Title: Statistical Physics Lecture - 120523

Speakers: Emilie Huffman

Collection: Statistical Physics 2023/24

Date: December 05, 2023 - 10:45 AM

URL: https://pirsa.org/23120013

$$Z \simeq \frac{1}{(\pi V)^{N/2}} \int^{\Lambda} D^N\varphi \; e^{-S(\varphi)}$$

$$S(\varphi) \simeq \frac{1}{2V} \sum_{k} (r+k^2)\,|\varphi_k|^2 - h\,\varphi_0$$

$$+ \frac{u}{4!\,V^3} \sum_{k_1,k_2,k_3,k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}$$

$$Z \simeq \frac{1}{(\pi V)^{N/2}} \int^{\Lambda} \mathcal{D}^N\varphi \; e^{-S(\varphi)}$$

$$S(\varphi) \simeq \overbrace{\frac{1}{2V} \sum_k (r+k^2)\,|\varphi_k|^2 - h\,\varphi_0}^{S_0}$$

$$+ \overbrace{\frac{u}{4!\,V^3} \sum_{k_1,k_2,k_3,k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}}^{S_I}$$

Let's rewrite Z as $S_0(\varphi^-) + S_0(\varphi^+)$

$$Z = \frac{1}{(\pi v)^{N/2}} \int^{\Lambda} \mathcal{D}^N \varphi \; e^{-S_0(\varphi) - S_I(\varphi)}$$

We will set $H = 0 \rightarrow$ don't need it to compute η and ν.

$$\int^{\Lambda} \mathcal{D}^N \varphi^- \; e^{-S'(\varphi^-)}$$

Focus on:

$$\underbrace{e^{-S_0(\varphi^-)} \int \hat{D}^N \varphi^+ \, e^{-S_0(\varphi^+)} \, e^{-S_I(\varphi^- + \varphi^+)}}_{e^{-S'(\varphi^-)}}$$

$$\propto e^{-S_0(\varphi^-)} \left(\frac{\int \hat{D}^N \varphi^+ \, e^{-S_0(\varphi^+)} \, e^{-S_I(\varphi^- + \varphi^+)}}{\int \hat{D}^N \varphi^+ \, e^{-S_0(\varphi^+)}} \right) \quad \leftarrow \text{avg. with } e^{-S_0(\varphi^+)} \text{ as prob. dist.}$$

$$= e^{-S_0(\varphi^-)} \left\langle e^{-S_I(\varphi^- + \varphi^+)} \right\rangle_+$$

This allows us to use the cumulant expansion.

$$S'(\varphi^-) \simeq -\ln\left(e^{-S_0(\varphi^-)} \left\langle e^{-S_I(\varphi^- + \varphi^+)} \right\rangle_+\right)$$

$$\ln(1+x) \simeq x - \frac{x^2}{2}$$

$$e^{-S_I(\varphi^- + \varphi^+)} \simeq 1 - S_I + \frac{S_I^2}{2}$$

$$\ln\Big(\big\langle 1 \underbrace{- S_I + \frac{S_I^2}{2}}_{x} \big\rangle_+\Big) \simeq \left\langle -S_I + \frac{S_I^2}{2} \right\rangle_+ - \frac{\langle S_I \rangle_+^2}{2}$$

$$\ln\langle e^{-S_I}\rangle_+ = -\langle S_I\rangle_+ + \frac{1}{2}\left(\langle S_I^2\rangle_+ - \langle S_I\rangle_+^2\right)$$

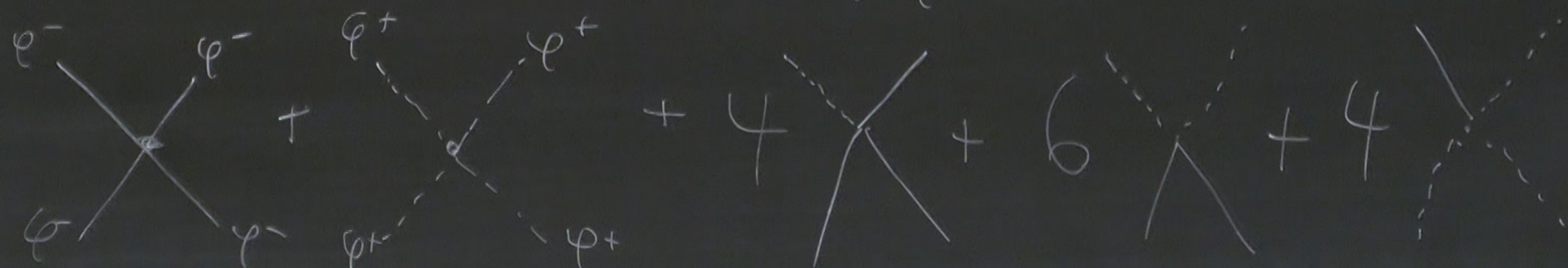
So we'll get terms like $\langle \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\rangle_+$

$$= \langle \varphi_{k_1}\varphi_{k_2}\rangle\langle\varphi_{k_3}\varphi_{k_4}\rangle + \langle\varphi_{k_1}\varphi_{k_3}\rangle\langle\varphi_{k_2}\varphi_{k_4}\rangle + \dots$$

