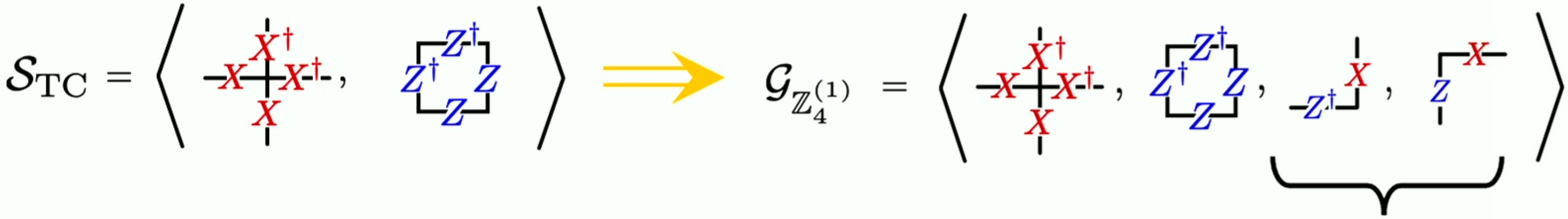
Construction of $\mathbb{Z}_4^{(1)}$ subsystem code

Gauging out:

Append string operators of anyon to stabilizer group (or gauge group) to form new gauge group

Construction:

Gauge out e^3m in $\mathbb{Z}_4$ toric code:



e^3m short string operators

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Construction of $\mathbb{Z}_4^{(1)}$ subsystem code

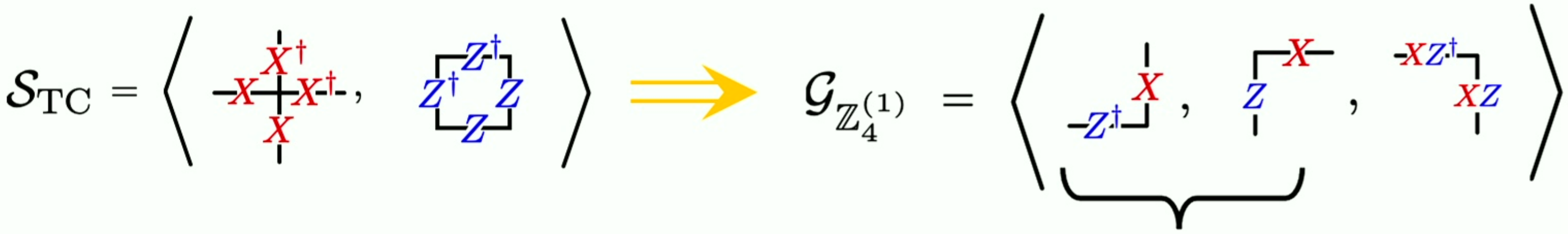
Gauging out:

Append string operators of anyon to stabilizer group (or gauge group) to form new gauge group

Construction:

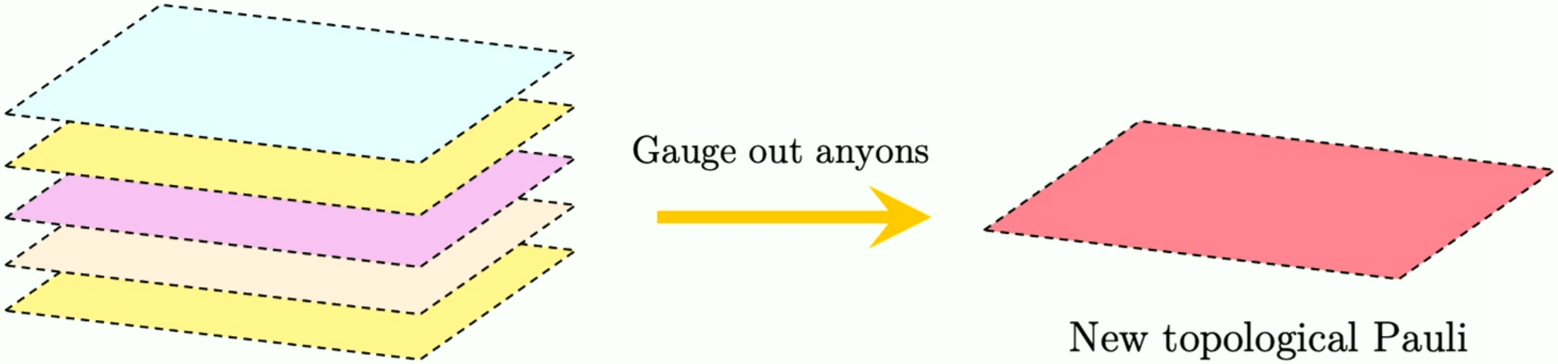
Gauge out e^3m in $\mathbb{Z}_4$ toric code:

