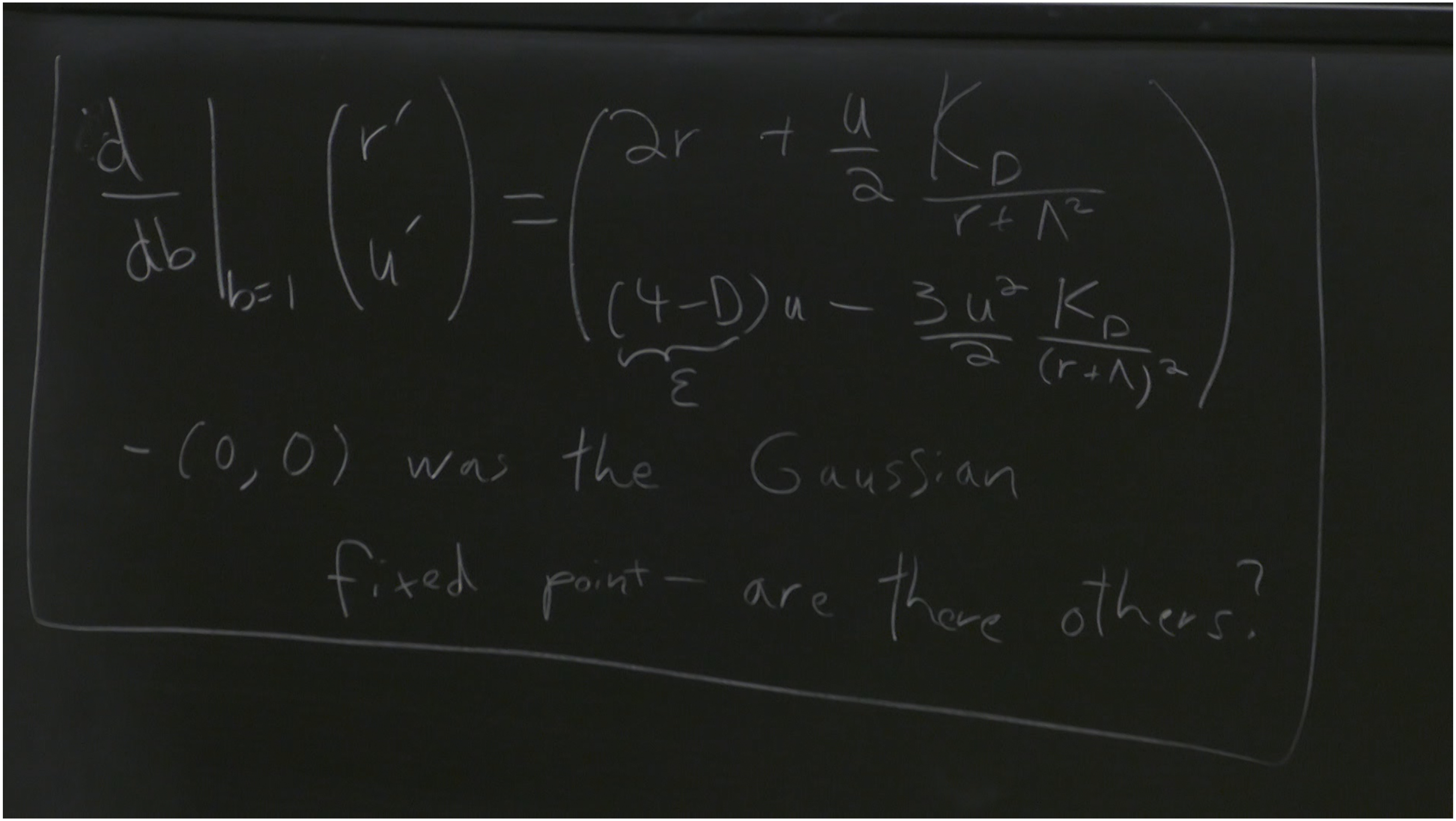
Title: Statistical Physics Lecture - 120623