Wick's Theorem

$$S_I(\varphi^- + \varphi^+) = \frac{u}{4!\,V^3}\sum_{k_1k_2k_3k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}$$

$\varphi^+\varphi^+\varphi^-\varphi^- \quad \varphi^+\varphi^-\varphi^+\varphi^- \quad \varphi^+\varphi^-\varphi^-\varphi^-$

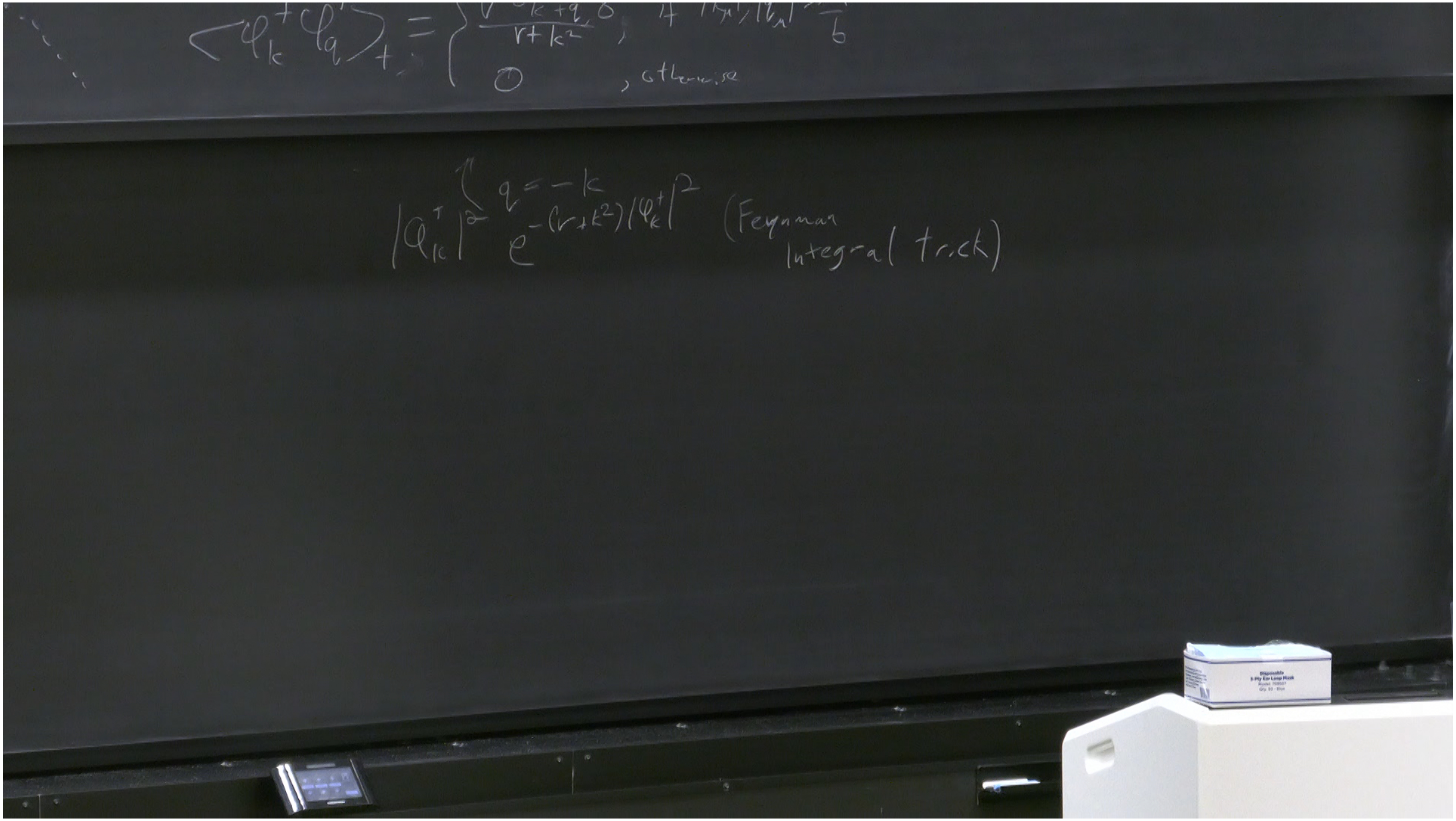


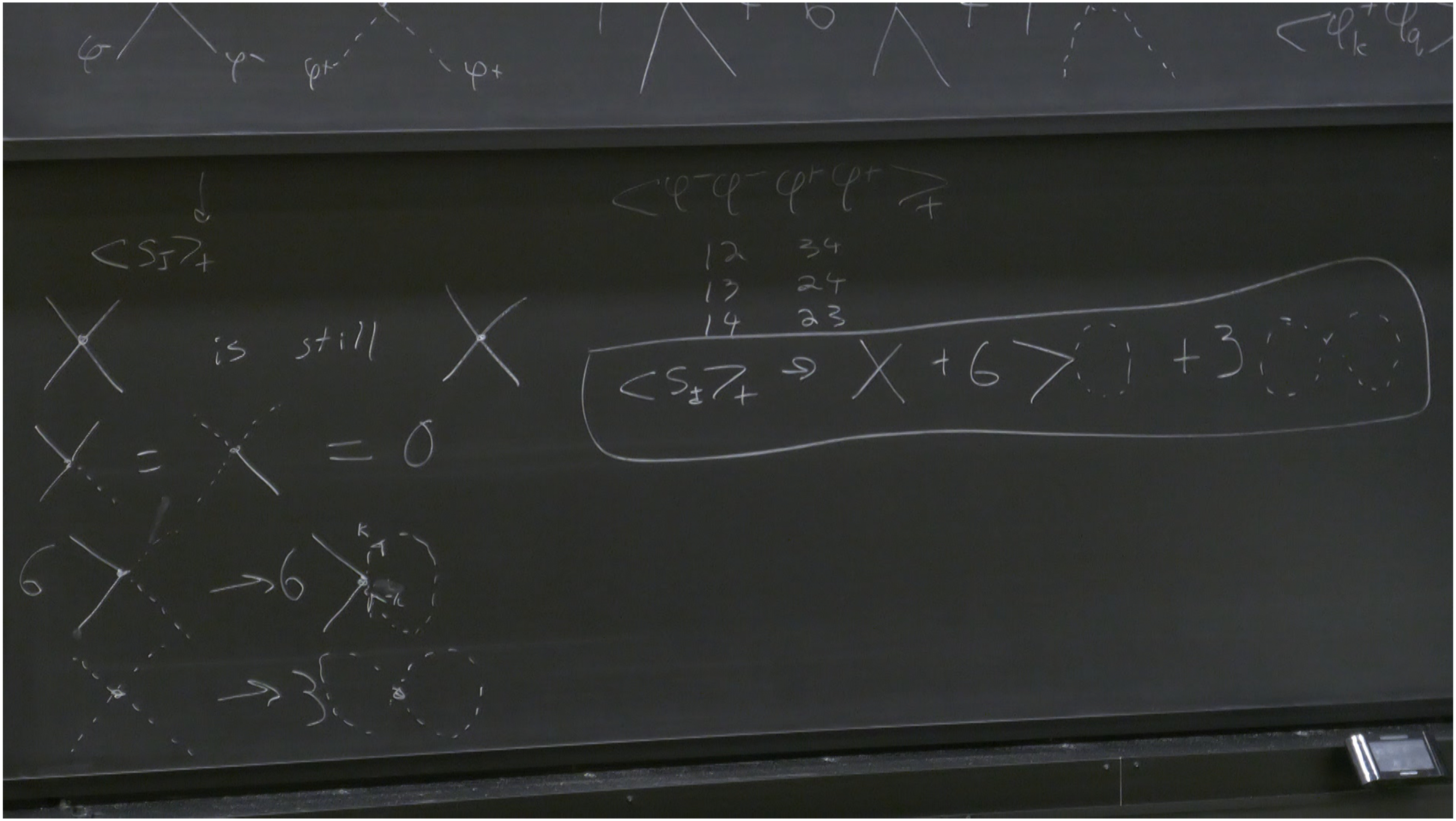
$$\varphi_{k_1} = \varphi^+_{k_1} + \varphi^-_{k_1}$$

Rules

- each solid external leg carries a φ^-_k
- each dashed external leg carries a φ^+_k
- each vertex involves a $\frac{u}{4!\,V^3}\,\delta_{k_1+k_2+k_3+k_4,\,0}$