Can simplify generators!



e^3m short string operators

General construction



Gauge out anyons

Stack of topological Pauli stabilizer codes

New topological Pauli subsystem code

Exhausts all Abelian anyon theories!

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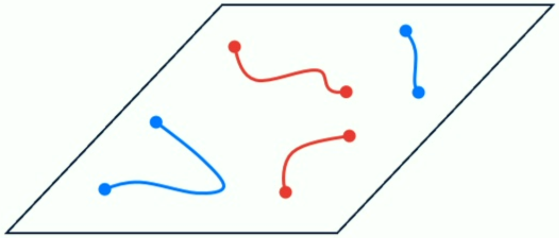
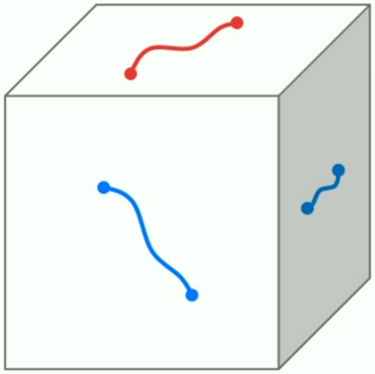
Main takeaways

2D topological Pauli stabilizer codes



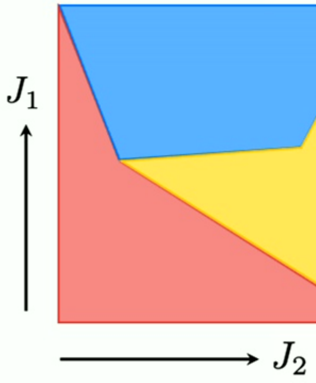
Anyon theories with gapped boundaries!

3D topological Pauli stabilizer codes



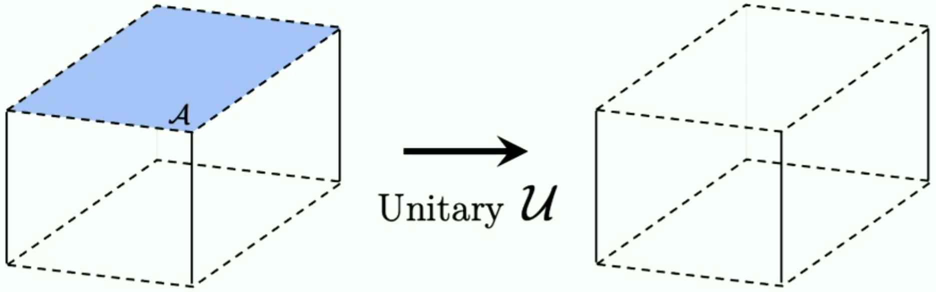
Modular anyon theories!

2D topological Pauli subsystem codes



Arbitrary anyon theories!

