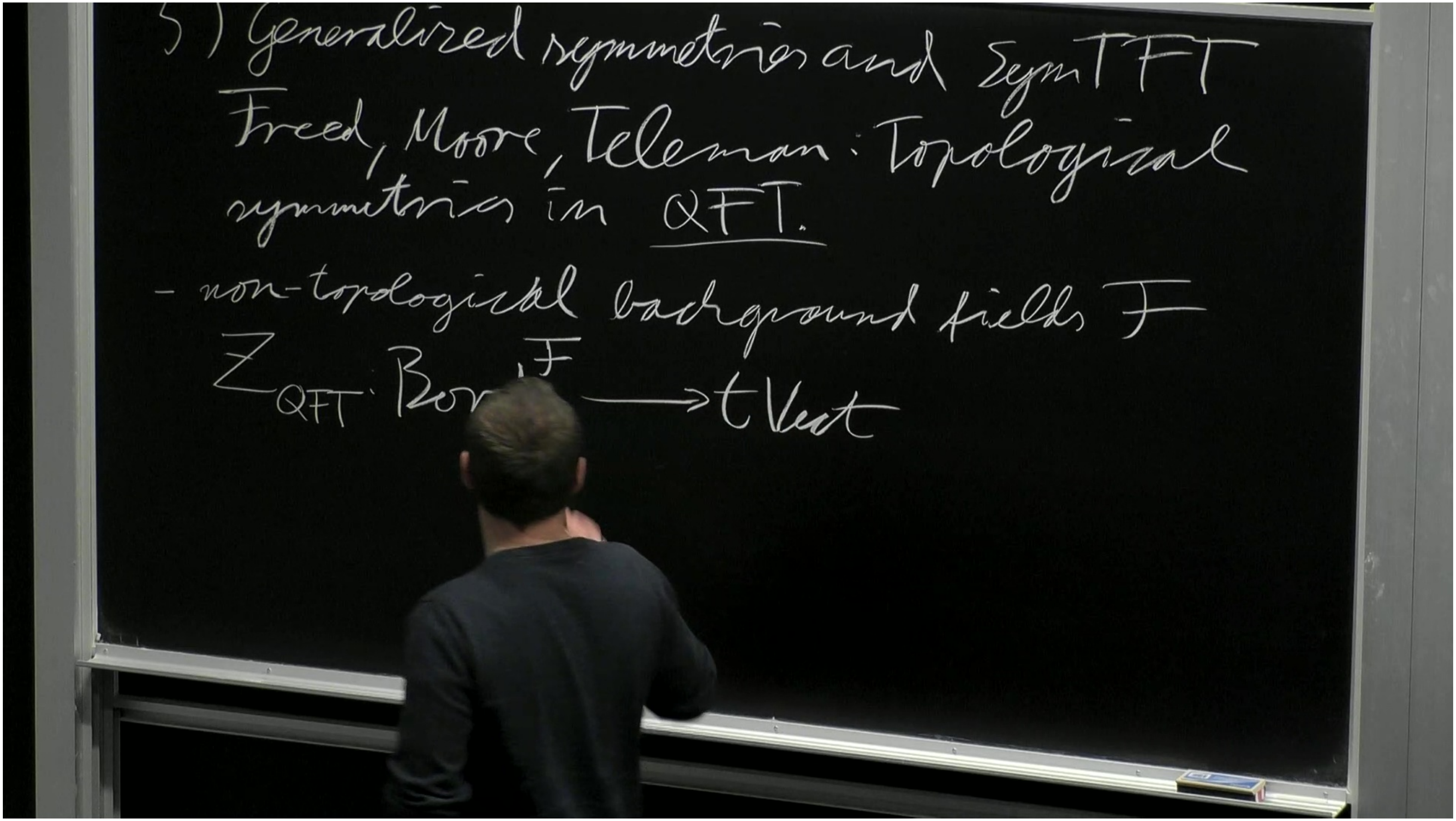
Title: Topological Quantum Field Theories Lecture 20231201

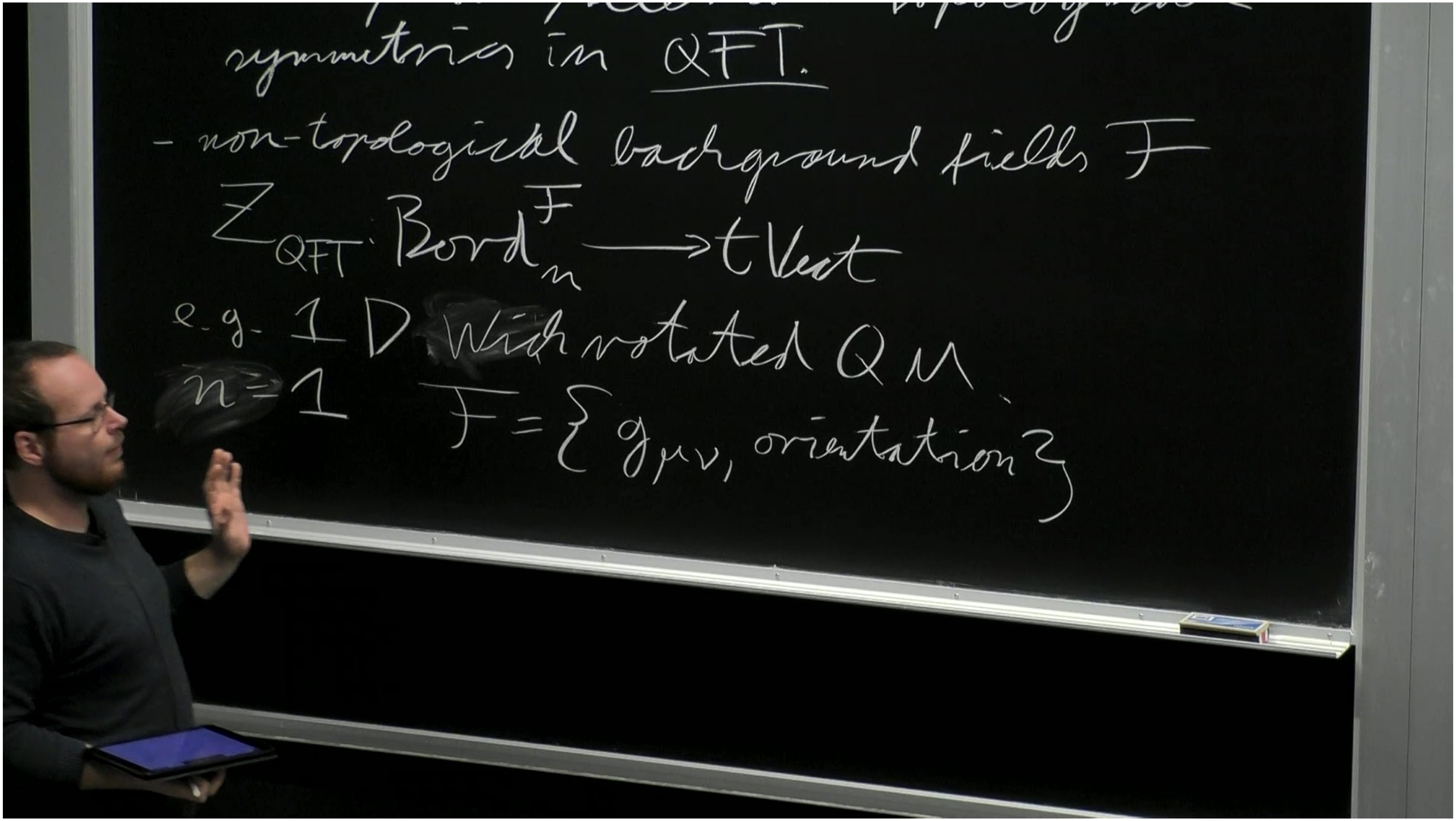
Speakers: Lukas Mueller

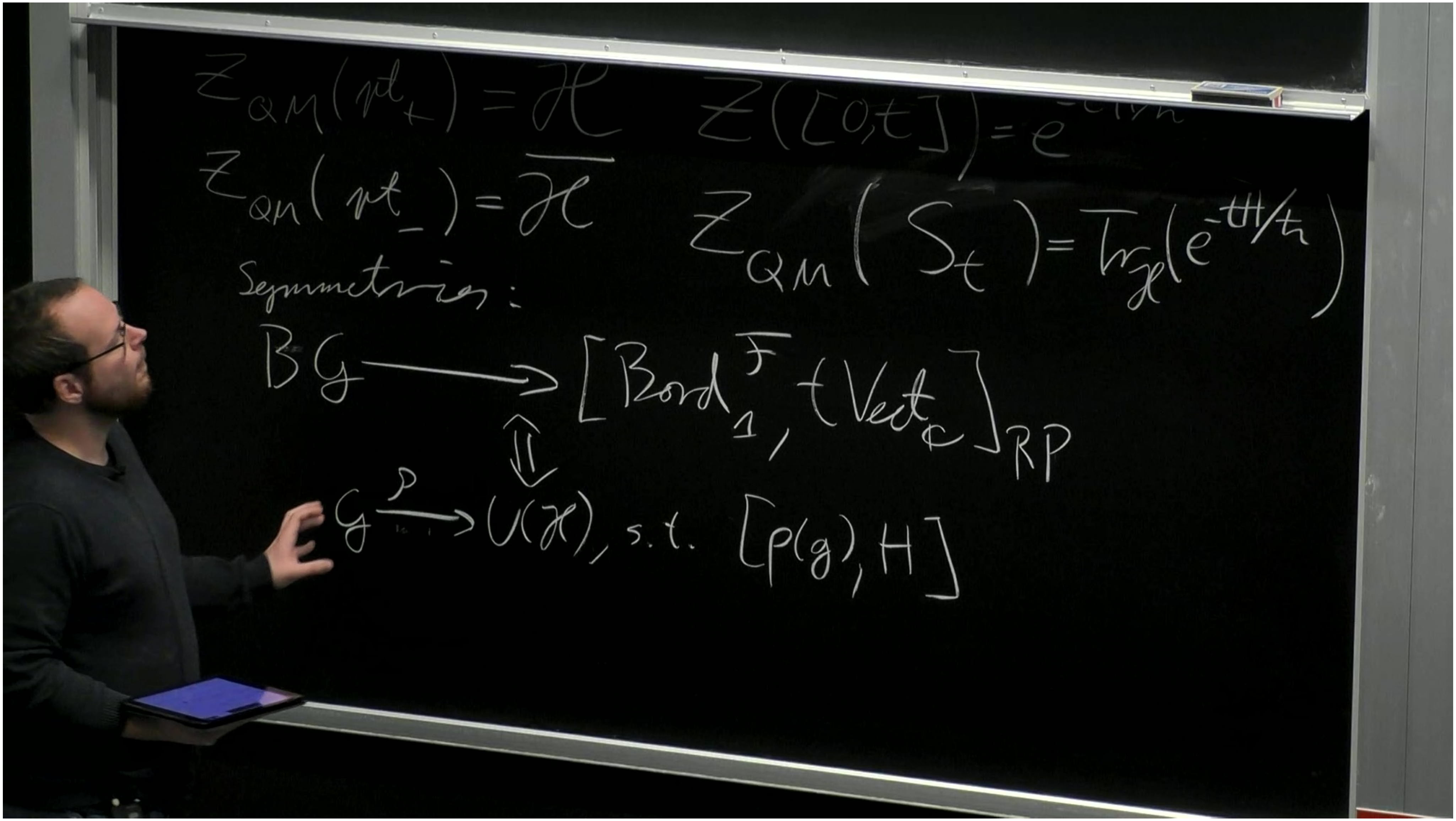
Collection: Topological Quantum Field Theories - mini-course