Speakers: Emilie Huffman

Collection: Statistical Physics 2023/24

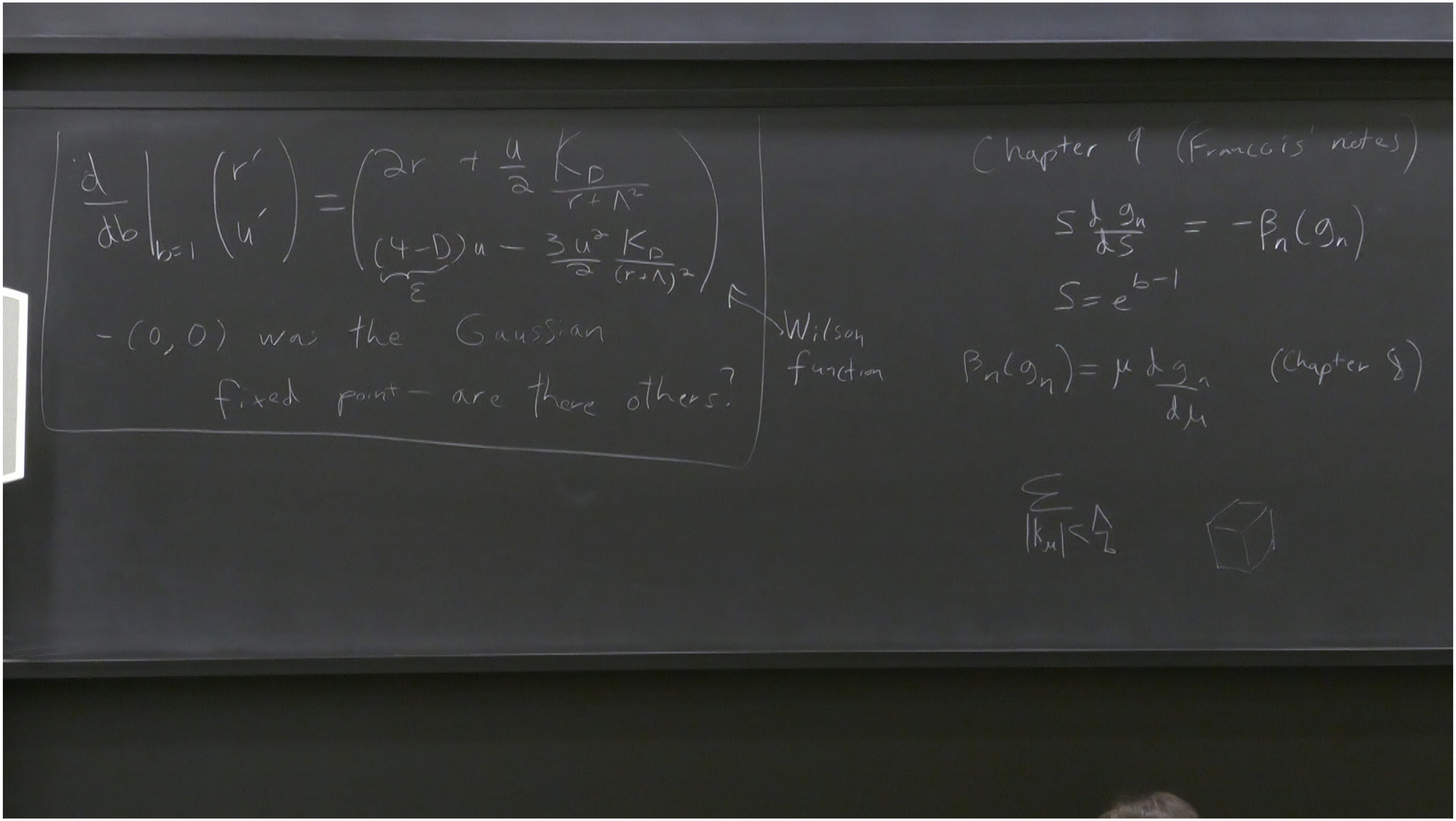
Date: December 06, 2023 - 10:45 AM

URL: https://pirsa.org/23120012



$$\left.\frac{d}{db}\right|_{b=1}\begin{pmatrix} r' \\ u' \end{pmatrix} = \begin{pmatrix} 2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2} \\ \underbrace{(4-D)}_{\varepsilon}u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda)^2} \end{pmatrix}$$

