

Title: Topological Quantum Field Theories Lecture 20231201

Speakers: Lukas Mueller

Collection: Topological Quantum Field Theories - mini-course

Date: December 01, 2023 - 2:00 PM

URL: <https://pirsa.org/23120016>

5) Generalized symmetries and SymTFT
Freed, Moore, Teleman: Topological
symmetries in QFT.

- non-topological background fields \mathcal{F}
 $Z_{\text{QFT}} \cdot \text{Bor } \mathcal{F} \longrightarrow \text{tVect}$

symmetries in QFT.

- non-topological background fields F

$$Z_{\text{QFT}} \cdot \text{Bord}_n^F \longrightarrow \text{tVect}$$

e.g. 1 D with rotated QM

$n=1$

$$F = \{g_{\mu\nu}, \text{orientation}\}$$

$$Z_{QM}(pt_+) = \mathcal{H}$$

$$Z([0, t]) = e^{-\int_0^t H dt}$$

$$Z_{QM}(pt_-) = \overline{\mathcal{H}}$$

$$Z_{QM}(S_t) = \text{Tr}_{\mathcal{H}}(e^{-tH/t_h})$$

Symmetries:

$$BG \longrightarrow [\text{Bord}_1^{\mathbb{F}}, (\text{Vect}_{\mathbb{C}})]_{RP}$$

$$G \xrightarrow{p} U(\mathcal{H}), \text{ s.t. } [p(g), H]$$

$$Z_{QM}(pt_+) = \mathcal{H}$$

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$$Z_{QM}(S_t) = \text{Tr}_{\mathcal{H}}(e^{-tH/t_h})$$

Symmetries:

$$BG \longrightarrow [\text{Bord}_1^{\mathbb{F}}, \text{Vect}_{\mathbb{C}}]_{RP}$$

$$G \xrightarrow{p} U(\mathcal{H}), \text{ s.t. } [p(g), H] = 0$$

$$Z_{QM}^g(\text{---} \underset{x}{g}) = e^{-tH/\hbar} p(g) \quad Z_{QM}^g(\text{---} \underset{x}{g} \text{---} \underset{x}{h}) = Z_{QM}^g(\text{---} \underset{x}{hg})$$

$$Z_{QM}^g(\text{O} \underset{x}{g}) = \text{Tr}_x \left(e^{-tH/\hbar} p(g) \right)$$

$$= Z_{QM}^g(\text{O} \underset{x}{g} \text{---} \underset{x}{h} \text{---} \underset{x}{g})$$

$$Z_{QM}^g(\bigcirc_{*g}) = \text{Tr}_x \left(e^{-tH/x} p(g) \right)$$

$$Z_{QM}(\bigcirc_{*h^{-1}hg}) = Z_{QM}(\bigcirc_{*hgh^{-1}})$$



$$Z_{QM}(\text{circle} * g) = \text{tr}_x(e^{-p(g)})$$

$$\parallel$$

$$Z_{QM}(\text{circle} * h^{-1} * g * h) = Z_{QM}(\text{circle} * h g h^{-1})$$

$$Z_{QM}^G : \text{Bord}_1^{F, G} \longrightarrow \text{Vect}$$

$$(\text{Ansatz: } p(g) p(h) = w(g, h) p(g \cdot h))$$

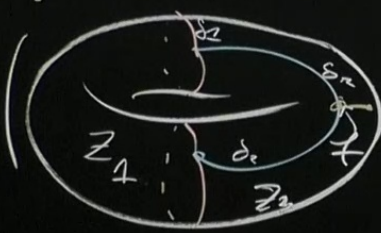


$$Z_{\text{QFT}} \left(\begin{array}{c} \text{[Diagram of a cylinder with a vertical dashed line and a parameter } g \text{]} \\ \Sigma \quad \Sigma \end{array} \right) = e^{-t H_{\text{QFT}} / \hbar} P(g)$$

Defects in Σ

$$QFT \left(\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) \right) = e^{P(g)}$$

Defects in QFT Σ

$$Z \left(\text{Diagram} \right) \in \mathbb{C}$$


Defects in QFT

$$Z(\text{Diagram}) \in \mathbb{C}$$

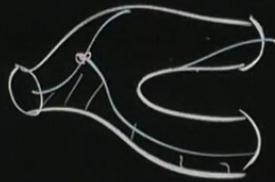
Examples -

- Wilson lines
- 't Hooft lines
- boundary condition

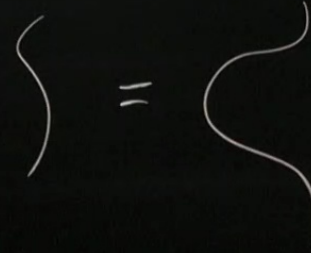
$$\mathbb{1} \left. \vphantom{\mathbb{1}} \right\} Z$$