Date: December 01, 2023 - 2:00 PM

URL: https://pirsa.org/23120016


5) Generalized symmetries and SymTFT
Freed, Moore, Teleman: Topological symmetries in QFT.
- non-topological background fields $\mathcal{F}$
Z_{QFT} : Bor $\mathcal{F}$ ⟶ tVect


symmetries in QFT.
- non-topological background fields F
Z_QFT : Bord_n^F ⟶ tVect
e.g. 1 D Wick rotated QM
n = 1   F = { g_μν, orientation }



$Z_{QM}(pt_+) = \mathcal{H}$
$Z([0,t]) = e^{-tH/\hbar}$
$Z_{QM}(pt_-) = \overline{\mathcal{H}}$
$Z_{QM}(S_t) = \mathrm{Tr}_{\mathcal{H}}(e^{-tH/\hbar})$
Symmetries:
$BG \longrightarrow [\mathrm{Bord}_1^F, t\mathrm{Vect}_{\mathbb{C}}]_{RP}$
$\Updownarrow$
$G \xrightarrow{\rho} U(\mathcal{H})$, s.t. $[\rho(g), H]$

$$Z_{QM}(pt_+) = \mathcal{H} \qquad Z([0,t]) = e^{-tH/\hbar}$$

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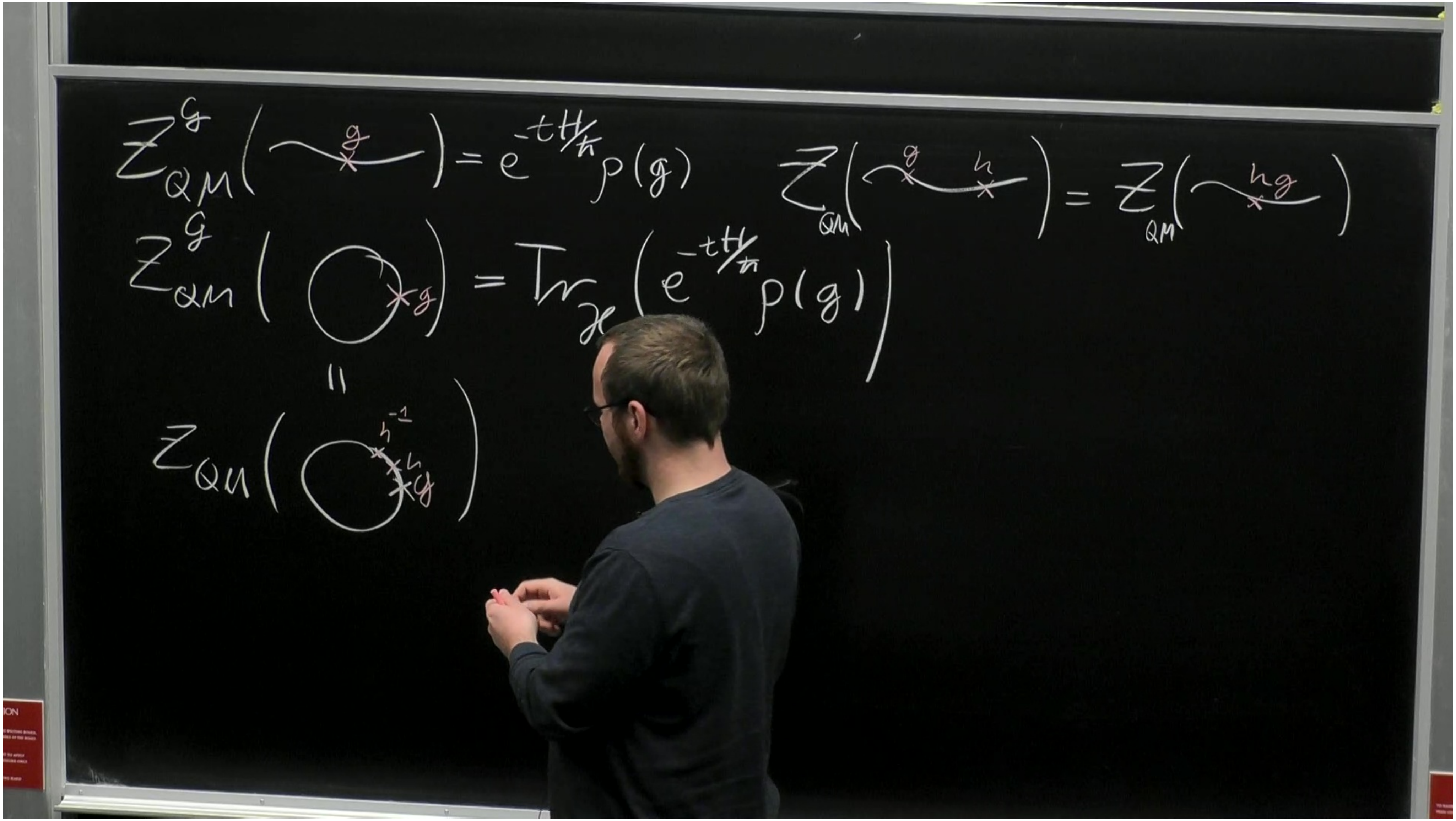
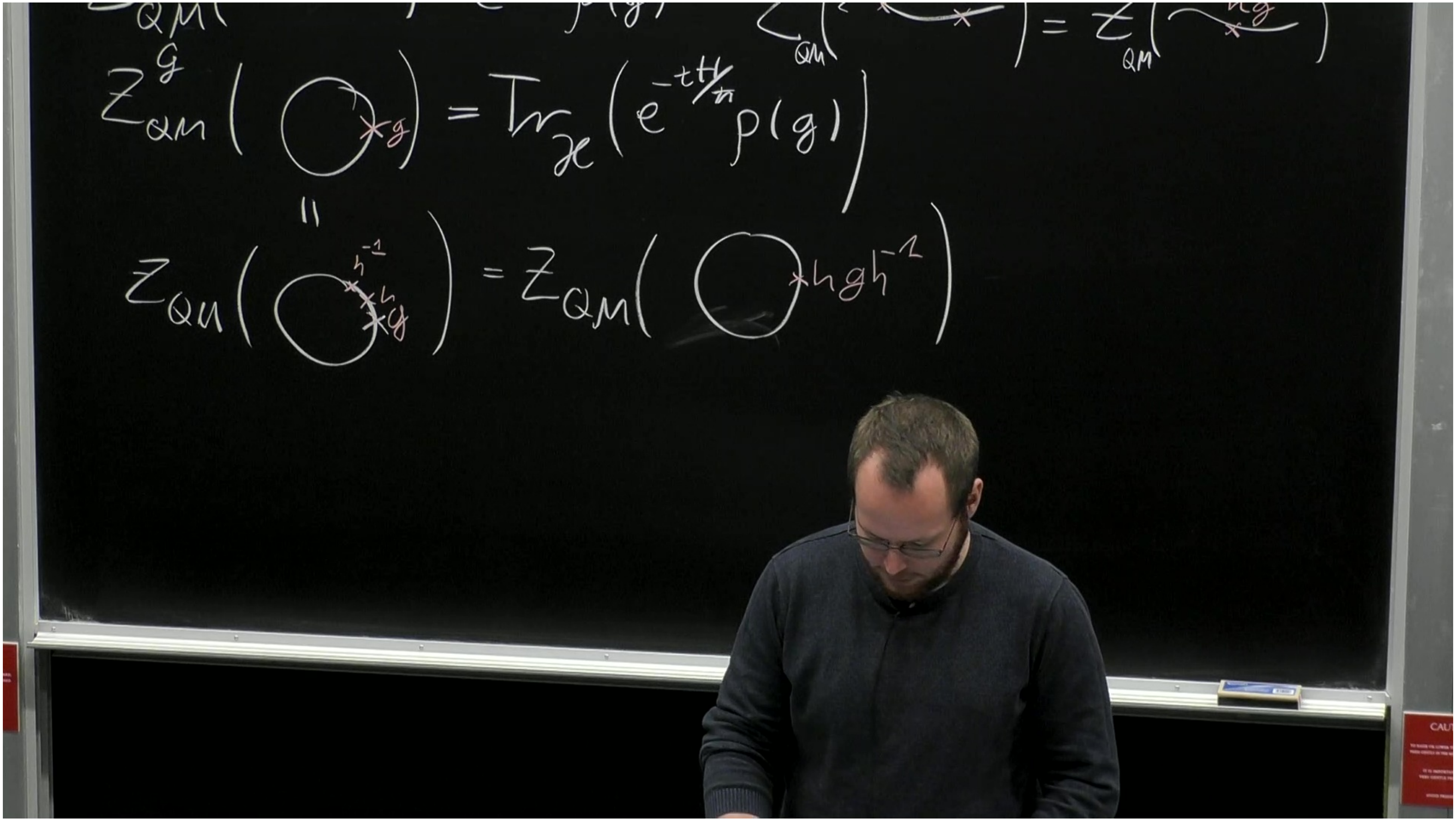
Symmetries:

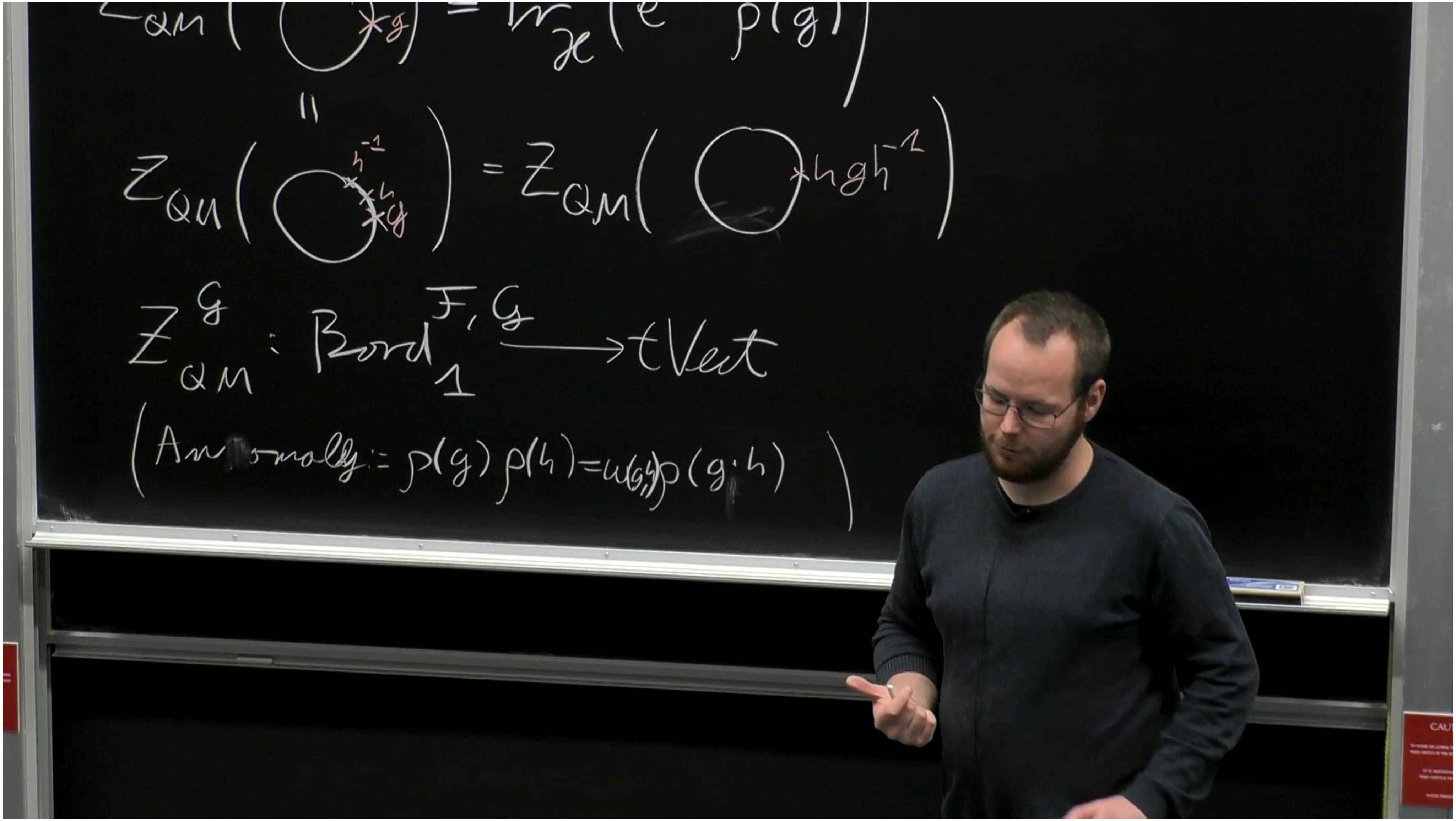
$$BG \longrightarrow [\mathrm{Bord}_1^{F}, t\mathrm{Vect}_{\mathbb{C}}]_{RP}$$

