Title: Uncertainty Relations for Metrology and Computation

Speakers: Jacob Bringewatt

Series: Perimeter Institute Quantum Discussions

Date: December 11, 2023 - 11:00 AM

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Abstract: Uncertainty relations are a familiar part of any introductory quantum mechanics course. In this talk, I will summarize how uncertainty relations have been re-interpreted and re-expressed in the language of information theory, leading to connections with the geometry of quantum state space and the limits of computational and information processing efficiency. As two particular examples, I will discuss how uncertainty relations allow one to design information-theoretically optimal measurement protocols for function estimation in networks of quantum sensors and how they enable one to bound the speed at which analog quantum computers can possibly perform optimization tasks. Based primarily on arXiv:2110.07613 and arXiv:2210.15687.

Zoom link https://pitp.zoom.us/j/98258695315?pwd=Q2pEcmg5MGhLWmFlR1FPako0NVFlQT09

Uncertainty Relations for Metrology and Computation

Jacob Bringewatt

Perimeter Institute

Dec. 11, 2023

Uncertainty relations in quantum mechanics

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}$$

position-momentum uncertainty relation

about non-commuting operators

about *simultaneous* measurements

$A = \hat{x},\ B = \hat{p} \implies \Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}$

$$\Delta E\Delta t \gtrsim \frac{1}{2}$$

energy-time uncertainty relation

no time operator

$\Delta t \sim$ lifetime of a state

$A = |\psi_0\rangle\langle\psi_0|,\ B = H \implies \Delta H t_\perp \geq \frac{\pi}{2}$

A unified approach

$$\Delta^2 A \Delta^2 B \geq \left|\frac{1}{2i}\langle[A, B]\rangle\right|^2$$

Robertson uncertainty relation

Uncertainty relations as performance limits

For metrology: how much information can we extract from a parameterized quantum state?

For computation: how quickly can we perform a computation or prepare a state?

Using how many/what resources (energy, entanglement, control parameters, time, ...)?

Quantum sensor networks

Quantum Sensing with Erasure Qubits

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(Dated: October 4, 2023)

PHYSICAL REVIEW A **103**, L030601 (2021)

Letter

Optimal measurement of field properties with quantum sensor networks

Timothy Qian,[1,2,3] Jacob Bringewatt,[1,2] Igor Boettcher,[2] Przemyslaw Bienias,[1,2] and Alexey V. Gorshkov[1,2]

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PHYSICAL REVIEW RESEARCH **5**, 033228 (2023)

Minimum-entanglement protocols for function estimation

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PHYSICAL REVIEW RESEARCH **3**, 033011 (2021)

Protocols for estimating multiple functions with quantum sensor networks: Geometry and performance

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Quantum annealing

PHYSICAL REVIEW LETTERS **130,** 140601 (2023)

Lower Bounds on Quantum Annealing Times

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On the stability of solutions to Schrödinger's equation short of the adiabatic limit

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Outline for rest of talk

I. Uncertainty relations as speed limits

Takeway: ultimate speed limits are connected to geometry of state space

II. Application: quantum sensor networks

Takeway: uncertainty relations give performance limits *and* resource requirements

III. Application: quantum annealing

Takeway: rigorous approaches to quantum annealing beyond the adiabatic regime

IV. Summary/outlook

Uncertainty relations as speed limits

Warm up: Mandelstam-Tamm style speed limit

$$\Delta^2 A \Delta^2 B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2$$

$$\Delta^2 A \equiv \langle A^2 \rangle - \langle A \rangle^2, \qquad \langle A \rangle \equiv \mathrm{Tr}(\rho A)$$