- each internal line carries a $\langle \varphi^+_k \varphi^+_q \rangle_+$, can be computed

$$\langle \varphi^+_k \varphi^+_q \rangle_+ = \begin{cases} \dfrac{V\,\delta_{k+q,\,0}}{r+k^2}, & \text{if } |k_\alpha|, |q_\alpha| > \dfrac{\Lambda}{b} \\ 0, & \text{otherwise} \end{cases}$$


, otherwise
q = -k
$|\varphi_k^\dagger|^2 e^{-(r+k^2)|\varphi_k^\dagger|^2}$
(Feynman integral trick)

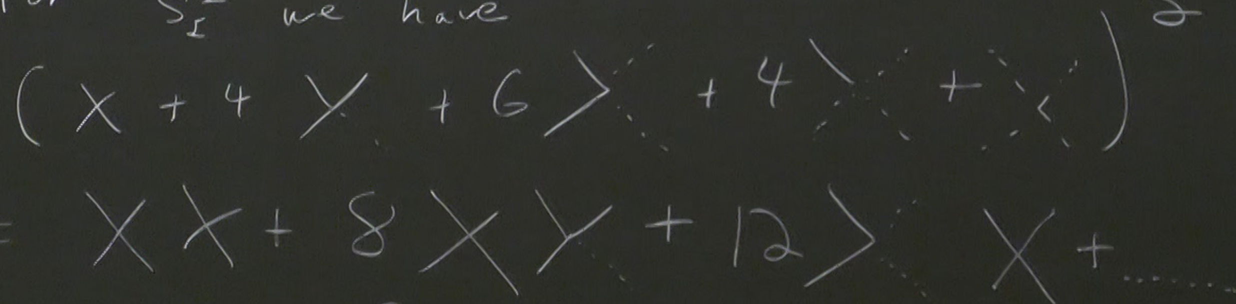


$\langle \varphi^- \varphi^- \varphi^+ \varphi^+ \rangle_+$
12 34
13 24
14 23
$\langle S_I \rangle_+$
is still
$= 0$
$\langle S_I \rangle_+ \to$ + 6 + 3