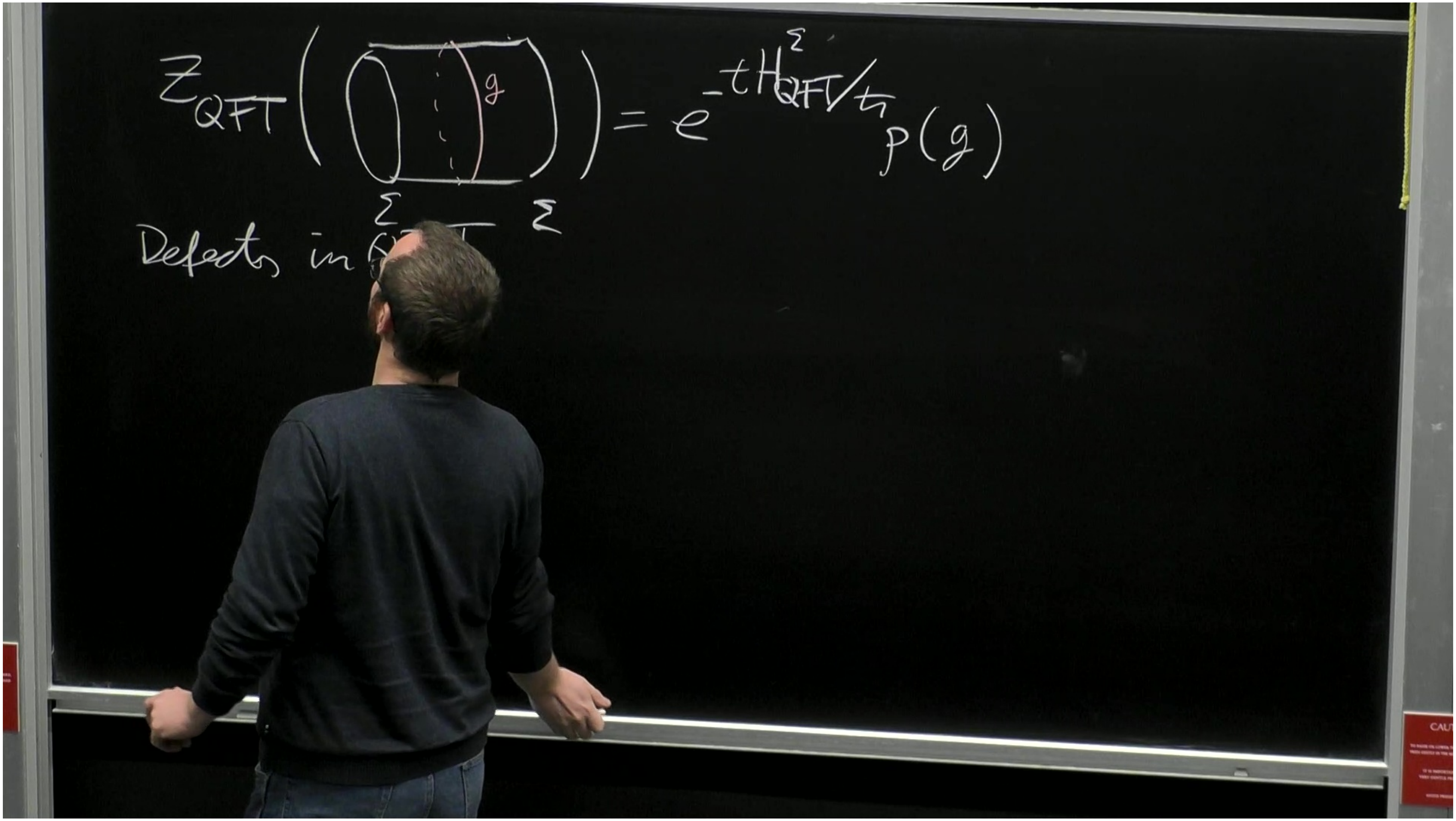
$$\Updownarrow$$

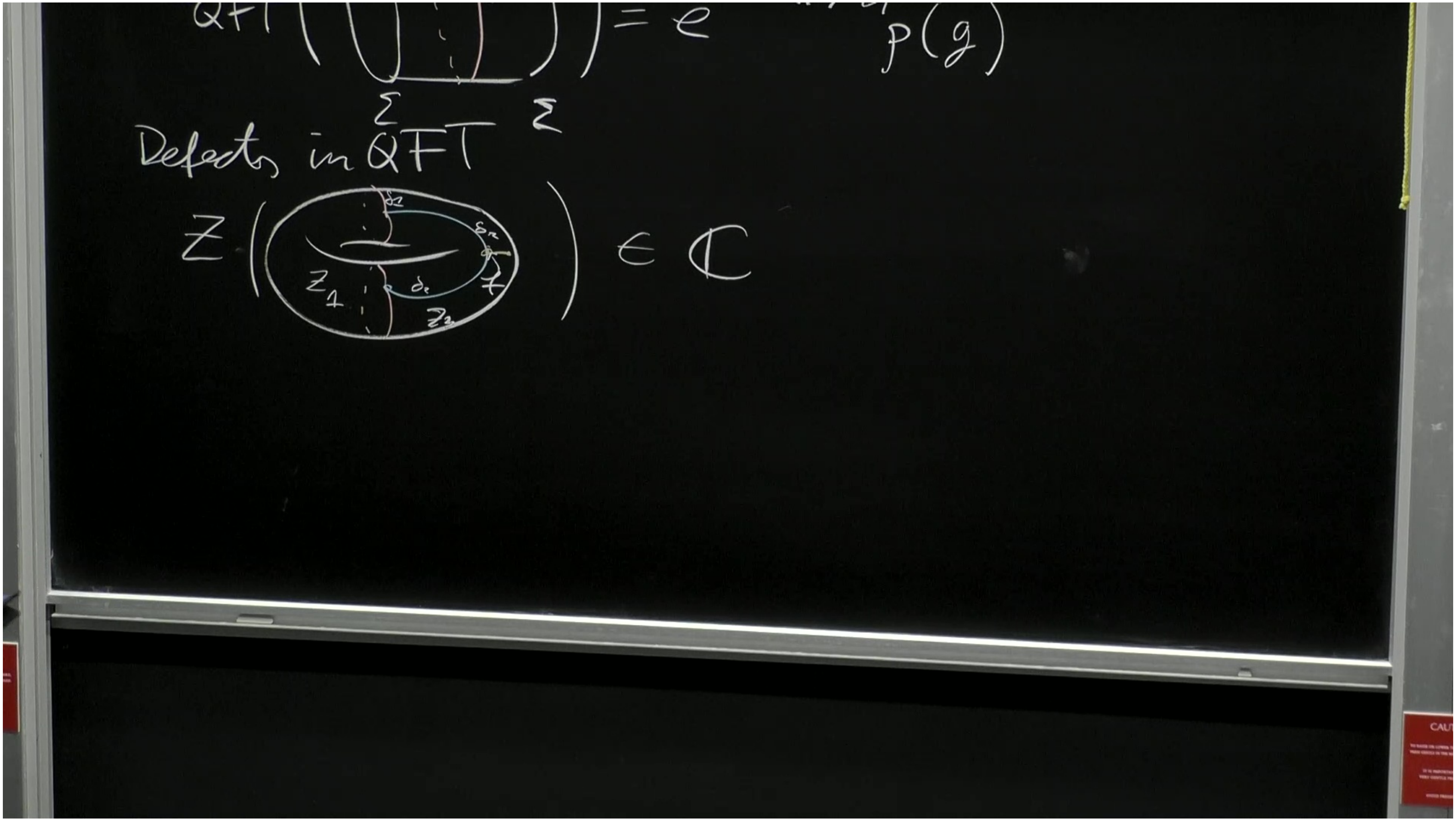
$$G \xrightarrow{\rho} U(\mathcal{H}), \text{ s.t. } [\rho(g), H] = 0$$