If $B = \hat{H}$, $\langle \partial A / \partial t \rangle = 0$ (A fixed):

$$-i \langle [A, H] \rangle = \frac{\partial \langle A \rangle}{\partial t}$$

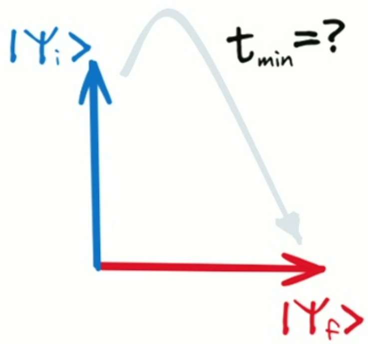
Ehrenfest theorem

Thus:

$$2 \Delta A \Delta H \geq \left| \frac{\partial \langle A \rangle}{\partial t} \right|$$

If $A = |\psi_i\rangle \langle \psi_i|$ integrate to get:

$$\Delta H t_{\perp} \geq \frac{\pi}{2}$$



Mandelstam-Tamm bound

A tighter speed limit

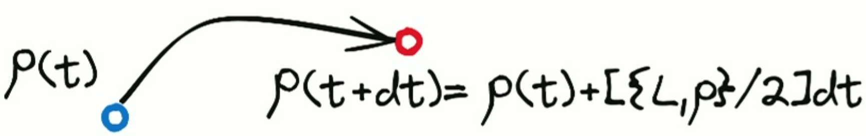
$$\Delta^2 A \Delta^2 B \geq \left| \frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle \right|^2 + \left| \frac{1}{2i}\langle [A,B]\rangle \right|^2$$

Speed limit from first term?

Schrödinger uncertainty relation

Let L generate time evolution via

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\{L,\rho\} = \frac{L\rho + \rho L}{2}$$



If $B = L$, $\langle dA/dt \rangle = 0$:

$$\Delta A \Delta L \geq \left| \frac{\partial \langle A \rangle}{\partial t} \right|$$

MT bound with $2\Delta H \to \Delta L$

The geometric nature of the speed limit

It can be shown that $2\Delta H \geq \Delta L$ so we have a tighter speed limit:

$$2\Delta A \Delta H \geq \Delta A \Delta L \geq \left| \frac{\partial \langle A \rangle}{\partial t} \right|$$