Functorial description

$$Z_{\text{QAT}}^{\text{def}} : \text{Bord}_n^{\mathcal{F}, \text{def}} \longrightarrow \text{Vect}$$

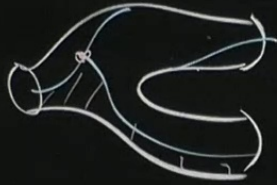


Def. A defect in topological \mathcal{A}

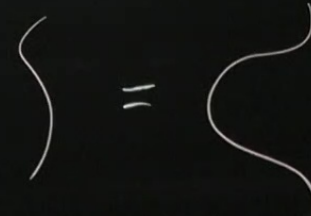


CAUTION
DO NOT TOUCH THE WRITING SURFACE
PLEASE REPORT TO THE STAFF IF THE BOARD IS DAMAGED
OR IF YOU NEED TO REPORT ANY OTHER PROBLEMS
PLEASE REPORT TO THE STAFF

$$\mathcal{Z}_{\text{QFT}} = \text{Zork}_m \longrightarrow \text{C. V. ...}$$



Def. A defect is topological
 \mathcal{A}



Example:

Let \mathcal{Z}_{QFT} be a field theory with $U(1)$ -symmetry.

$$\bar{j}_1 \in \Omega^1(M) \quad Q = \int_{\mathbb{R}^{n-2}} * \bar{j} \quad \begin{cases} d(*\bar{j}) = 0 \\ \mathbb{R}^{n-2} \times [0,1] \end{cases}$$



CAUTION
 DO NOT STAND ON CHALKBOARD
 IT IS UNSAFE TO STAND ON CHALKBOARD
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$$\mathbb{R}^{n-1}$$

$$\mathbb{R} \times [0, 1]$$

$\int_{M^{n-1}} * \bar{j}_1$ is a topological operator.

$$\exp(2\pi i \alpha \left(\int_{M^{n-1}} * \bar{j}_1 \right))$$

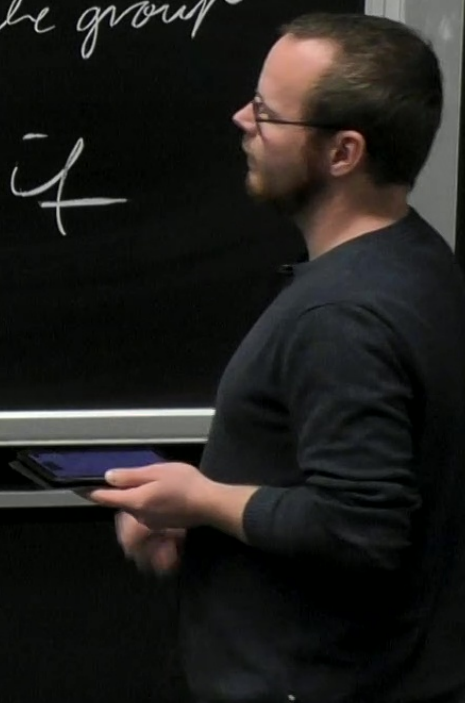
$$\left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right) \rightarrow \left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right) = \left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right) = \left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right)$$

$U(1)$ -symmetry \Rightarrow codim-1 defects whose fusion implements the group structure

CAUTION
DO NOT TOUCH THE WRITING BOARD.
READ INSTRUCTIONS ON THE BOARD.
IT IS PROHIBITED TO LEAVE
YOUR OWNERSHIP MARKS.
PLEASE RESPECT THIS!

$\int_{M^{n-1}} * \tilde{j}_2$ is a topological operator.
 $\exp(2\pi i \alpha \left(\int_{M^{n-1}} * \tilde{j}_2 \right)) \rightarrow \left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right) = \left(\begin{array}{c} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{array} \right) = \dots$
 $U(1)$ -symmetry \Rightarrow codim=1 defects whose fusion implements the group structure

Def A topological defect is invertible if
 $\exists D^{-1}$ s.t. $\left(\begin{array}{c} D \\ D^{-1} \\ 1 \end{array} \right) = \left(\begin{array}{c} D^{-1} \\ D \\ 1 \end{array} \right) = \left(\begin{array}{c} 1 \\ D \\ D \end{array} \right) = \left(\begin{array}{c} 1 \\ 1 \\ D \end{array} \right)$



CAUTION
 DO NOT TOUCH THE BOARD SURFACE
 WITH OBJECTS OR YOUR HANDS
 TO AVOID DAMAGE TO THE BOARD
 AND YOURSELF
 THANK YOU

$\{g_{\mu\nu}, \text{orientation}\}$

Beline ("Theorem" for QFTs)

$\{\text{Symmetries}\} \Leftrightarrow \{\text{invertible codim-1 defects}\}$

$$\text{circle with } \gamma \text{ and } * \text{ inside} \stackrel{g}{=} * \text{ on } \gamma \quad p(g) := \int_{\tau} \text{square with } \gamma \text{ and } \tau \text{ labels}$$

Generalized symmetries
higher form symmetries
- non-invertible (categorical) symmetries

Assume all symmetries are finite.