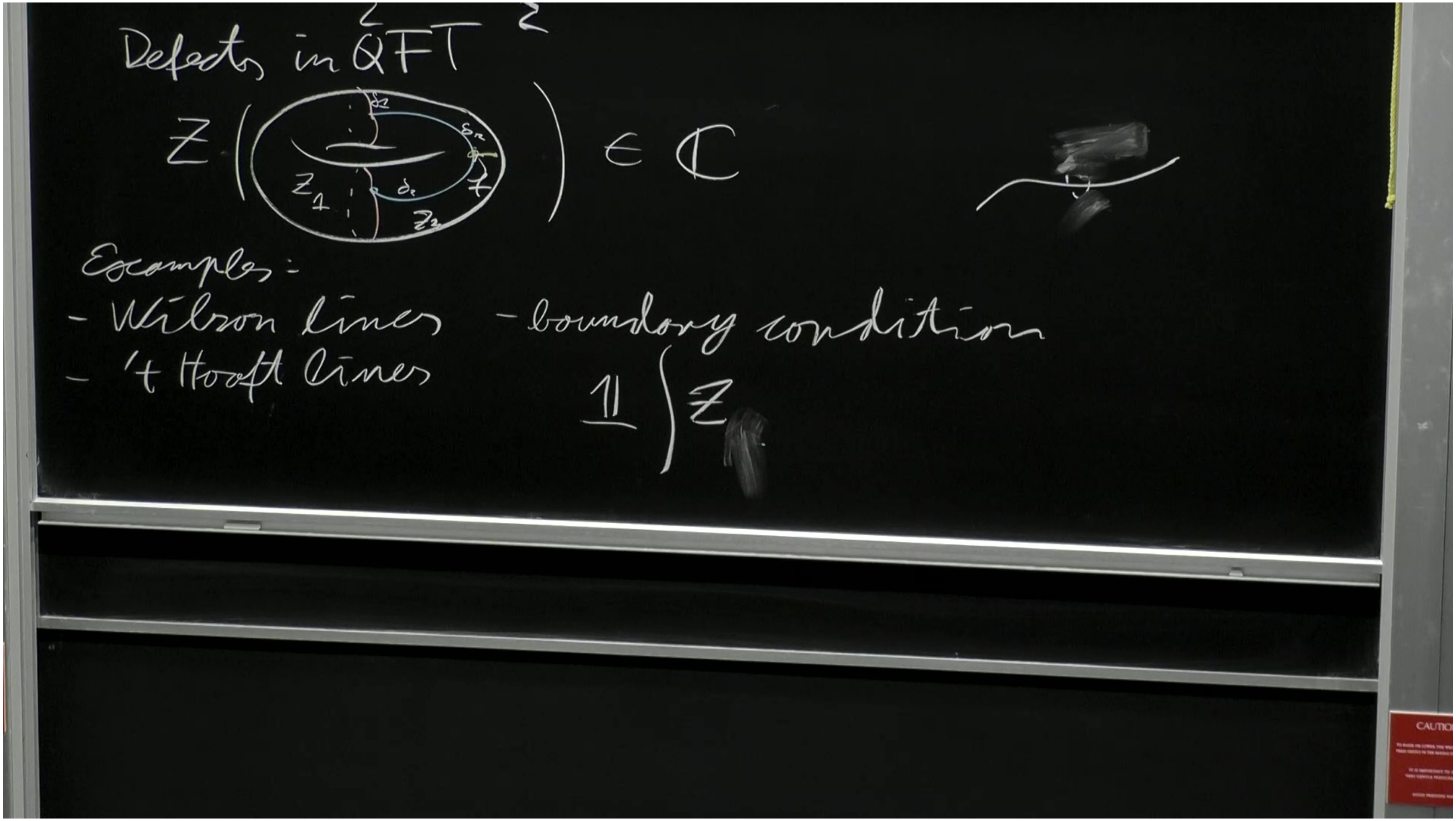

$Z^g_{QM}(\ g \ ) = e^{-tH/\hbar}\rho(g)$
$Z_{QM}(\ g \ \ h \ ) = Z_{QM}(\ hg \ )$
$Z^g_{QM}(\ g \ ) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-tH/\hbar}\rho(g)\right)$
$\shortparallel$
$Z_{QM}(\ h^{-1} \ h \ g \ )$

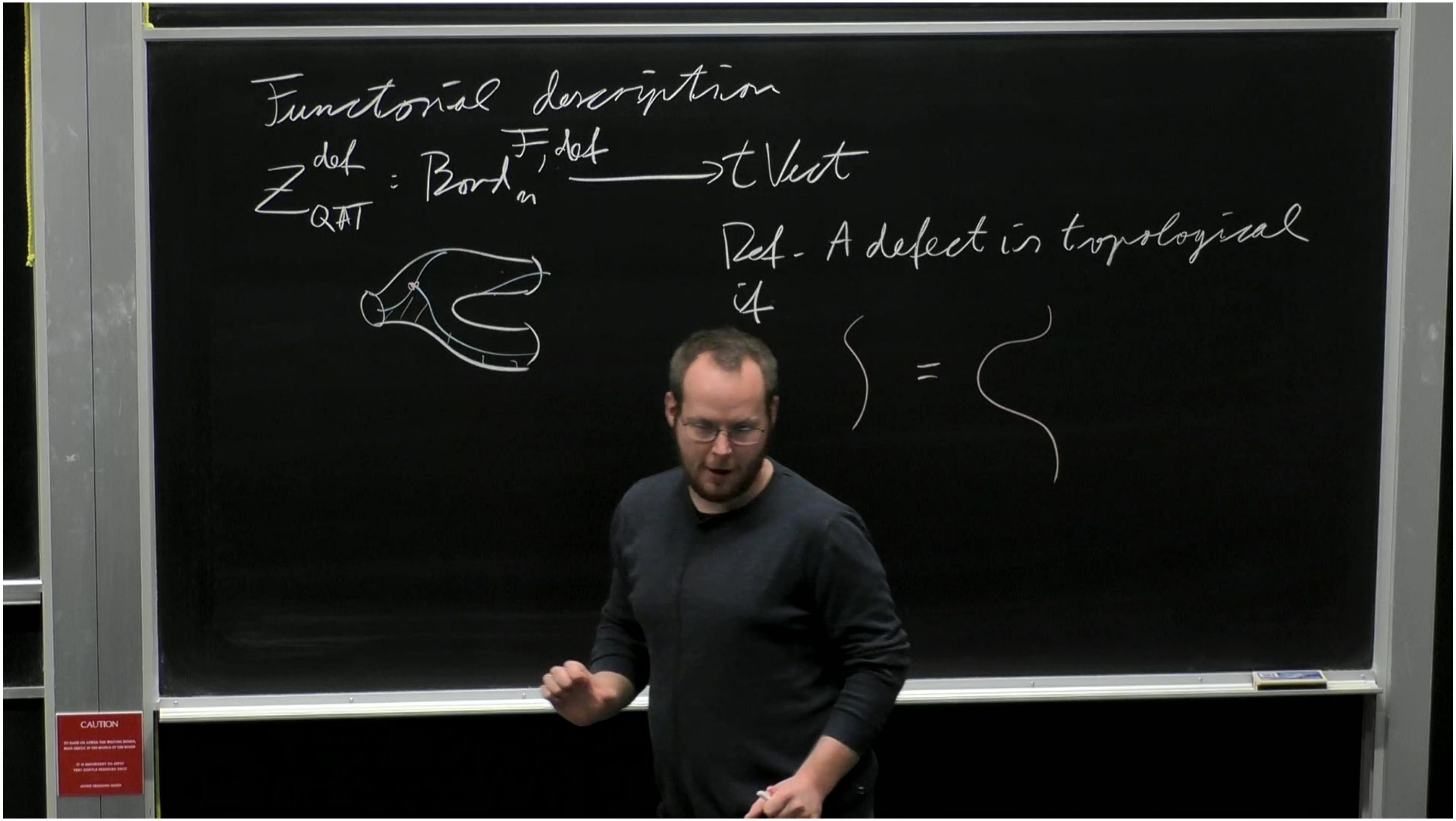

$Z^{g}_{QM}(\bigcirc \times g) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-tH/\hbar}\rho(g)\right)$
$\Vert$
$Z_{QM}(\bigcirc\ h^{-1}, h, g) = Z_{QM}(\bigcirc \times hgh^{-1})$


Z_QM : Bord_1^{F,G} → tVect
(Anomaly: ρ(g) ρ(h) = ω(g,h) ρ(g·h))

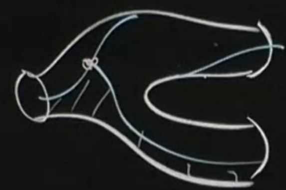

$Z_{QFT}\left(\quad \right) = e^{-tH^{\Sigma}_{QFT}/\hbar}\rho(g)$
g
Σ
Σ
Defects in


QFT( ) = e ρ(g)
Σ Σ
Defects in QFT
Z( ) ∈ ℂ
Z1 Z2

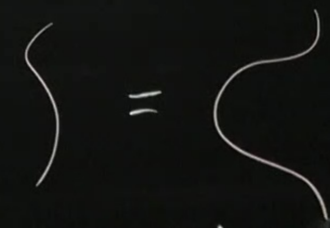

Defects in QFT
Z ( ) ∈ ℂ
Z1
Z2
Examples:
- Wilson lines
- 't Hooft lines
- boundary condition
1 | Z


Functorial description
$Z^{def}_{QFT} : Bord_n^{F, def} \longrightarrow tVect$
Def. A defect is topological
if
=