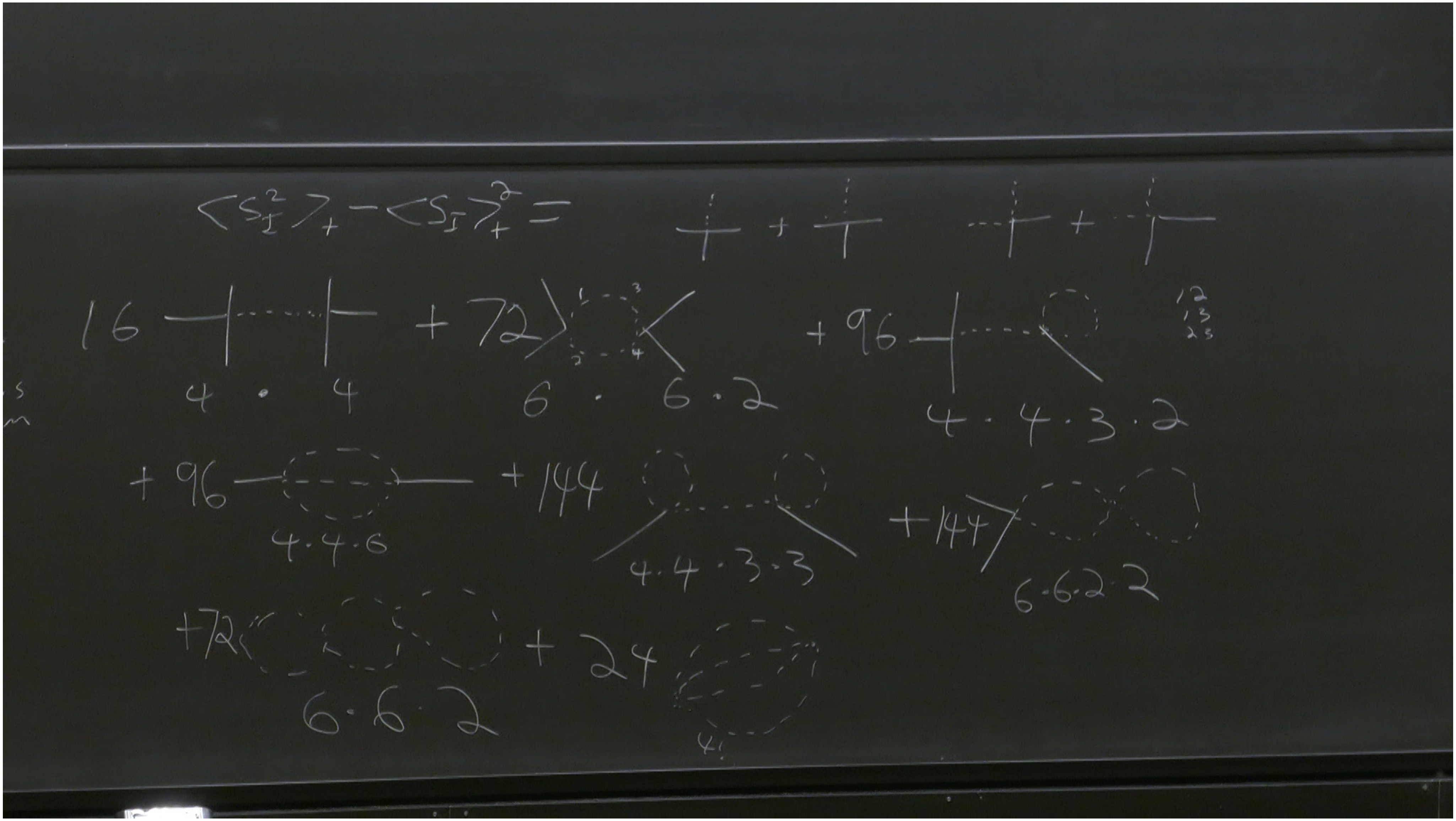
What about about $\langle S_I^2 \rangle - \langle S_I \rangle_+^2$

we will remove all disconnected diagrams with this term

For S_I^2 we have

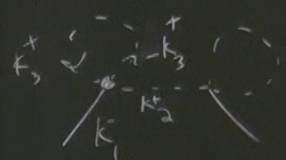


$- \langle S_I^2 \rangle_+ - \langle S_I \rangle_+^2$ is only the connected diagrams for $\langle S_I^2 \rangle$


$\langle S_I^2\rangle_+ - \langle S_I\rangle_+^2 =$
16
4 · 4
+ 72
6 · 6 · 2
+ 96
4 · 4 · 3 · 2
12
13
23
+ 96
4 · 4 · 6
+ 144
4 · 4 · 3 · 3
+ 144
6 · 6 · 2 · 2
+ 72
6 · 6 · 2
+ 24
4!