Further comments on 3D stabilizer codes



3D Pauli stabilizer code

Trivial paramagnet

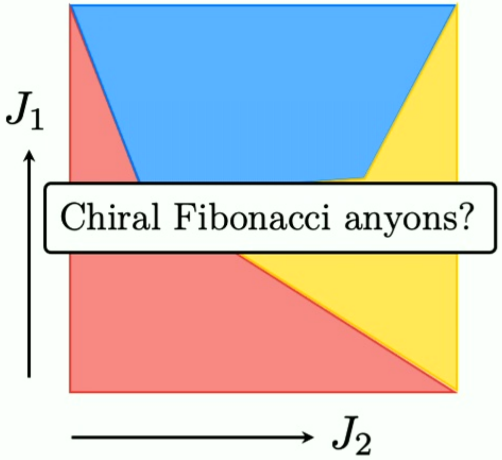
$$H = -\sum_e Z_e + \text{h.c.}$$

- $\mathcal{U}$ is locality preserving – i.e., maps local operators to local operators via conjugation
- If $\mathcal{A}$ does not admit a gapped boundary, then $\mathcal{U}$ is not a finite-depth circuit
- First non-Clifford example and conjectured classification of locality-preserving unitaries

W. Shirley, Y.-A. Chen, A. Dua, TDE, N. Tantivasadakarn, D.J. Williamson 2021

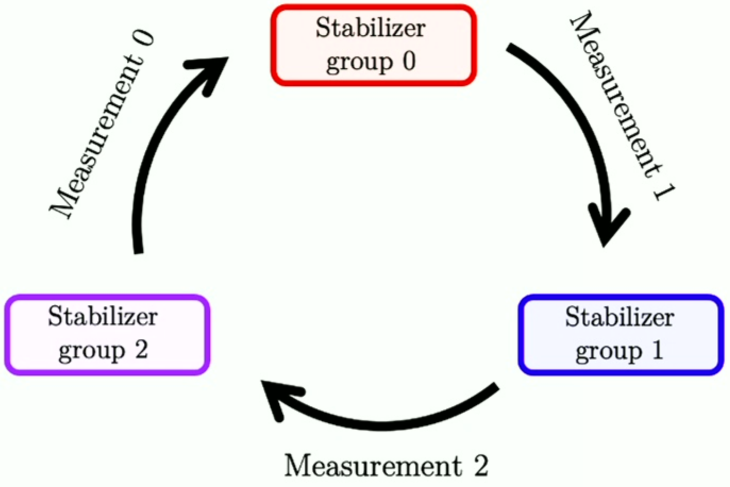
Further comments on subsystem codes

Explored phase diagram of $\mathbb{Z}_3^{(1)}$ subsystem code



- Evidence of Fibonacci anyons, but recent calculation suggests local minimum of numerics

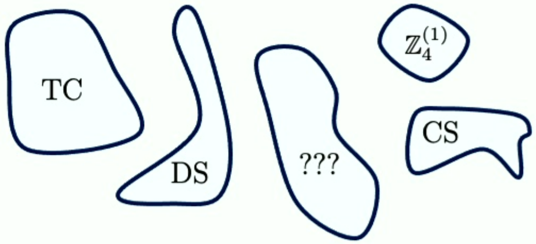
Dynamical codes from scheduled measurements of gauge operators



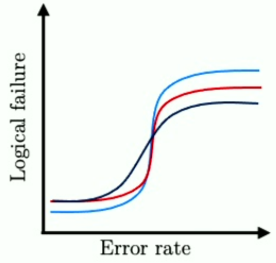
L.ei-Mei Chen, TDE, Meng Cheng, Peng Ye, Ji-Yao Chen 2023; A. Dua, J. Sullivan, N. Tantivasadakarn, TDE 2023

Future directions

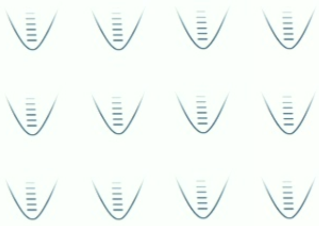
Future directions



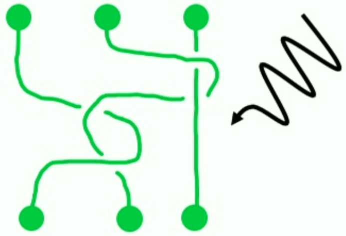
Classification of local quantum error-correcting codes (QECC)



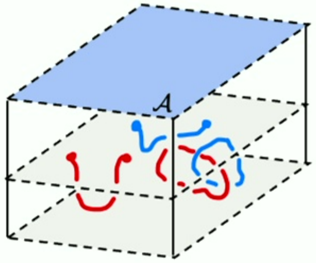
Diagnostics of new QECC and identifying ideal platforms