$Z_{QFT} : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$



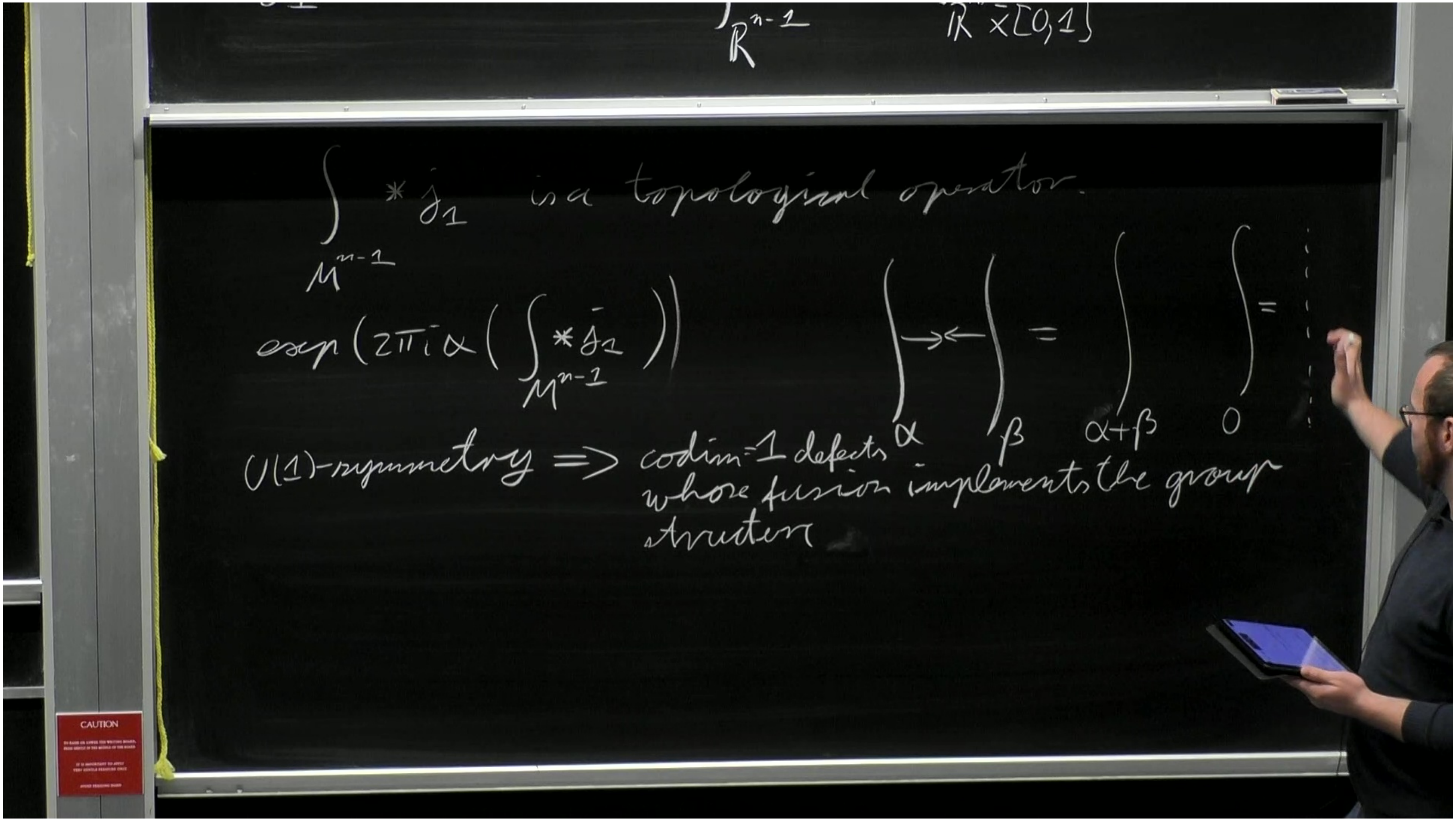
Def. A defect is topological if



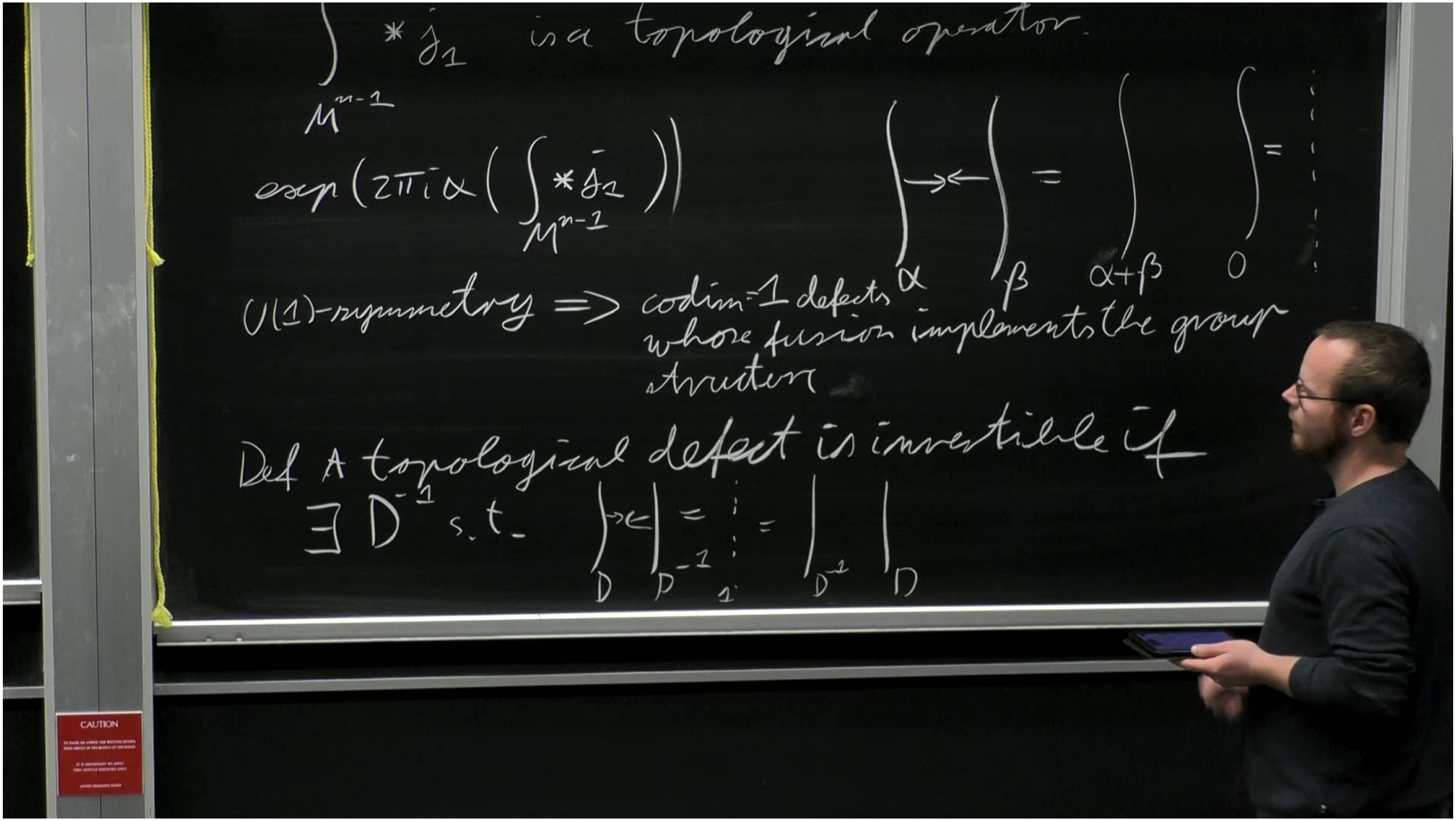
Example:

Let Z_{QFT} be a field theory with $U(1)$-symmetry.

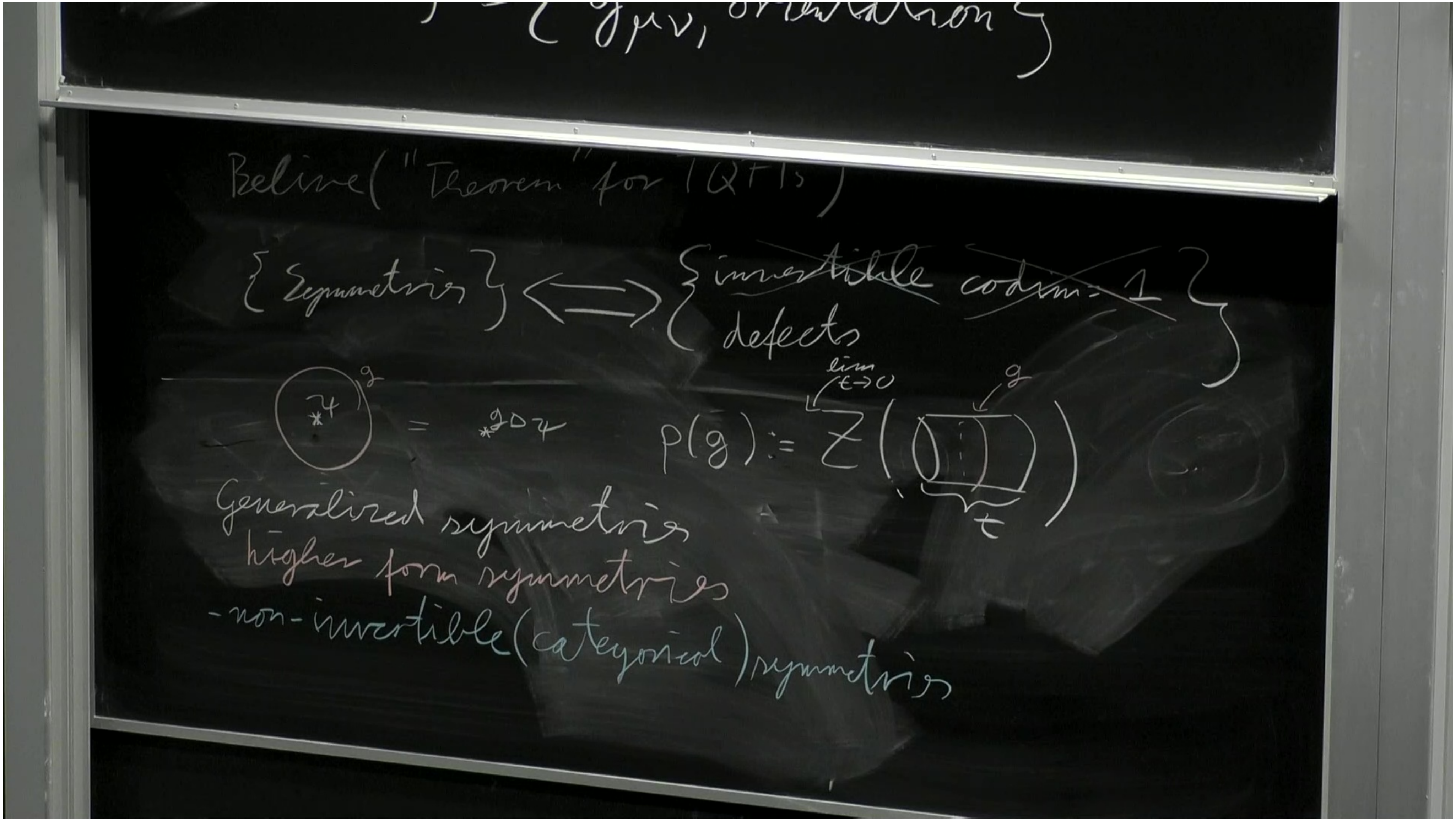
$j_1 \in \Omega^1(M)$ $\quad Q = \int_{\mathbb{R}^{n-1}} *j$ $\quad \int_{\mathbb{R}^{n-1}\times[0,1]} d(*j) = 0$



$\int_{\mathbb{R}^{n-1}}$
$\mathbb{R}^{n-1} \times [0,1]$
$\int_{M^{n-1}} * j_1$ is a topological operator.
$\exp\left(2\pi i \alpha \left(\int_{M^{n-1}} * j_1 \right)\right)$
α β $=$ $\alpha+\beta$ $=$ 0 $=$
U(1)-symmetry $\Longrightarrow$ codim=1 defects whose fusion implements the group structure



$\int_{M^{n-1}} * j_1$ is a topological operator.
$\exp\left(2\pi i \alpha \left(\int_{M^{n-1}} * \bar{j}_1 \right)\right)$
α β $=$ $\alpha+\beta$ $=$ 0
U(1)-symmetry $\Rightarrow$ codim-1 defects whose fusion implements the group structure
Def A topological defect is invertible if
$\exists\, D^{-1}$ s.t.
D D^{-1} $=$ 1 $=$ D^{-1} D
CAUTION


{ gμν, orientation }
Belive ("Theorem" for TQFTs)
{Symmetries} ⟺ {invertible codim=1 defects}
lim ε→0
ρ(g) := Z( )
Generalized symmetries
higher form symmetries
-non-invertible (categorical) symmetries

Assume all symmetries are finite.

$G \rightsquigarrow \mathbb{C}[G]$ abstract symmetry

G-rep $\rightsquigarrow$ $\mathbb{C}[G]$-modules concrete realizations.