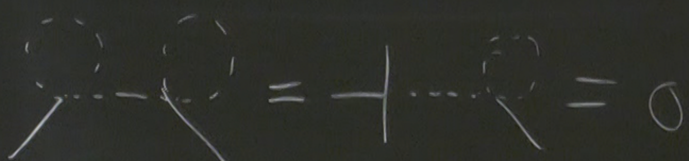
Now .... is for $k \equiv |k_\mu^+| > \frac{\Lambda}{b}$

——— is for $|k_\mu^-| < \frac{\Lambda}{b}$

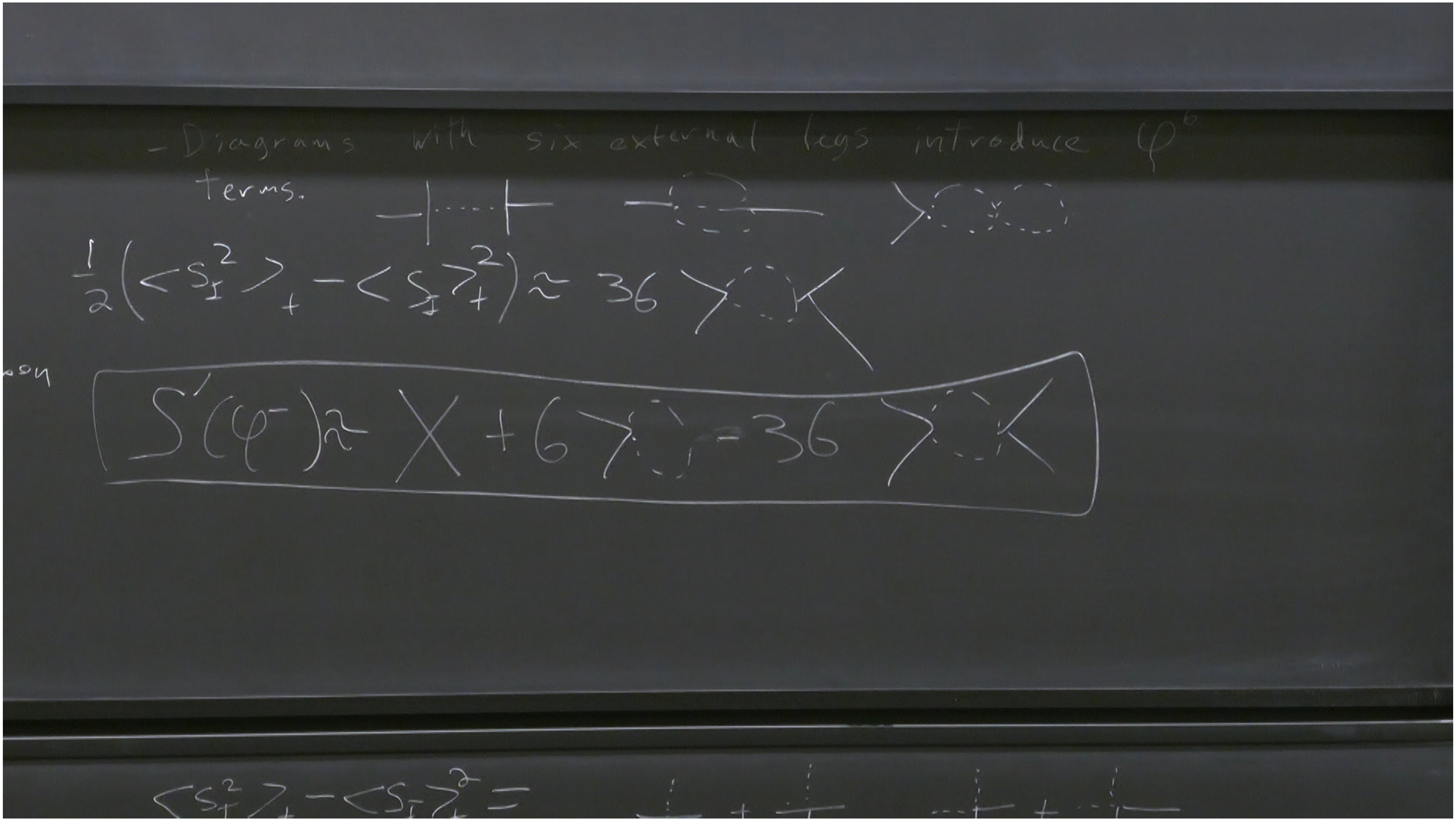


involves two vertices with $k_1^- - k_2^+ + \underbrace{k_3^+ - k_3^+}_{0} = 0$

$k_1^- - k_2^+ = 0$ ✗ can't happen



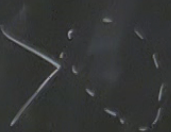
We will also drop diagrams with no external legs — no φ^--dependence


- Diagrams with six external legs introduce φ^6 terms.
1/2 (<S_I^2>_+ − <S_I>_+^2) ≈ 36
S'(φ^-)^2 + 6 − 36
<S_I^2>_+ − <S_I>_+^2 =