- (0, 0) was the Gaussian fixed point — are there others?



$$\boxed{\frac{d}{db}\Big|_{b=1}\begin{pmatrix} r' \\ u' \end{pmatrix} = \begin{pmatrix} 2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2} \\ \underbrace{(4-D)}_{\varepsilon}u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda)^2} \end{pmatrix}}$$

Wilson function

- (0,0) was the Gaussian fixed point – are there others?

Chapter 9 (Francois' notes)

$$s\frac{d g_n}{ds} = -\beta_n(g_n)$$

$$s = e^{b-1}$$

$$\beta_n(g_n) = \mu\frac{d g_n}{d\mu}$$ (Chapter 8)

$$\sum_{|k_\mu|<\frac{\Lambda}{b}}$$

$$\frac{d}{db}\bigg|_{b=1}\begin{pmatrix} r' \\ u' \end{pmatrix} = \begin{pmatrix} 2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2} \\ \underbrace{(4-D)}_{\varepsilon}u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda)^2} \end{pmatrix} \overset{?}{=} 0$$

Wilson function

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$$s\frac{d g_n}{ds} = -\beta_n(g_n)$$

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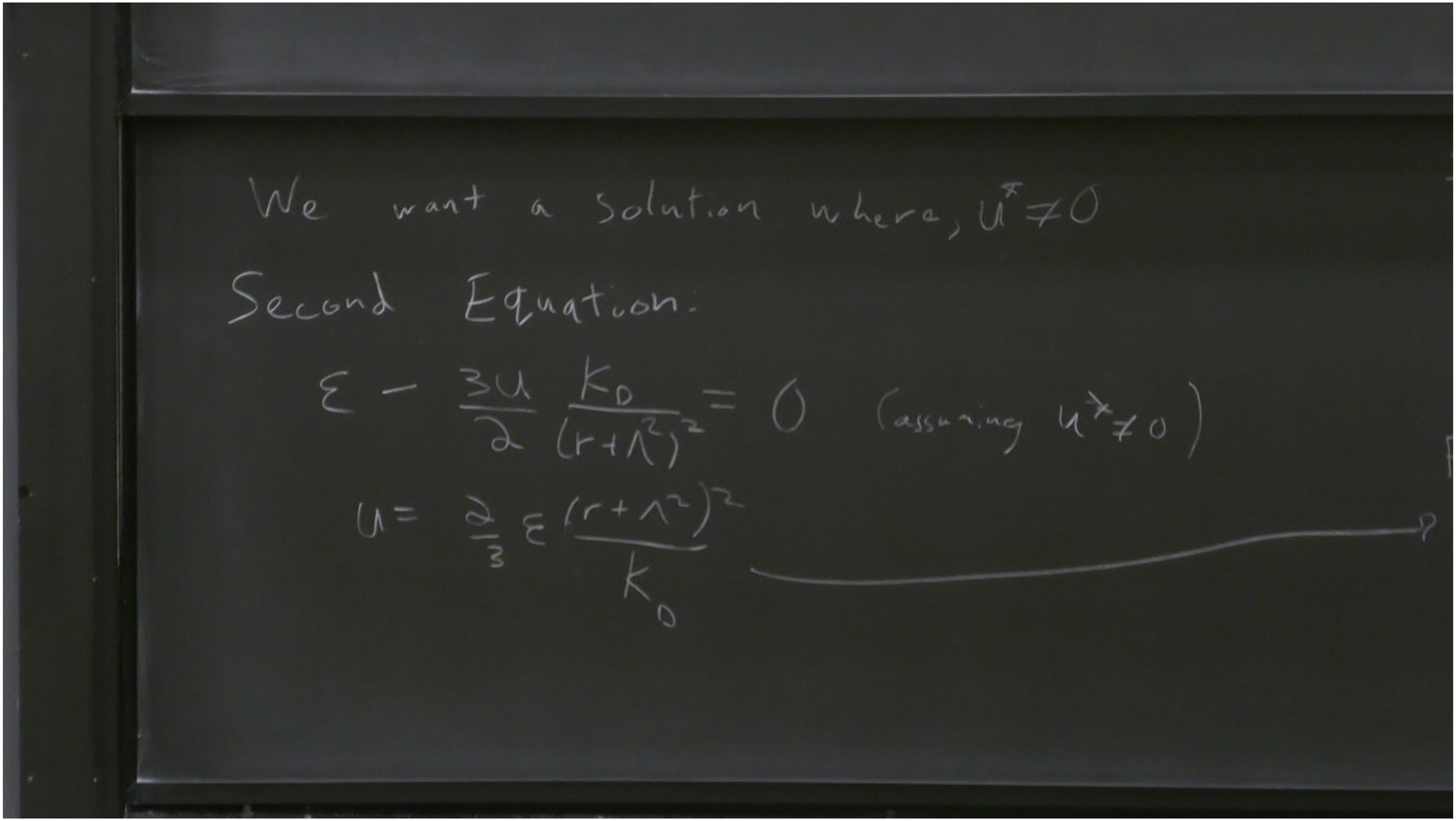
$$\beta_n(g_n) = \mu\frac{d g_n}{d\mu} \quad \text{(Chapter 8)}$$