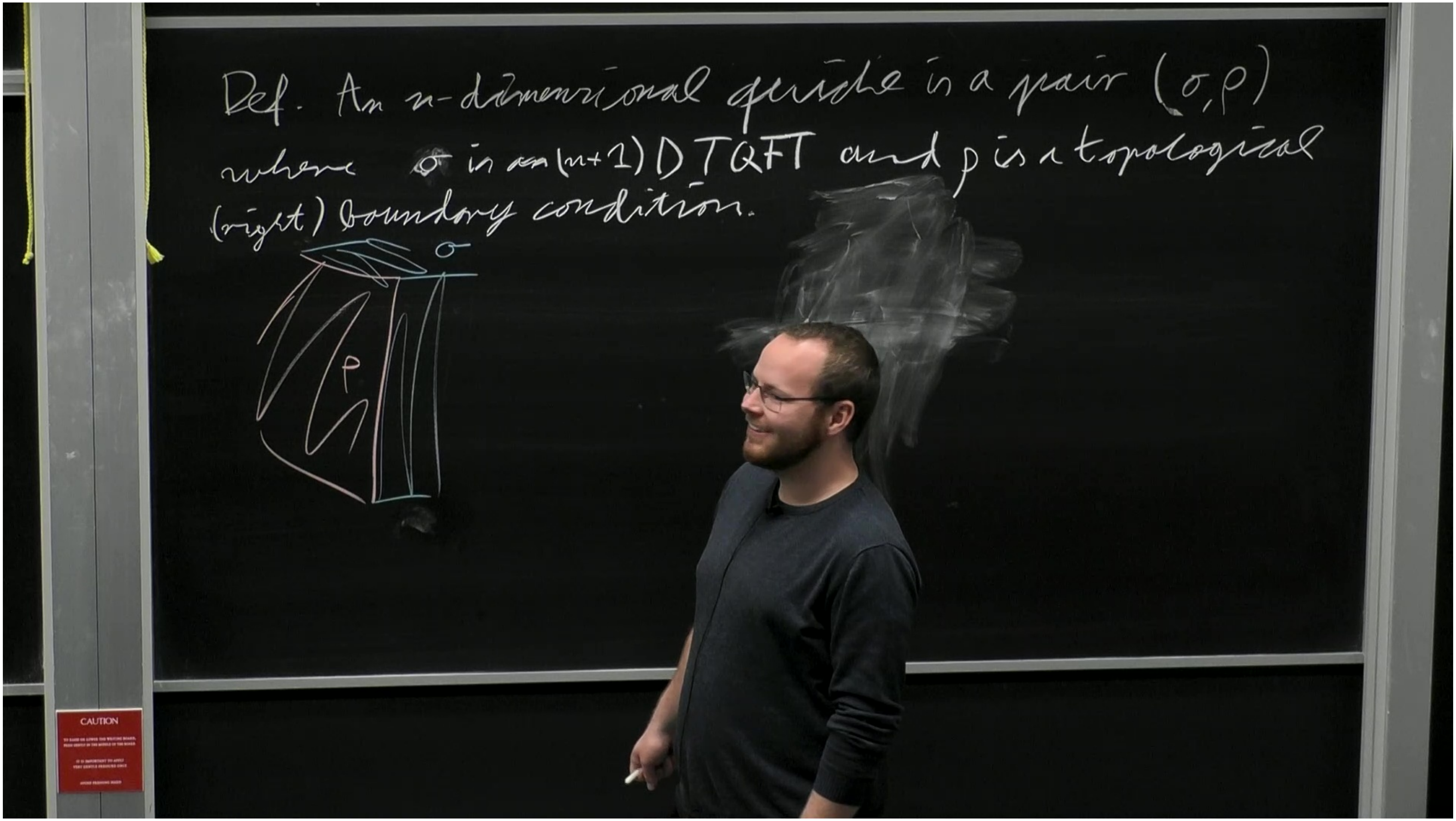
A an algebra. $R = A_A$ regular right module

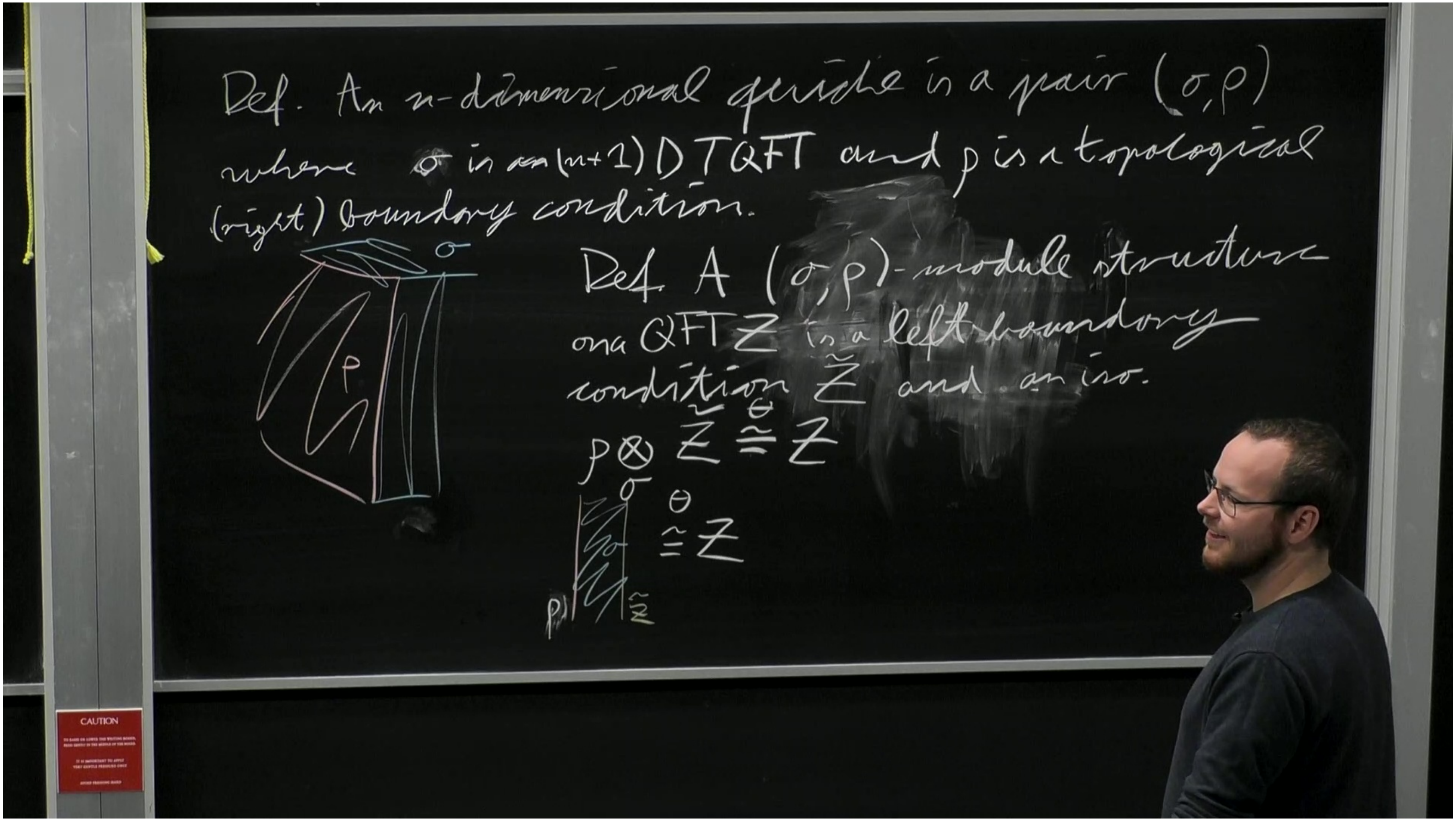
(A, R) is an abstract symmetry type.

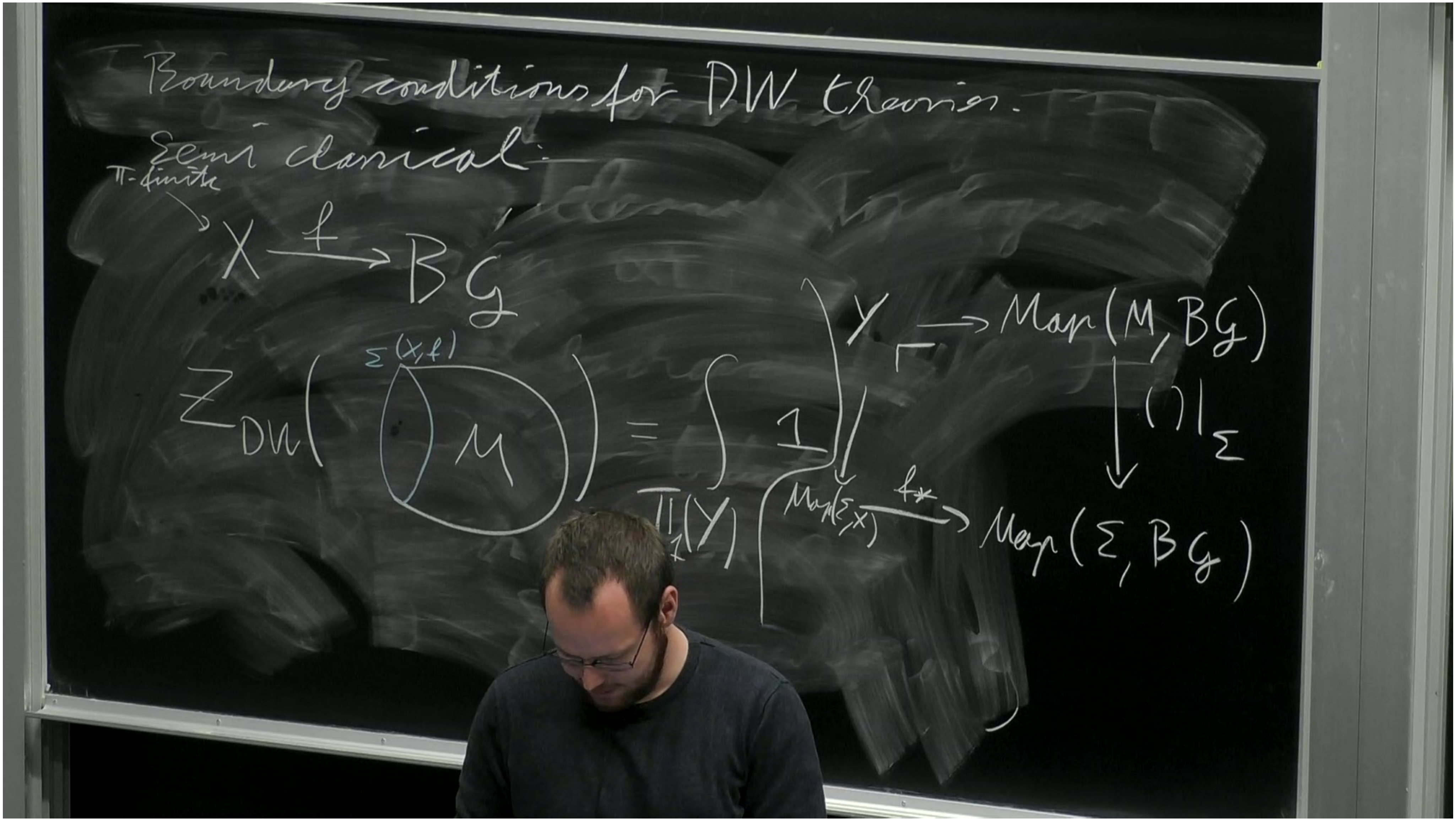
(A, R) acts on V: left A module L and $R \underset{A}{\otimes} L \cong V$