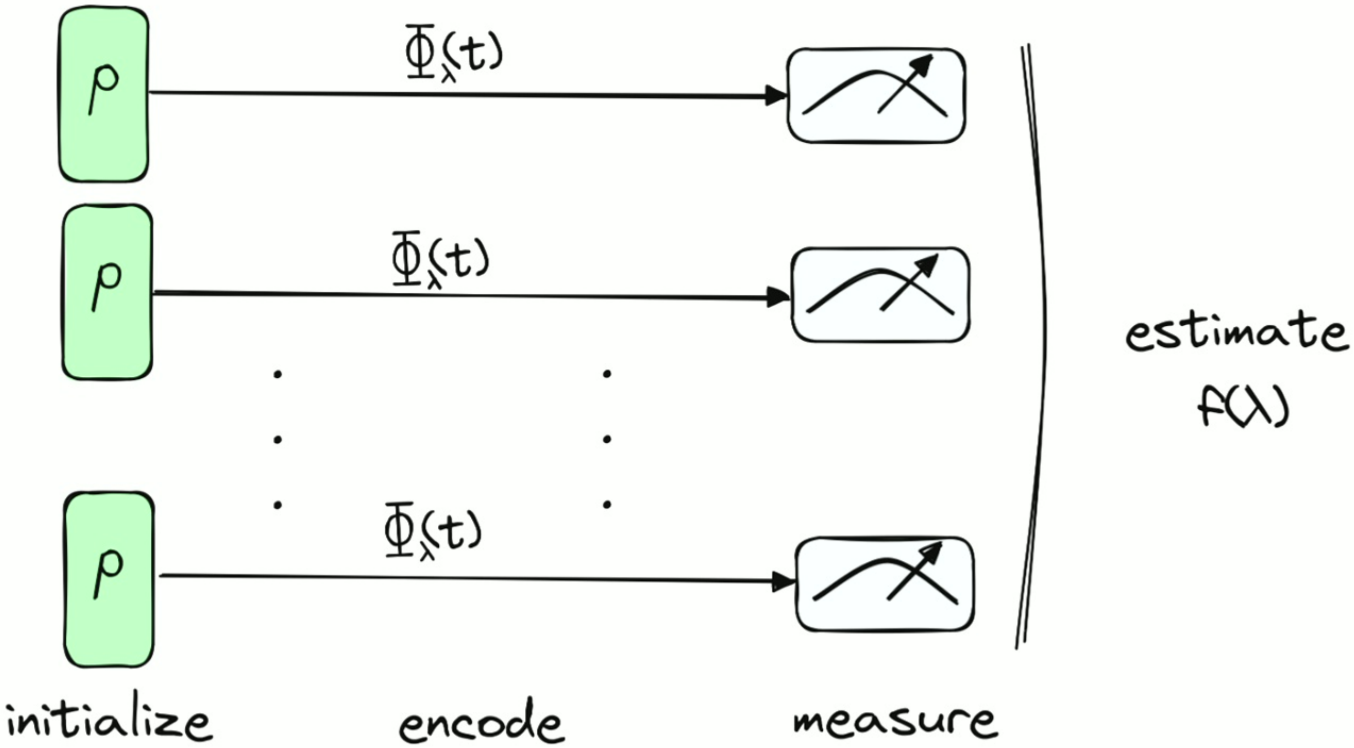
Quantum Fisher Information

$$\mathcal{F}(t) \equiv \Delta^2 L$$

Geometric interpretation: natural notion of distance between $\rho(t)$ and $\rho(t + dt)$ in projective Hilbert space.

Holds for *any* parameter $\lambda \implies$ precision limits for metrology

Application I: Precision limits for quantum metrology



The quantum Cramér-Rao bound

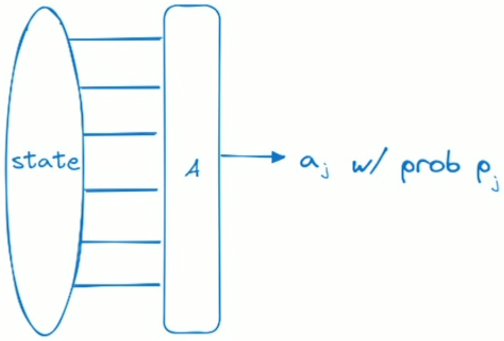
Goal: a lower bound on precision of estimating unknown parameter λ encoded in state

A Proof Sketch

Suppose $\exists$ A such that $\langle A \rangle = \lambda$

Estimate λ via ν experimental repititions as $\widetilde{\lambda} = \langle A \rangle_\nu$

For large number of repetitions ν: $\langle A \rangle_\nu \longrightarrow \langle A \rangle$ and $\Delta\widetilde{\lambda} = \frac{\Delta A}{\sqrt{\nu}}$

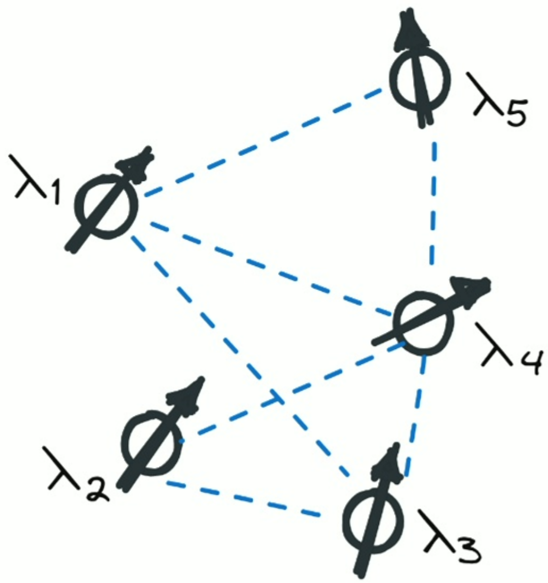


Uncertainty relation becomes:

$$\Delta A \sqrt{\mathcal{F}} \geq \left| \frac{\partial \langle A \rangle}{\partial \lambda} \right| \implies \Delta\widetilde{\lambda} \geq \frac{1}{\sqrt{\nu}\sqrt{\mathcal{F}}}$$

quantum Cramér-Rao bound

Function estimation in a quantum sensor network



how precisely can I measure a function $q(\boldsymbol{\lambda})$?

how do I design optimal protocols to acheive this?

what resources (i.e. entanglement) do those protocols require?