Computing the Diagrams:

$$= \sum_{\substack{k_1 k_2 \\ k_3 k_4}} \varphi^-_{k_1} \varphi^-_{k_2} \varphi^-_{k_3} \varphi^-_{k_4} \frac{u}{4!\,V^3} \delta_{k_1+k_2+k_3+k_4,0}$$

$$\langle \varphi^+_k \varphi^+_q \rangle$$

6 

$$= \sum_{k_1,k_2} \varphi^-_{k_1} \varphi^-_{k_2} \, 6 \, \delta_{k_1+k_2,0} \frac{u}{4!\,V^3} \sum_q \frac{V}{r+q^2}$$

$$= \frac{1}{2V} \sum_k |\varphi^-_k|^2 \frac{u}{2} \underbrace{\frac{1}{V} \sum_{|q_\mu| > \frac{\Lambda}{b}} \frac{1}{r+q^2}}_{I_1(b)}$$

$$I_+ \; \frac{}{2}\left(\langle S_I\rangle_+ - \langle S_I\rangle_+^2\right)$$

$$I_1(b) \simeq \frac{S_{D-1}}{(2\pi)^D}\int_{\Lambda/b}^{\Lambda}\frac{q^{D-1}\,dq}{r+q^2}$$

solid angle

$k_3 + k_4, 0$

$\langle \varphi_k^+ \varphi_q^+ \rangle$

$$\frac{V}{r+q^2}$$

$$\left(\int D^N\varphi^+ \, e^{-S_0(\varphi^+)} \right) \quad \text{prob. dist.}$$

$$= e^{-S_0(\varphi^-)} \left\langle e^{-S_I(\varphi^- + \varphi^+)} \right\rangle_+$$

$$\ln\left(\left\langle 1 - S_I + \frac{S_I^2}{2} \right\rangle_+\right) \approx \left\langle -S_I + \frac{S_I^2}{2} \right\rangle_+$$

$$\ln\langle e^{-S_I}\rangle_+ = -\langle S_I\rangle_+ + \frac{1}{2}\left(\langle S_I^2\rangle_+ - \ldots\right.$$

Computing the Diagrams:

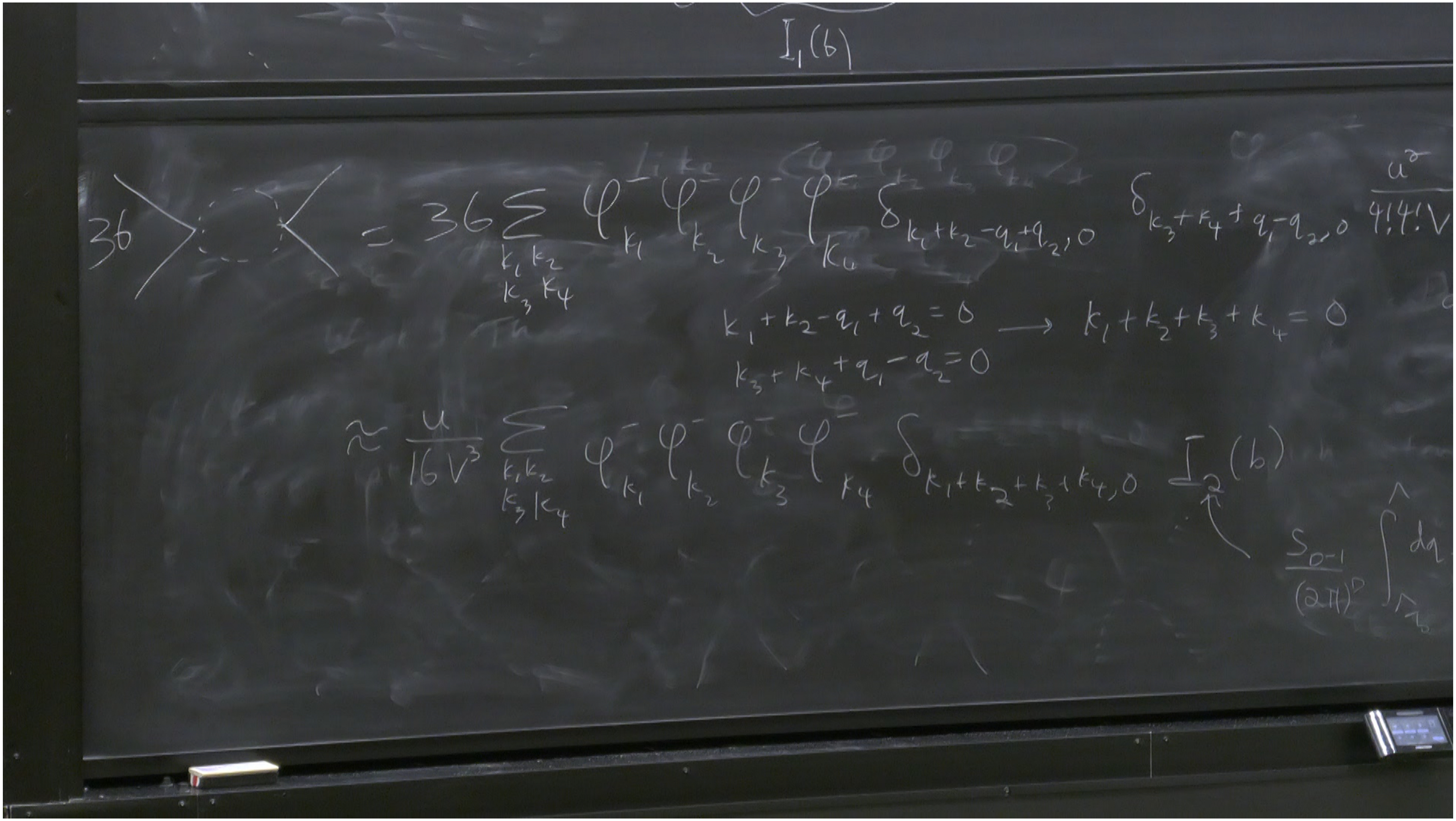
$$= \sum_{k_1 k_2 k_3 k_4} \varphi^-_{k_1} \varphi^-_{k_2} \varphi^-_{k_3} \varphi^-_{k_4} \frac{u}{4!V^3} \delta_{k_1+k_2+k_3+k_4,0}$$