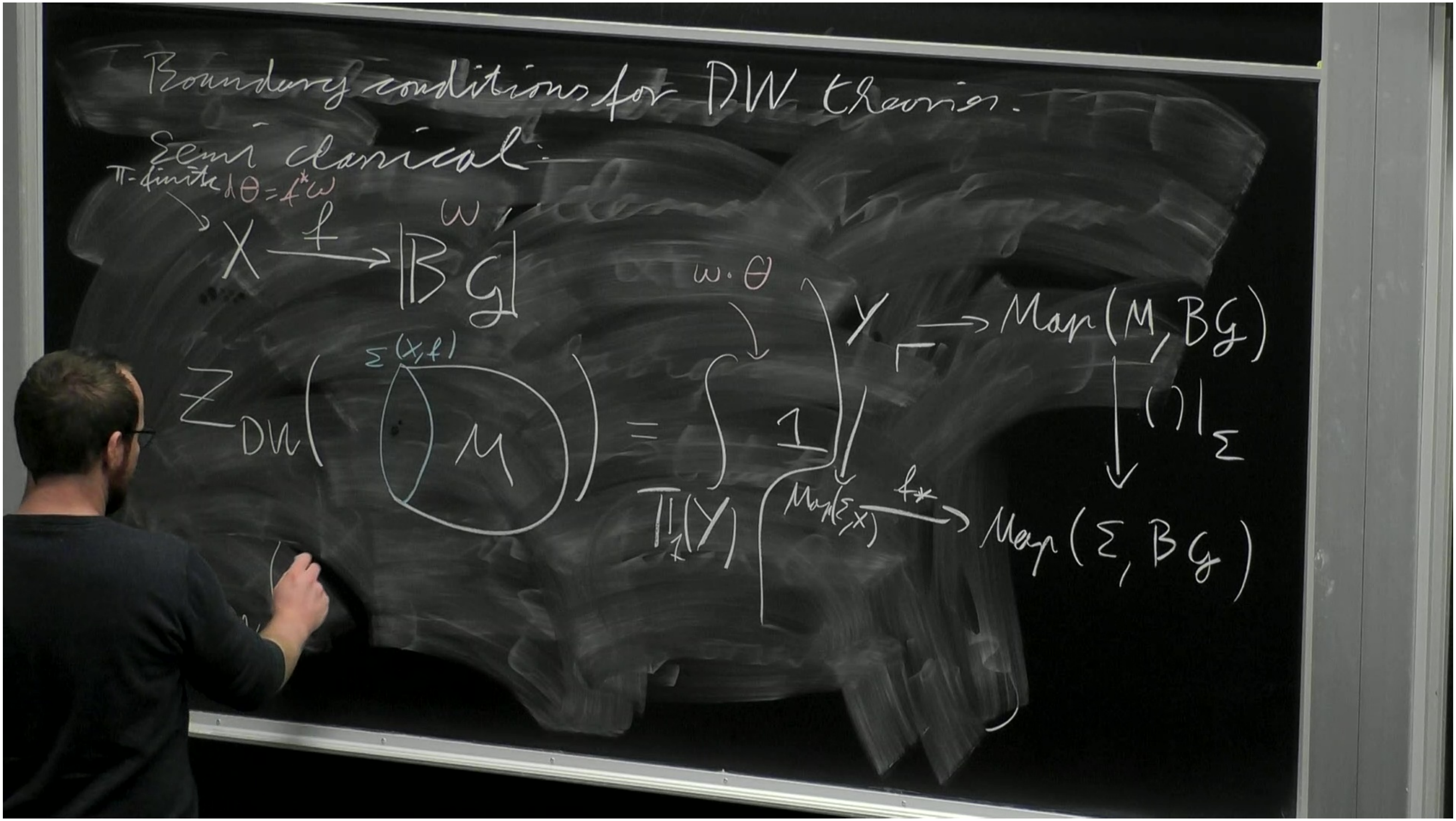
$R \quad A \quad L \quad \otimes \quad V$

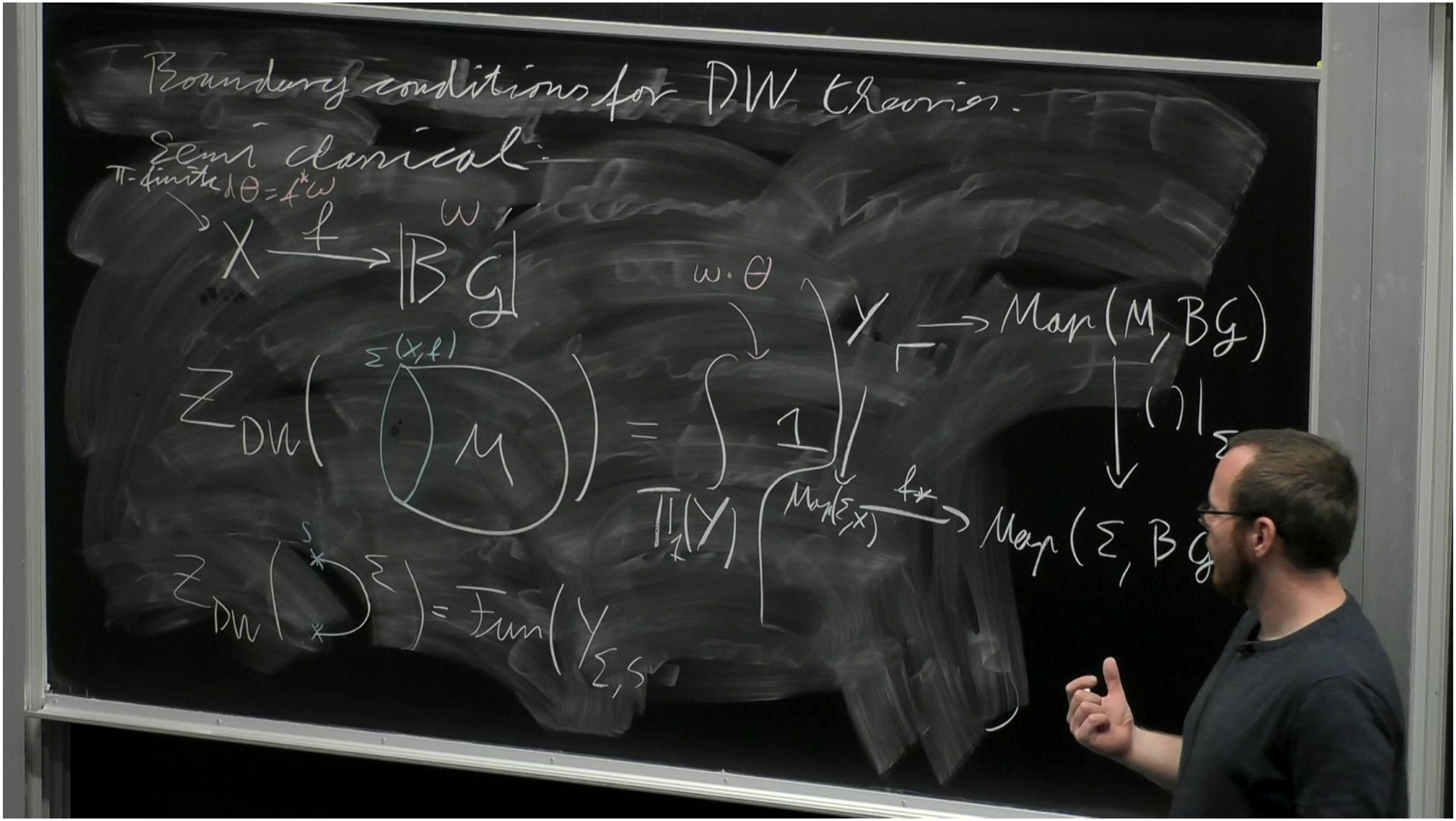
$\cong$


Def. An n-dimensional quiche is a pair (σ, ρ)
where σ is an (n+1)D TQFT and ρ is a topological
(right) boundary condition.
σ
ρ
CAUTION

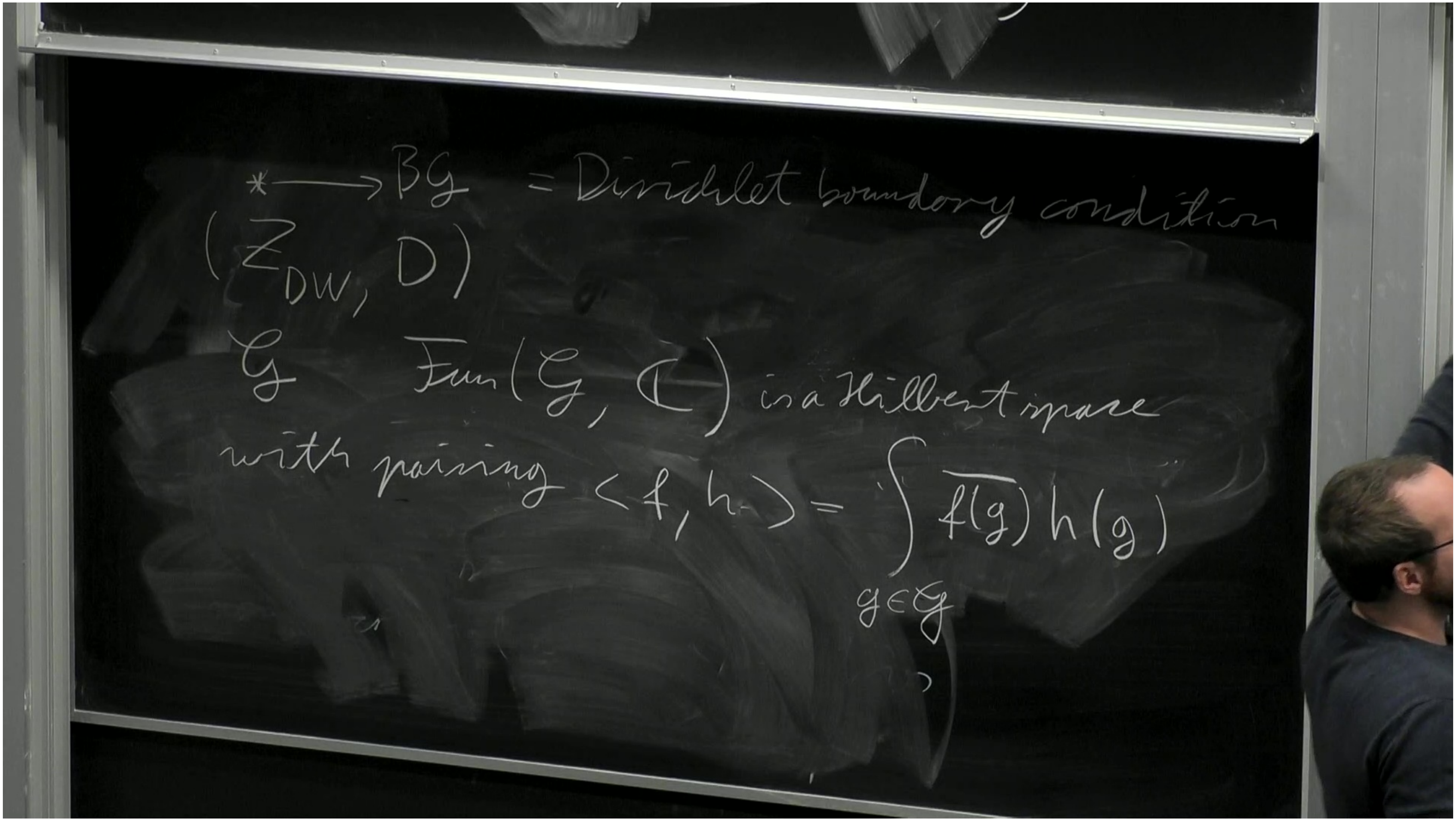

Def. An n-dimensional quiche is a pair (σ, ρ)
where σ is an $(n+1)$D TQFT and ρ is a topological
(right) boundary condition.
Def. A (σ,ρ)-module structure
on a QFT Z is a left boundary
condition $\tilde{Z}$ and an iso.
$\rho \otimes_{\sigma} \tilde{Z} \overset{\theta}{\cong} Z$
σ
ρ
$\tilde{Z}$
$\overset{\theta}{\cong} Z$