Function estimation for qubit sensors

couple independent local parameters $\boldsymbol{\lambda} \in \mathbb{R}^d$ to a network of d qubit sensors via

$$H = \frac{1}{2}\sum_{j=1}^{d} \sigma_j^z \lambda_j + H_c$$

task: measure a linear function $q(\boldsymbol{\lambda}) = \boldsymbol{\alpha} \cdot \boldsymbol{\lambda}$. [Eldredge et. al. **PRA** (2018)]

Optimal performance for this problem: $\Delta\tilde{q} \geq \frac{\|\boldsymbol{\alpha}\|_\infty}{t}$

Also:

- analytic functions $q(\boldsymbol{\lambda})$ [Qian et. al. **PRA** (2019)]
- dependent field amplitudes $\boldsymbol{\lambda}(\boldsymbol{\theta}) \in \mathbb{R}^d$ [Qian, **JB**, et. al. **PRA** (2021)]
- multiple functions [**JB** et. al. **PRR** (2021)]
- algebraic approach to protocols, minimizing entanglement [Ehrenberg, **JB** et. al. **PRR** (2023)]
- non-commuting generators, interferometers, applications to geophysics, ... [in prep.]

Deriving the function estimation precision bound

apply quantum Cramér-Rao bound optimized over fixing extra d.o.f.

$$\mathrm{Var}(\tilde{q}) \geq \max_{\text{fixing extra d.o.f.}} \frac{1}{\mathcal{F}(q)} \geq \max_{\text{fixing extra d.o.f.}} \frac{1}{t^2 \|\hat{g}_q\|_s^2}$$

$$\|\cdot\|_s = (\text{max eigenvalue}) - (\text{min eigenvalue})$$ [Boixo *et. al.* **PRL** (2008)]

$\hat{g}_q$ is defined with respect to extra d.o.f.

$$H = \frac{1}{2}\sum_{j=1}^{d} \sigma_j^z \lambda_j + H_c \longrightarrow H = \frac{1}{2}\underbrace{(\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^z)}_{\hat{g}_q}\underbrace{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda})}_{q} + \frac{1}{2}\sum_{j>1}\hat{g}_{q_j} q_j + H_c$$

Optimal choice of fixing extra d.o.f. corresponds to optimizing over $\boldsymbol{\beta}$ s.t. $\boldsymbol{\beta}\cdot\boldsymbol{\alpha} = 1$, yielding

$$\mathrm{Var}(\tilde{q}) \geq \frac{\|\boldsymbol{\alpha}\|_\infty^2}{t^2}$$

A function-dependent Heisenberg scaling

entangled

$$\mathrm{Var}(\tilde{q}) \geq \frac{\|\boldsymbol{\alpha}\|_\infty^2}{t^2}$$

unentangled

$$\mathrm{Var}(\tilde{q}) \geq \frac{\|\boldsymbol{\alpha}\|_2^2}{t^2}$$

$$\text{precision gain} = \frac{\|\boldsymbol{\alpha}\|_2}{\|\boldsymbol{\alpha}\|_\infty}$$

best case

$$\boldsymbol{\alpha} = (1, 1, \cdots, 1)^T$$

$$\text{precision gain} = \sqrt{d}$$

worst case

$$\boldsymbol{\alpha} = (1, 0, \cdots, 0)^T$$

$$\text{precision gain} = 1$$

From bounds to protocols

For the bound to be saturable, the quantum Fisher information matrix must be of the form:

$$\mathcal{F}(\boldsymbol{q}) = \begin{pmatrix} \frac{t^2}{\|\boldsymbol{\alpha}\|_\infty^2} & 0 & \cdots \\ 0 & & \\ \vdots & & \end{pmatrix} \implies \mathcal{F}(\boldsymbol{\lambda}) = \begin{pmatrix} \leftarrow & t^2 \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|_\infty} & \rightarrow \\ & & \end{pmatrix}$$