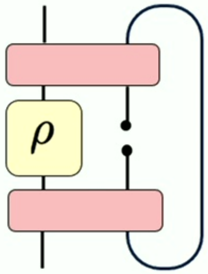
Develop QECCs for continuous variable or fermionic systems



Fault tolerance of non-Abelian quantum computation



Measurement-based and single-shot codes from boundary codes



Mixed-state topological orders

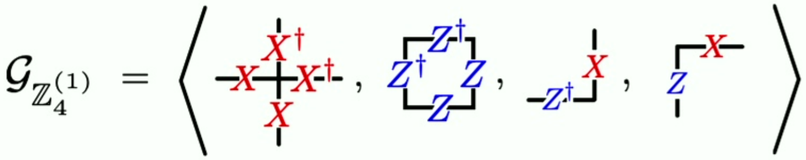
Condensation vs. Gauging out

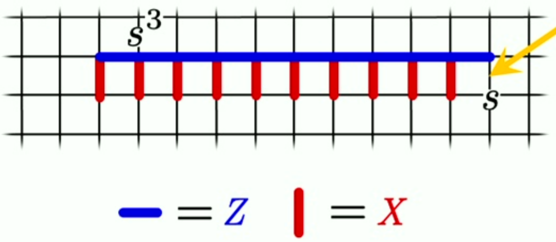
Condensation of a (Stabilizer code)	Gauging out a (Subsystem code)
• $\theta(a) = 1$ (a is a boson)	• $\theta(a)$ is arbitrary
• Add short string operators of a to **stabilizer** group and remove non-commuting stabilizers	• Add short string operators of a to **gauge** group
• Anyons become identified	• No identification of anyons
• Anyons become confined	• Anyons become confined

$\mathbb{Z}_4^{(1)}$ subsystem code

Anyons: Violations of gauge operators that cannot be created by local operators

(Subsystem codes)





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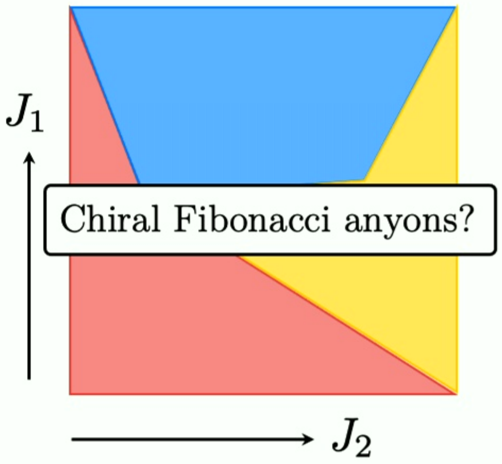

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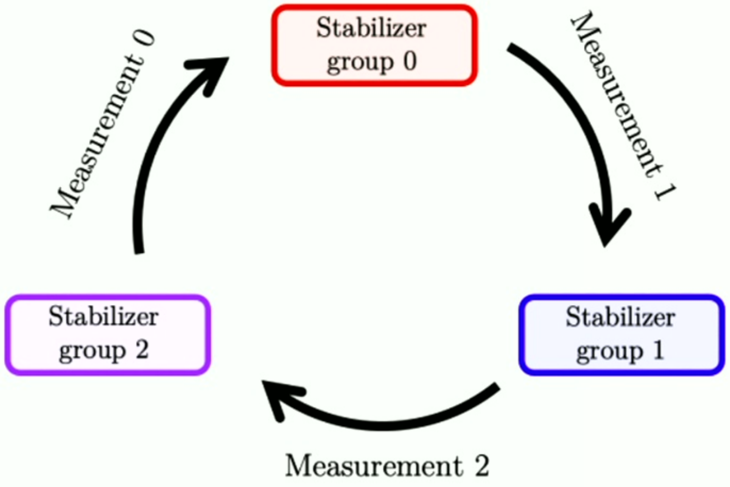

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