$$6$$ 

$$= \sum_{k_1,k_2} \varphi^-_{k_1} \varphi^-_{k_2} \, 6 \, \delta_{k_1+k_2,0} \frac{u}{4!V^3} \sum_q \frac{V}{r+q^2} \qquad \langle \varphi^+_k \varphi^+_q \rangle$$

$$= \frac{1}{2V} \sum_k |\varphi^-_k|^2 \frac{u}{2} \underbrace{\frac{1}{V} \sum_{|q_\mu| > \frac{\Lambda}{b}} \frac{1}{r+q^2}}_{I_1(b)}$$

$$I_1(b) \approx \underbrace{\frac{S_{D-1}}{(2\pi)^D}}_{\text{solid angle}} \int_{\Lambda/b}^{\Lambda} \frac{q^{D-1}\,dq}{r+q^2}$$


$I_1(b)$
$36 \;=\; 36 \sum_{k_1 k_2 k_3 k_4} \varphi^-_{k_1} \varphi^-_{k_2} \varphi^-_{k_3} \varphi^-_{k_4} \, \delta_{k_1+k_2-q_1+q_2,0} \, \delta_{k_3+k_4+q_1-q_2,0} \, \frac{u^2}{4!4!V}$
$k_1+k_2-q_1+q_2=0 \longrightarrow k_1+k_2+k_3+k_4=0$
$k_3+k_4+q_1-q_2=0$
$\approx \frac{u}{16V^3} \sum_{k_1 k_2 k_3 k_4} \varphi^-_{k_1} \varphi^-_{k_2} \varphi^-_{k_3} \varphi^-_{k_4} \, \delta_{k_1+k_2+k_3+k_4,0} \, I_2(b)$
$\frac{S_{D-1}}{(2\pi)^D} \int dq$

$I_1(b)$

$$\varphi^-_{k_2}\varphi^-_{k_3}\varphi^-_{k_4}\,\delta_{k_1+k_2-q_1+q_2,0}\,\delta_{k_3+k_4+q_1-q_2,0}\,\frac{u^2}{4!4!V^6}\sum_{q_1 q_2}\frac{V}{r+q_1}\frac{V}{r+q_2}$$

$q_1 = q_2$

$$k_1+k_2-q_1+q_2=0 \longrightarrow k_1+k_2+k_3+k_4=0$$
$$k_3+k_4+q_1-q_2=0$$

$$\varphi^-_{k_2}\varphi^-_{k_3}\varphi^-_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}\,I_2(b)$$

$$\frac{S_{D-1}}{(2\pi)^D}\int_{\Lambda/b}^{\Lambda} dq\,\frac{q^{D-1}}{(r+q^2)^2}$$

$$\frac{\;}{2V}\sum_k |\varphi_k|^2 \quad \frac{u}{2}\underbrace{\frac{1}{V}\sum_{|q_\mu|>\frac{\Lambda}{b}}\frac{1}{r+q^2}}_{I_1(b)}$$

36 

$$= 36\sum_{\substack{k_1k_2\\k_3k_4}}\varphi^-_{k_1}\varphi^-_{k_2}\varphi^-_{k_3}\varphi^-_{k_4}\,\delta_{k_1+k_2-q_1+q_2,0}\,\delta_{k_3+k_4+q_1-q_2,0}\,\frac{u^2}{4!4!V^6}\sum_{q_1q_2}\frac{V}{r+q_1}\frac{V}{r+q_2}$$

$$k_1+k_2-q_1+q_2=0 \longrightarrow k_1+k_2+k_3+k_4=0 \qquad q_1=q_2$$

$$k_3+k_4+q_1-q_2=0$$

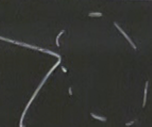
$$\approx \frac{u}{16V^3}\sum_{\substack{k_1k_2\\k_3k_4}}\varphi^-_{k_1}\varphi^-_{k_2}\varphi^-_{k_3}\varphi^-_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}\; I_2(b)$$

$$I_2(b) = \frac{S_{D-1}}{(2\pi)^D}\int_{\Lambda/b}^{\Lambda} dq\,\frac{q^{D-1}}{(r+q^2)^2}$$

- Diagrams with six external legs introduce φ^6 terms.