I.e. must exist a choice of fixing extra d.o.f. that gives no useful information

Heavily constrains the allowed probe states/control operations

$$\mathcal{F}(\boldsymbol{\lambda})_{jk} = \frac{1}{2}\langle\{\mathcal{H}_j, \mathcal{H}_k\}\rangle - \langle\mathcal{H}_j\rangle\langle\mathcal{H}_k\rangle, \qquad \mathcal{H}_j = -iU^\dagger(\partial_j U)$$

Next: Pick states from the allowed families subject to relevant constraints (entanglement, number/type of control operations, etc.)

Minimum entanglement protocols

Example: minimize amount of entanglement used at any point during the encoding process

Theorem

Suppose that

$$k - 1 < \frac{\|\boldsymbol{\alpha}\|_1}{\|\boldsymbol{\alpha}\|_\infty} \leq k.$$

Any optimal protocol requires at least, but no more than, k-partite entangled states.

Proof approach:

- no more than: provide an explicit protocol using k-partite entangled states
- requires at least: proof by contradiction assuming existence of a protocol using $(k-1)$-partite entangled states violates saturability conditions

Taking stock

Robertson uncertainty relation $\rightarrow$ quantum Fisher info. $\rightarrow$ Cramér-Rao bnd. $\rightarrow$ metrology

Uncertainty relations allow us to:

- understand the limits of metrological performance
- derive optimal protocols
- understand resource requirements (i.e. entanglement)

Next: These same tools can be applied to analyze quantum annealing.

Application II: Limits of computation for quantum annealing

Task: starting from ground state of H_0 prepare ground state of H_1

$$H(t) = (1 - g(t))H_0 + g(t)H_1, \qquad g(0) = 0,\ g(t_f) = 1,$$

adiabatic regime

adiabatic theorem guarantees success if slow enough interpolation

$$t_f \sim \frac{1}{\Gamma^2} \quad (\Gamma := \text{min. eigenvalue gap})$$

sufficient, but not necessary condition

[Jansen, Ruskai, Seiler **J. Math. Phys.** (2007)]
[**JB** *et. al.* **PRA** (2018,2019,2022)]
[**JB** *et. al.* **PRL** (2021)]

general annealing bounds

[García-Pintos, Brady, **JB**, Liu, **PRL** (2023)]

first general, rigorous bounds beyond adiabatic regime

comes down to upper bounding

$$\left| \frac{d\langle H_1 \rangle_t}{dt} - \frac{d\langle H_0 \rangle_t}{dt} \right|$$

A loose bound

Assume: ground state energy of H_0, $H_1 = 0$.

Then, from Ehrenfest's theorem + a bit of algebra

$$i\mathrm{Tr}(\rho_t[H_1, H_0]) = \frac{d\langle H_0\rangle_t}{dt} - \frac{d\langle H_1\rangle_t}{dt}$$

From Robertson uncertainty relation

$$2\Delta H_0 \Delta H_1 \geq \left| \frac{d\langle H_0\rangle_t}{dt} - \frac{d\langle H_1\rangle_t}{dt} \right|$$

Integrating from 0 to t_f and the fact that $\Delta H_j \leq 2\,\|H_j\|$ gives a bound on annealing time t_f

$$t_f \geq \frac{2(\langle H_0\rangle_{t_f} + \langle H_1\rangle_0 - \langle H_1\rangle_{t_f})}{\|H_0\|\,\|H_1\|}$$