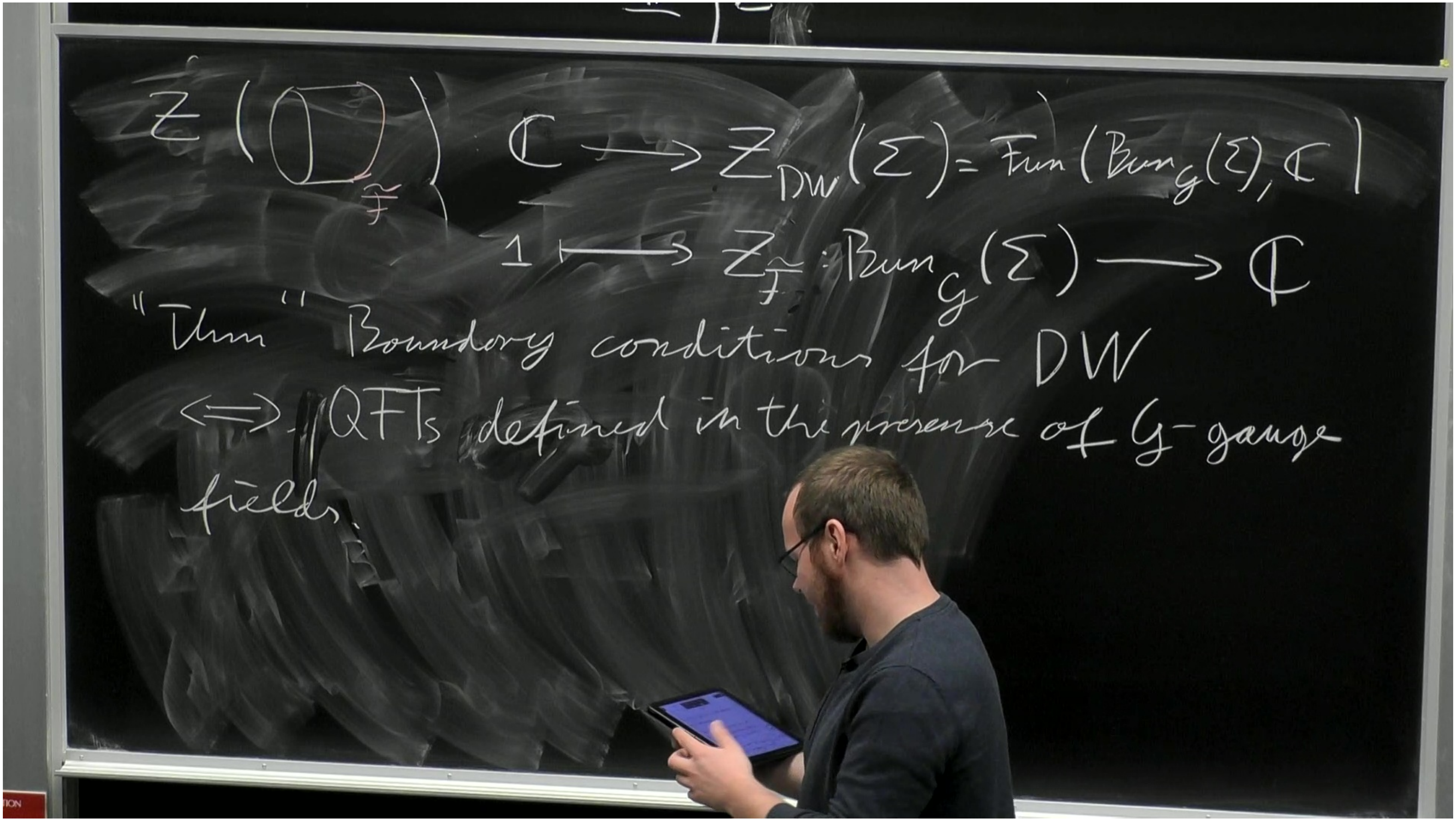

Boundary conditions for DW theories.
Semi classical:
π-finite
X → BG
f
Z_DW
Σ(X,f)
M
= ∫ 1
π_1(Y)
Y
Map(M, BG)
( )|_Σ
Map(Σ, X)
f_*
Map(Σ, BG)


Boundary conditions for DW theories.
Semi classical:
π-finite
dθ = f*ω
X
f
|BG|
ω
ω · θ
Y
Map(M, BG)
Z_DW
Σ(X,f)
M
=
π_1(Y)
1
( )|_Σ
Map(Σ, X)
f_*
Map(Σ, BG)

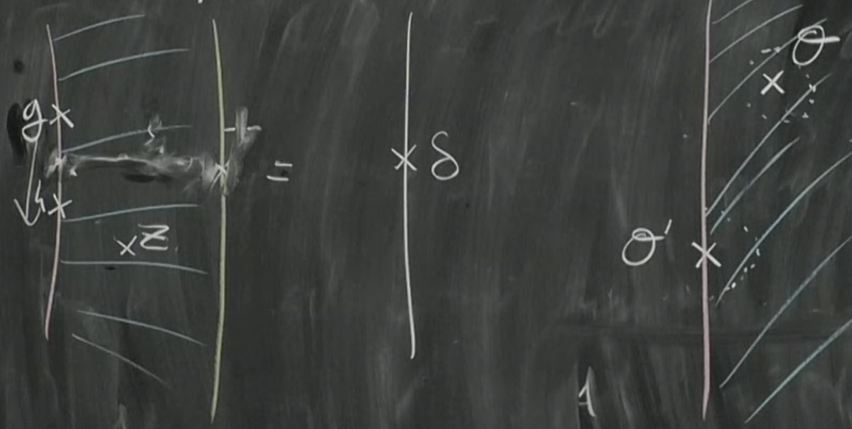


Boundary conditions for DW theories.
Semi classical:
π-finite
dθ = f*ω
X
f
|BG|
ω
ω · θ
Z_DW
Σ(X,f)
M
Y
Map(M, BG)
( )|_Σ
Π_1(Y)
Map(Σ,X)
f_*
Map(Σ, BG
Fun(Y_Σ,S


$* \longrightarrow BG$ = Dirichlet boundary condition
(Z_{DW}, D)
$\mathcal{G}$ $\mathrm{Fun}(\mathcal{G}, \mathbb{C})$ is a Hilbert space
with pairing $\langle f, h \rangle = \int_{g \in \mathcal{G}} \overline{f(g)}\, h(g)$

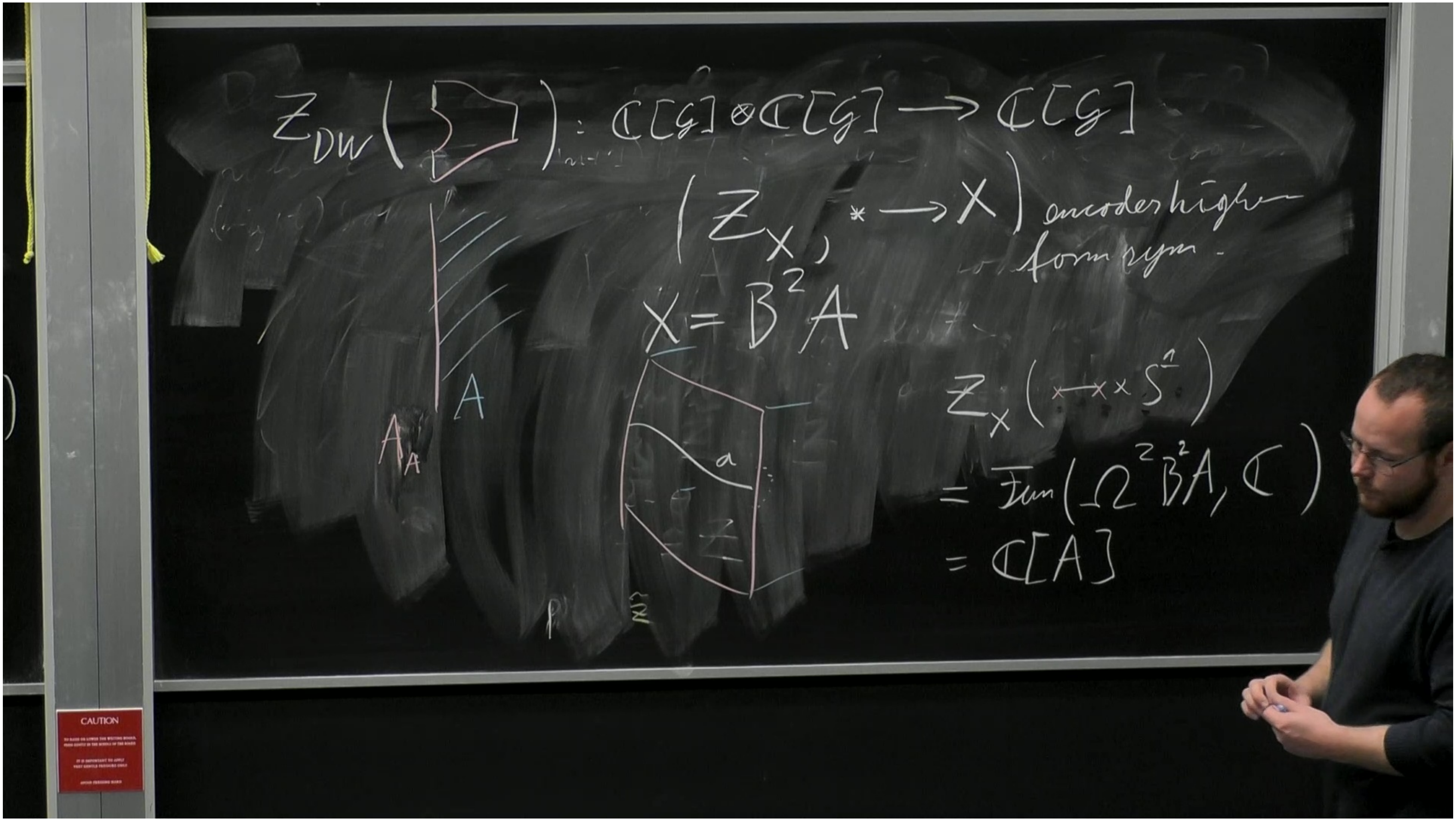

Z( )
ℂ → Z_DW(Σ) = Fun(Bun_G(Σ), ℂ)
1 ↦ Z_F̃ : Bun_G(Σ) → ℂ
"Thm" Boundary conditions for DW
⟺ QFTs defined in the presence of G-gauge fields.

Cor. 1D QM with G symmetry is a $\left(Z_{DW}, * \xrightarrow{i} BG\right)$-module.



$$\mathcal{O} \in Z_{DW}(S^1) = Z(\mathbb{C}[G])$$

$$\mathcal{O}' \in Z_{DW}(\times\!\!\longrightarrow\!\!\times) = \mathrm{Fun}(Y, \mathbb{C}) = \mathbb{C}[G]$$



$Z_{DW}(\;\;) : \mathbb{C}[G] \otimes \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$
$(Z_X, * \longrightarrow X)$ encodes higher form sym.
$X = B^2 A$
A
A_A
a
$Z_X(\times S^n)$
$= \mathrm{Fun}(\Omega^2 B^2 A, \mathbb{C})$
$= \mathbb{C}[A]$
CAUTION