$G \rightsquigarrow \llbracket G \rrbracket$ abstract symmetries
 $G\text{-rep} \rightsquigarrow \llbracket G \rrbracket\text{-modules}$ concrete realizations.

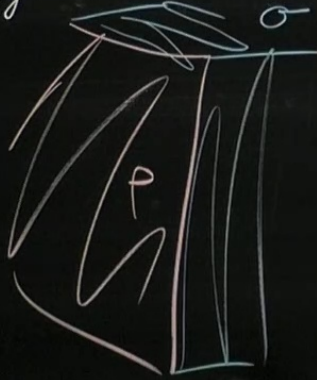
A an algebra, $R = A_A$ regular right module

(A, R) is an abstract symmetry type.

(A, R) acts on V : let A module L and $R \otimes_A L \cong V$

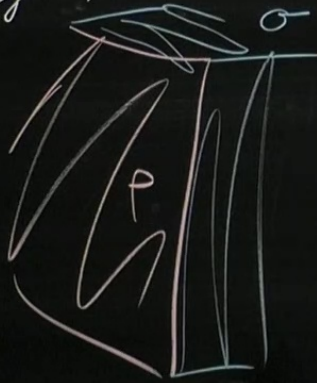
$$\begin{array}{ccc} & R & A \\ \downarrow & \times & \downarrow \\ & L & \otimes & V \\ & \downarrow & \downarrow & \downarrow \\ & & \cong & V \end{array}$$

Def. An n -dimensional gerbe is a pair (σ, ρ)
where σ is an $(n+1)$ -D TQFT and ρ is a topological
(right) boundary condition.



CAUTION
DO NOT TOUCH THE BOARD SURFACE
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Def. An n -dimensional gerbe is a pair (σ, ρ) where σ is an $(n+1)$ -D TQFT and ρ is a topological (right) boundary condition.



Def. A (σ, ρ) -module structure on a QFT \mathcal{Z} is a left boundary condition $\tilde{\mathcal{Z}}$ and an iso.

$$\rho \otimes \tilde{\mathcal{Z}} \cong \mathcal{Z}$$

$$\begin{array}{c} \sigma \\ \downarrow \\ \rho \otimes \tilde{\mathcal{Z}} \\ \cong \\ \mathcal{Z} \end{array}$$



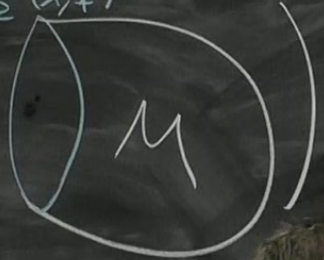
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 PLEASE CONTACT THE STAFF
 IF YOU NEED TO USE IT

Boundary conditions for DW theories.

Semi classical:

π -finite

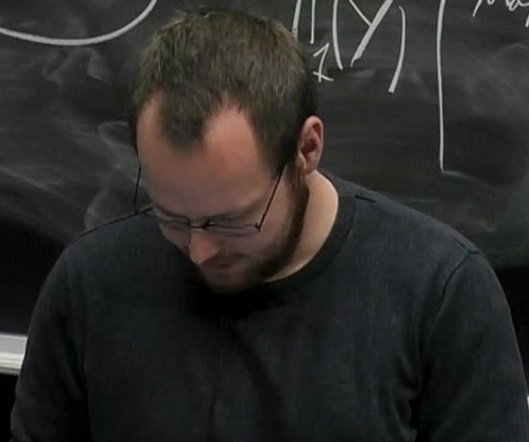
$$X \xrightarrow{f} BG$$

$$Z_{DW}(\Sigma^{(X,f)}, M) = \int \mathbb{1}$$


$$Y \xrightarrow{f} \text{Map}(M, BG)$$

$$\mathbb{1} \downarrow \text{Map}(\Sigma, X)$$

$$\text{Map}(\Sigma, BG)$$



Boundary conditions for DW theories.

Semi-classical:

π -finite $\downarrow \theta = \mathcal{F}^* \omega$

$$X \xrightarrow{\mathcal{F}} B\mathcal{G}$$

$$Z_{DW}(\Sigma(x, \mathcal{F}), M)$$

$$\int_{\mathbb{T}_1(Y)}$$

$$Y \xrightarrow{\Gamma} \text{Map}(M, B\mathcal{G})$$

$$\downarrow \int_{\text{Map}(\Sigma, X)}$$

$$\text{Map}(\Sigma, B\mathcal{G})$$

Boundary conditions for DW theories.