A sequence of tighter bounds

Theorem (A Lower Bound on Annealing Times)

Let H_0 and H_1 be a pair of Hamiltonians with smallest eigenvalue zero. Then an annealing schedule of time t_f obeys

$$t_f \geq \tau_1 \geq \tau_2 \geq \tau_3, \qquad \text{where} \quad \tau_j = \left[\frac{\langle H_0 \rangle_{t_f} + \langle H_1 \rangle_0 - \langle H_1 \rangle_{t_f}}{\| [H_0, H_1] \|} \right] \times \frac{1}{\eta_j}$$

$$\underbrace{\eta_1 = \frac{1}{2t_f} \int_0^{t_f} C_1(\rho_t) dt}_{\text{coherence as a resource}} \qquad \underbrace{\eta_2 = \frac{1}{t_f} \int_0^{t_f} \sqrt{1 - \sum_j p_{j,t}^2} dt}_{\text{useful for adiabatic annealing}} \qquad \underbrace{\eta_3 = 1}_{\text{indep. of schedule}}$$

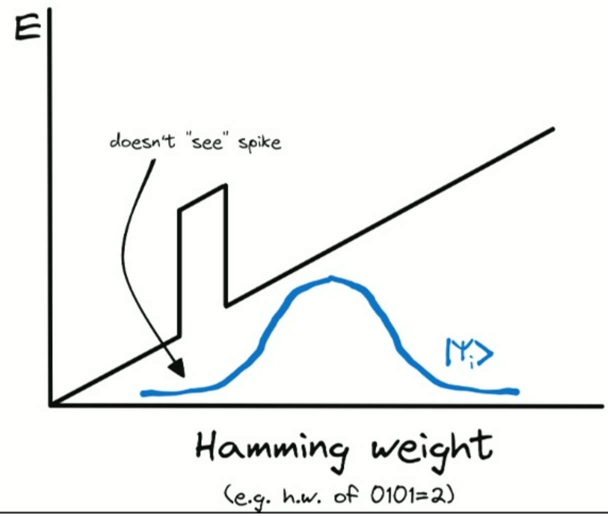
$\langle H_1 \rangle_{t_f} \sim$ fidelity, $\quad C_1$: measure of coherence, $\quad p_{j,t}$: population in energy eigenbasis

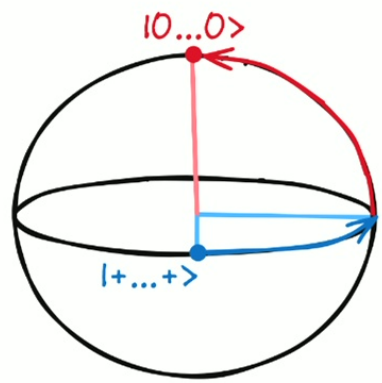
Saturable by adiabatic and non-adiabatic annealing schedules

For instance:

- unstructured search via locally optimized adiabatic algorithm [Roland and Cerf (2002)]
- Hamming spike via numerically optimized algorithm (diabatic cascade) or QAOA [Muthukrishnan *et. al.* **PRX** (2016); Bapat and Jordan **Quantum Inf. Comput.** (2019)]

$$H_0 = \sum_j \frac{I - X_j}{2}, \quad H_1 = \sum_j \frac{I - Z_j}{2} + (\text{spike})$$





$$|\psi_f\rangle \approx e^{-iM_x\pi/4} e^{-iM_z\pi/4} |\psi_i\rangle$$

$$\tau_3 \sim 1 \checkmark$$

Bounds with catalyst Hamiltonians

Corollary

Given an annealing schedule of time t_f specified by the time-dependent Hamiltonian

$$\widetilde{H}(t) = H(t) + \sum_{a=1}^{N_C} f_t^a H_C^a,$$

where $\{H_C^a\}$ are arbitrary control Hamiltonians with schedules $\{f_t^a\} \geq 0$ such that $f_0^a = f_{t_f}^a = 0$,

$$t_f \geq \frac{\langle H_0 \rangle_{t_f} + \langle H_1 \rangle_0 - \langle H_1 \rangle_{t_f}}{\left\| [H_1, H_0] \right\| + \sum_{a=1}^{N_C} \left\| [H_1 - H_0, H_C^a] \right\|}.$$