$\frac{1}{2}\left(\langle S_I^2 \rangle_+ - \langle S_I \rangle_+^2\right) \simeq 36$ 

$S'(\varphi^-) \simeq$  $+ 6$  $- 36$  $+ S_0(\varphi^-)$

Rescaling: $N' = \frac{N}{b^D}$ $V' = \frac{V}{b^D}$ $k' = kb$ $\varphi'_k = z\varphi_k$

$$S'(\varphi^-) \approx \frac{b^{-D}}{2V'} \sum_{k'} \big(\underbrace{r + b^{-2}k'}_{S_0(\varphi^-)} + \frac{u}{2} I_1(b)\big) z^{-2} |\varphi'_{k'}|^2 + \frac{b^{-3D}}{4! V^3}$$

$b^{-D} z^{-2} = b^2$

$z = b^{(2+D)/2}$

RGT:
$$\boxed{\begin{cases} r' = b^2\left(r + \frac{u I_1(b)}{2}\right) \\ u' = b^{4-D}\left(u - \frac{3}{2}u^2 I_2(b)\right) \end{cases}}$$

What is the

$\frac{\partial r'}{\partial b}$, $\frac{\partial u'}{\partial b}$ $\frac{\partial I_1(b)}{\partial b}$,

Leibniz rule:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x, b(x))$$

$$+ \frac{b^{-3D}}{4!\,V^3} \sum_{k'} \left(u - \frac{3}{2} u^2 I_2(b) \right) z^{-4} \varphi'_{k_1} \varphi'_{k_2} \varphi'_{k_3} \varphi'_{k_4} \delta_{k_1' + k_2' + k_3' + k_4', 0}$$

$= b^2$

$b^{(2+D)/2}$

is the β-function?

$$\frac{\partial I_1(b)}{\partial b}, \quad \frac{\partial I_2(b)}{\partial b}$$

$$t = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt$$

$$
\begin{cases}
\dfrac{dr'}{db} = 2b\left(r + \dfrac{u}{2}\dfrac{K_D}{r+\Lambda^2 b^{-2}}\right) \\[2ex]
\dfrac{du'}{db} = (4-D)\,b^{3-D}\left(u - \dfrac{3u^2}{2}\dfrac{K_D}{(r+\Lambda^2 b^{-2})}\right)
\end{cases}
$$

$$
K_D = \frac{S_{D-1}\,\Lambda^D\, b^{-1-D}}{(2\pi)^D}
$$

Now what are the fixed points?

$(r^*, u^*) = (0,0)$ is one ← Gaussian fixed point

t

$s = e^{b-1}$ ← is 1 for no rescaling

$$\begin{cases} s\dfrac{dr}{ds}\Big|_{b=1} = 2(r + \frac{1}{2}K_D\Lambda^{-2}u - \frac{1}{2}K_D u r \Lambda^{-4} \\ s\dfrac{du}{ds} = -\dfrac{3u^2}{2}\dfrac{K_D}{(r+\Lambda^2)^2} \end{cases}$$

$\boxed{D=4}$

$u \to g$

$r \to t$

$V = \frac{t}{2}\phi$

$s\frac{d}{ds}t = -\beta_t($

$s\frac{d}{ds}g = -\beta$

the fixed points?

Francois' Notes

is one $\leftarrow$ Gaussian fixed point

$$V = \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4$$

is 1 for no rescaling

$$s\frac{d}{ds}t = -\beta_t(t,g) = 2t + g - tg$$

$$K_D \Lambda^{-2} u - \frac{1}{2} K_D u r \Lambda^{-4} \qquad u \to g$$

$$r \to t$$

$$s\frac{d}{ds}g = -\beta_g(t,g) = -3g^2$$

$$\frac{K_D}{(r+\Lambda^2)^2}$$

$$dx \int_{a(x)} f(x,$$

$$\frac{1}{r+\Lambda^2} = \Lambda^{-2} \frac{1}{\frac{r}{\Lambda^2}+1} \sim \Lambda^{-2}\left(1-\frac{r}{\Lambda^2}\right)$$

$$\begin{cases} \dfrac{dr'}{db} = 2b\left(r + \dfrac{u}{2}\dfrac{K_D}{r+\Lambda^2 b^{-2}}\right) \\ \dfrac{du'}{db} = (4-D)b^{3-D}\left(u - \dfrac{3u^2}{2}\dfrac{K_D}{(r+\Lambda^2 b^{-2})}\right) \end{cases}$$

$$K_D = \frac{S_{D-1}\Lambda^D b^{-1-D}}{(2\pi)^D}$$

Now

$(r^*,$

t

S

$$\begin{cases} S\dfrac{dr'}{dS} \\ S\dfrac{du'}{dS} \end{cases}$$

Leibniz rule:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\cdot\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt$$

Now what are the fixed points?

$(r^*, u^*) = (0,0)$ is one $\leftarrow$ Gaussian fixed point

t

$s = e^{b-1}$ $\leftarrow$ is 1 for no rescaling

$$\begin{cases} s\dfrac{dr}{ds}\Big|_{b=1} = 2(r + \tfrac{1}{2}K_0\Lambda^{-2}u - \tfrac{1}{2}K_0 u r\Lambda^{-4} \\ s\dfrac{du}{ds} = -\dfrac{3u^2}{2}\dfrac{K_0}{(r+\Lambda^2)^2} \end{cases}$$

$(D=4)$

$u \to g$

$r \to t$

Francois' Notes

$$V = \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4$$

$$s\frac{d}{ds}t = -\beta_t(t,g) = 2t + g - tg$$

$$s\frac{d}{ds}g = -\beta_g(t,g) = -3g^2$$