Semi-classical:

π -finite $\downarrow \theta = A^* \omega$

$$X \xrightarrow{f} B\mathbb{G}$$

$$Z_{DW}(\Sigma(x, \ell), M) = \int_{\mathbb{Z}(Y)}$$

$$Z_{DW}(\Sigma(x, s), \Sigma) = \text{Fun}(Y_{\Sigma, s})$$

$\omega \cdot \theta$

$$Y \xrightarrow{\Gamma} \text{Map}(M, B\mathbb{G})$$

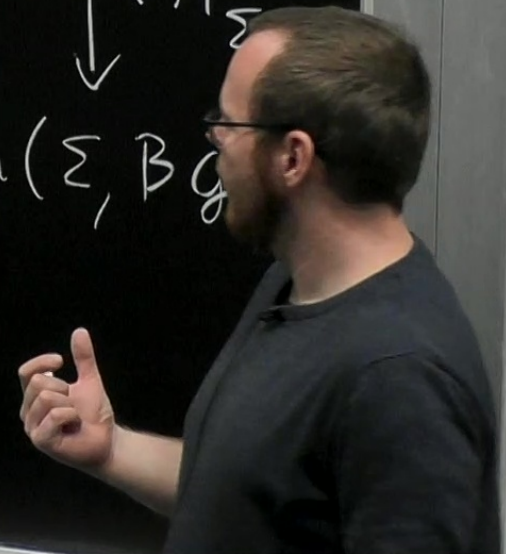
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$$\text{Map}(\Sigma, X)$$

f^*

$$\text{Map}(\Sigma, B\mathbb{G})$$

$(\Gamma)_{\Sigma}$



* \rightarrow B_G = Dirichlet boundary condition
(Z_{DW}, D)

\mathcal{G} $\text{Fun}(\mathcal{G}, \mathbb{C})$ is a Hilbert space
with pairing $\langle f, h \rangle = \int_{g \in \mathcal{G}} \overline{f(g)} h(g)$

$$Z(\mathbb{D})$$

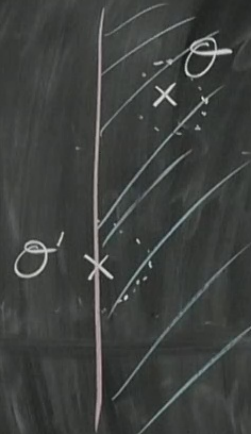
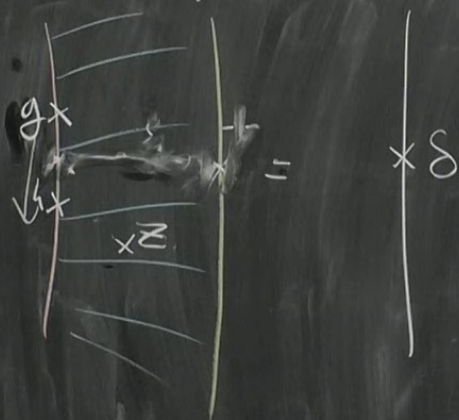
$$\mathbb{C} \longrightarrow Z_{\text{DW}}(\Sigma) = \text{Fun}(\text{Bun}_G(\Sigma), \mathbb{C})$$

$$1 \longmapsto Z_{\mathbb{F}} : \text{Bun}_G(\Sigma) \longrightarrow \mathbb{C}$$

"Thin" Boundary conditions for DW

\Leftrightarrow QFTs defined in the presence of G -gauge fields

Cor. 1D QM with G symmetry is a
 $(Z_{DW}, * \xrightarrow{B_G})$ -module.



$$\theta \in Z_{DW}(S^1)$$

$$\cong Z(\mathbb{C}[G])$$

$$\theta' \in Z_{DW}(x \rightarrow x) = \text{Fun}(Y, \mathbb{C}) = \mathbb{C}[G]$$

$$Z_{DW}(\cdot) : \mathbb{C}[G] \otimes \mathbb{C}[G] \rightarrow \mathbb{C}[G]$$

$(Z_X, * \rightarrow X)$ encodes high form sym.

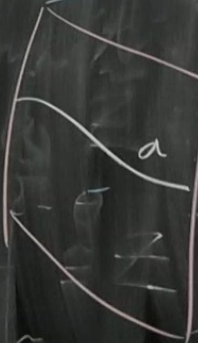
$$X = B^2 A$$

$$Z_X(x \rightarrow x S^1)$$

$$= \text{Fun}(\Omega^2 B^2 A, \mathbb{C})$$

$$= \mathbb{C}[A]$$

A
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CAUTION
NO BURNING OR OTHER HOT SURFACES
OR FLAMMABLE LIQUIDS
OR GASES PERMITTED