- Relevant for shortcuts to adiabaticity
- if τ_3 is tight, catalysts cannot help

Another approach to bounds

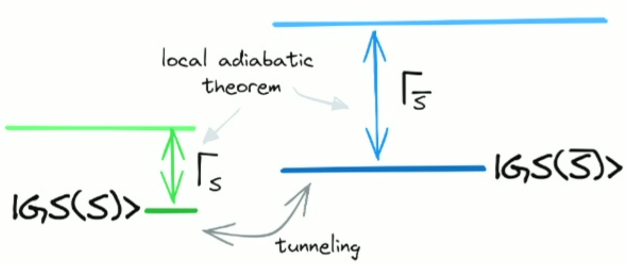
On the stability of solutions to Schrödinger's equation short of the adiabatic limit

Jacob Bringewatt[1,2], Michael Jarret[3,4,5], T. C. Mooney[1,2]

our theorem

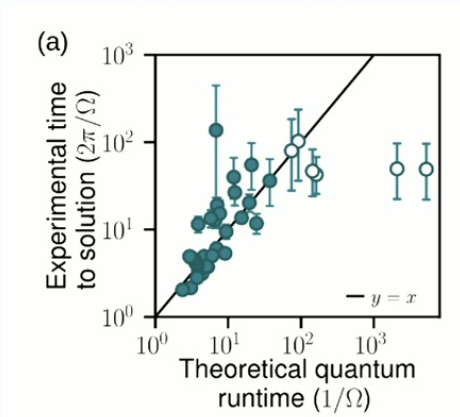
$$\text{error} \sim \underbrace{\frac{1}{t_f \Gamma_S^2}}_{\text{local adiab.}} + \underbrace{\sqrt{h t_f}}_{\text{tunneling}}$$

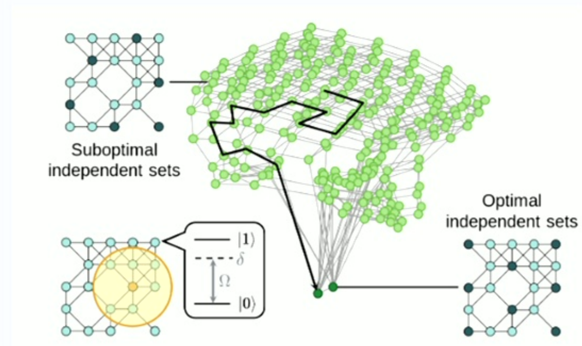
$$|GS\rangle \approx |GS(S)\rangle + |GS(\bar{S})\rangle$$



MIS in Rydberg atom arrays [Ebadi et. al. Science (2022), Cain et. al. (2023)]

Task: Find largest set of vertices in unit-disk graph with no shared edges (NP-hard)





$$|\text{GS}\rangle \approx |\text{opt. indep. sets}\rangle + |\text{nearly. opt. sets}\rangle$$

Summary and outlook

Takeaway # 1: Uncertainty relations a tool for understanding limits of measurement and computation and the resources required to reach them.

Takeaway #2: Rigorous performance bounds for quantum annealing beyond adiabatic regime.

Key question

What resources to achieve the annealing bounds?

Quantum sensor networks

- non-commuting generators
- secure, delegated sensing
- connections to Hamiltonian learning
- specific implementations (e.g. gravimetry)

Quantum annealing

- full understanding of saturability
- classical speed limits and the sign problem
- fast vs. robust annealing?
- applying intermediate timescale adiabatic theorem to understand experimental and numerical results

Thanks

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