$$\sum_{|k_\mu| < \frac{\Lambda}{b}}$$

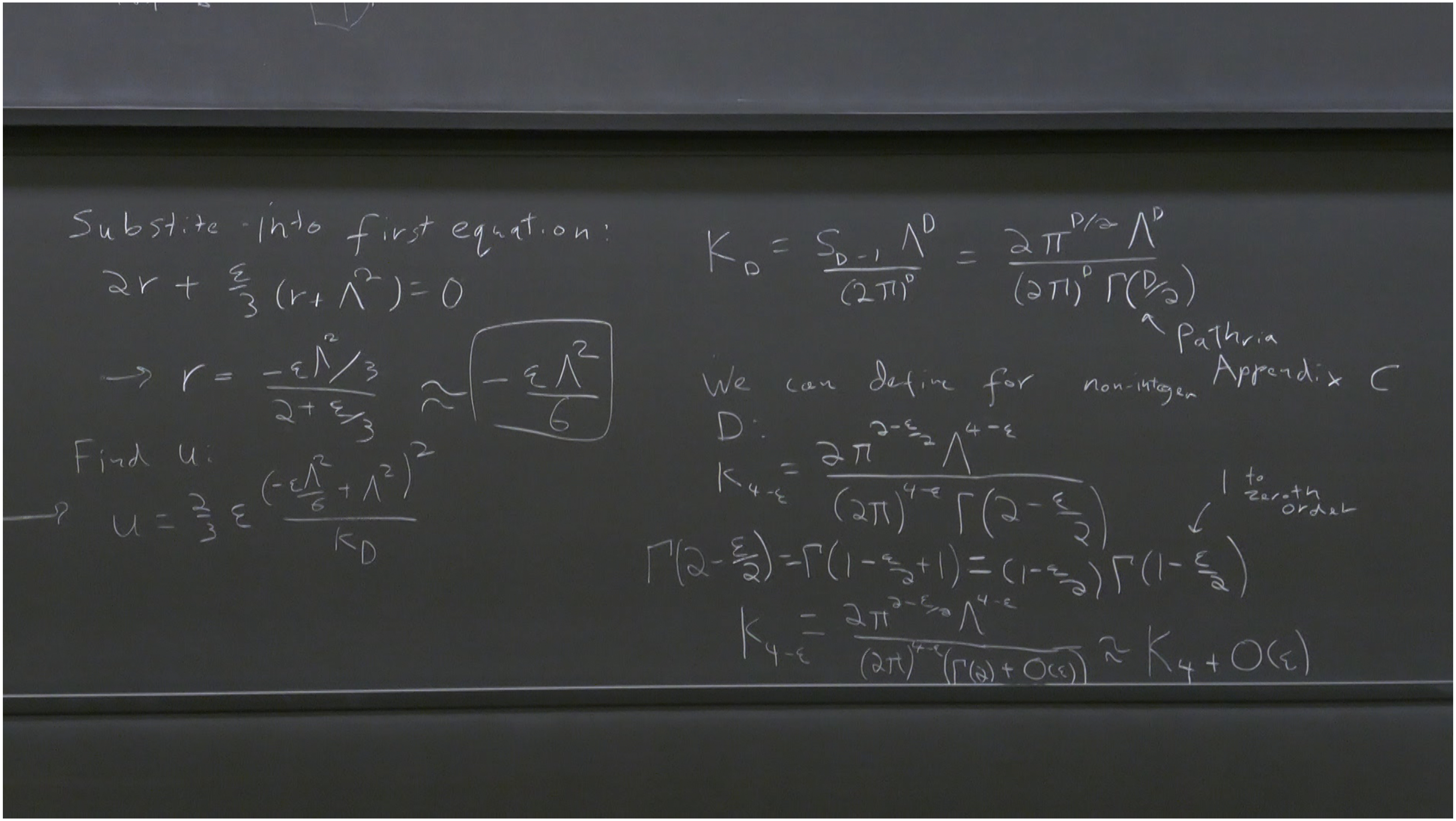


We want a solution where, $u^* \neq 0$

Second Equation:

$$\varepsilon - \frac{3u}{2}\frac{K_D}{(r+\Lambda^2)^2} = 0 \quad (\text{assuming } u^* \neq 0)$$

$$u = \frac{2}{3}\varepsilon\frac{(r+\Lambda^2)^2}{K_D}$$



Substite into first equation:

$$2r + \frac{\varepsilon}{3}(r + \Lambda^2) = 0$$

$$\longrightarrow r = \frac{-\varepsilon\Lambda^2/3}{2 + \varepsilon/3} \approx \boxed{-\frac{\varepsilon\Lambda^2}{6}}$$

Find u:

$$\longrightarrow u = \frac{2}{3}\varepsilon \frac{\left(-\frac{\varepsilon\Lambda^2}{6} + \Lambda^2\right)^2}{K_D}$$

$$K_D = \frac{S_{D-1}\Lambda^D}{(2\pi)^D} = \frac{2\pi^{D/2}\Lambda^D}{(2\pi)^D\,\Gamma(D/2)}$$

↖ Pathria Appendix C

We can define for non-integer D:

$$K_{4-\varepsilon} = \frac{2\pi^{2-\varepsilon/2}\Lambda^{4-\varepsilon}}{(2\pi)^{4-\varepsilon}\,\Gamma\left(2-\frac{\varepsilon}{2}\right)}$$

$$\Gamma\left(2-\frac{\varepsilon}{2}\right) = \Gamma\left(1-\frac{\varepsilon}{2}+1\right) = \left(1-\frac{\varepsilon}{2}\right)\Gamma\left(1-\frac{\varepsilon}{2}\right)$$

1 to zeroth order

$$K_{4-\varepsilon} = \frac{2\pi^{2-\varepsilon/2}\Lambda^{4-\varepsilon}}{(2\pi)^{4-\varepsilon}\left(\Gamma(2) + O(\varepsilon)\right)} \approx K_4 + O(\varepsilon)$$

$$u \simeq \frac{2}{3}\varepsilon \frac{\left(-\frac{\varepsilon\Lambda^2}{6}+\Lambda^2\right)^2}{K_4+O(\varepsilon)} \simeq \boxed{\frac{2}{3}\frac{\Lambda^4}{K_4}\varepsilon}$$

So we have another fixed point at:

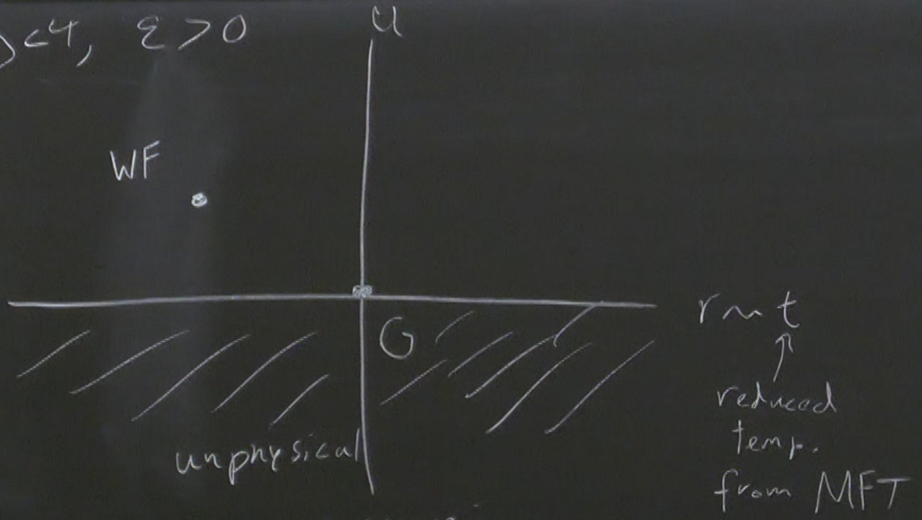
$$(r^*, u^*) = \left(-\frac{\varepsilon}{6}\Lambda^2, \frac{2}{3}\frac{\Lambda^4}{K_4}\varepsilon\right)$$

↑close to $D=4$

Wilson-Fisher Fixed point

→ merges with the Gaussian Fixed point at $D=4$

$D<4,\ \varepsilon>0$

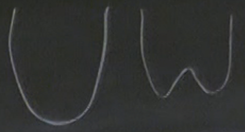


Theory in real space:

$$(\nabla\phi)^2 + r\phi^2 + u\phi^4$$

MFT → set derivs to zero

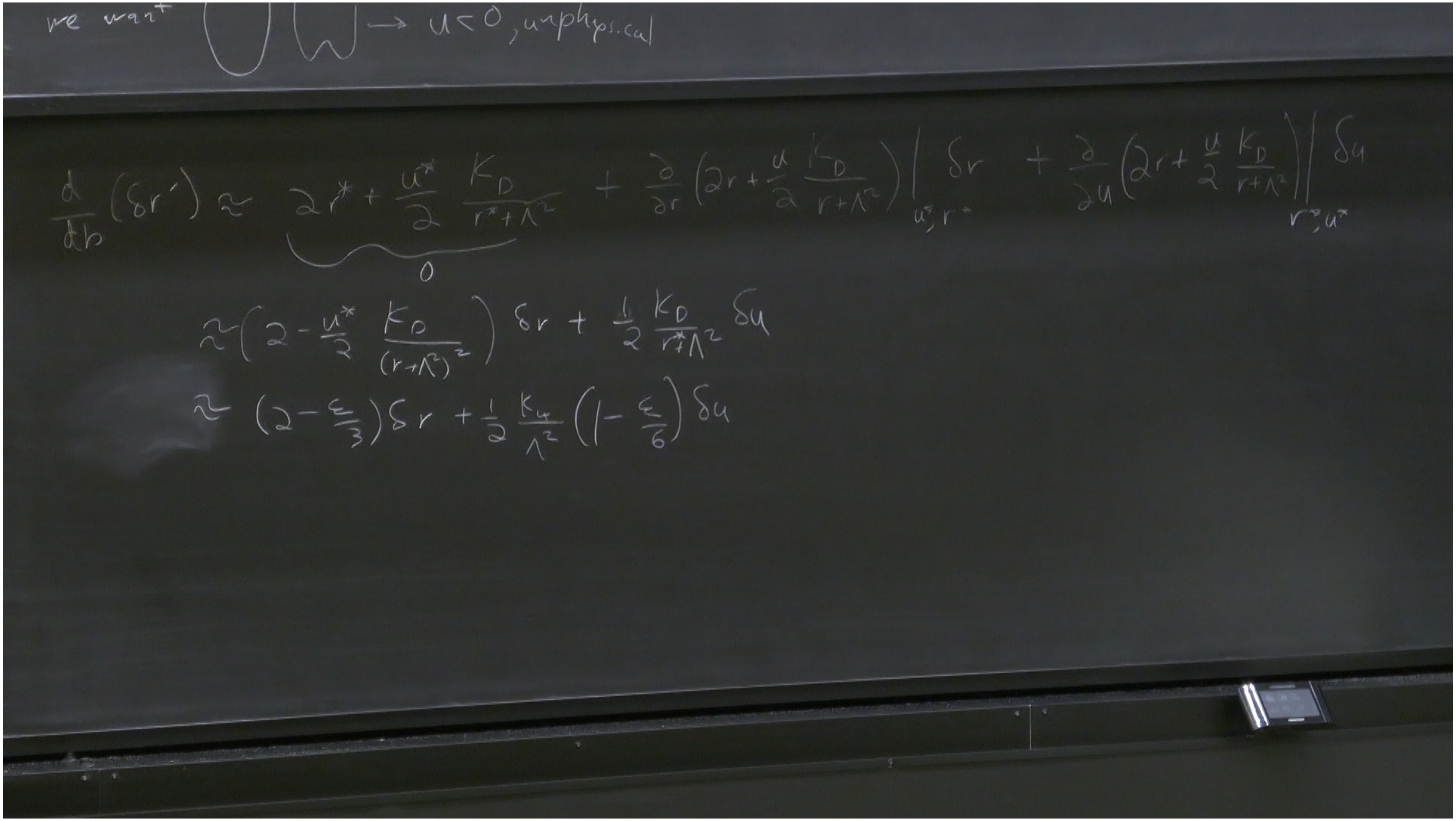
↪ minimize $r\phi^2 + u\phi^4$

we want  → $u<0$, unphysical

Let's Linearize the flow equations near WF.

$$\begin{cases} \delta r = r - r^* \\ \delta u = u - u^* \end{cases} \qquad \begin{cases} \delta r' = r' - r^* \\ \delta u' = u' - u^* \end{cases} \qquad \delta K' = M \, \delta K$$

$$\frac{d}{db}\begin{pmatrix} \delta r' \\ \delta u' \end{pmatrix} = \frac{d}{db}\begin{pmatrix} r' - r^* \\ u' - u^* \end{pmatrix} = \frac{d}{db}\begin{pmatrix} r' \\ u' \end{pmatrix}$$



we want $\rightarrow u<0$, unphysical

$$\frac{d}{db}(\delta r') \approx \underbrace{2r^* + \frac{u^*}{2}\frac{K_D}{r^*+\Lambda^2}}_{0} + \frac{\partial}{\partial r}\left(2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2}\right)\Bigg|_{u^*,r^*} \delta r + \frac{\partial}{\partial u}\left(2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2}\right)\Bigg|_{r^*,u^*} \delta u$$

$$\approx \left(2 - \frac{u^*}{2}\frac{K_D}{(r+\Lambda^2)^2}\right)\delta r + \frac{1}{2}\frac{K_D}{r^*+\Lambda^2}\delta u$$

$$\approx \left(2 - \frac{\varepsilon}{3}\right)\delta r + \frac{1}{2}\frac{K_4}{\Lambda^2}\left(1 - \frac{\varepsilon}{6}\right)\delta u$$

$$\frac{d}{db}(\delta r') \approx \underbrace{2r^* + \frac{u^*}{2}\frac{K_D}{r^*+\Lambda^2}}_{0} + \frac{\partial}{\partial r}\left(2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2}\right)\Bigg|_{u^*, r^*} \delta r + \frac{\partial}{\partial u}\left(2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2}\right)\Bigg|_{r^*, u^*} \delta u$$

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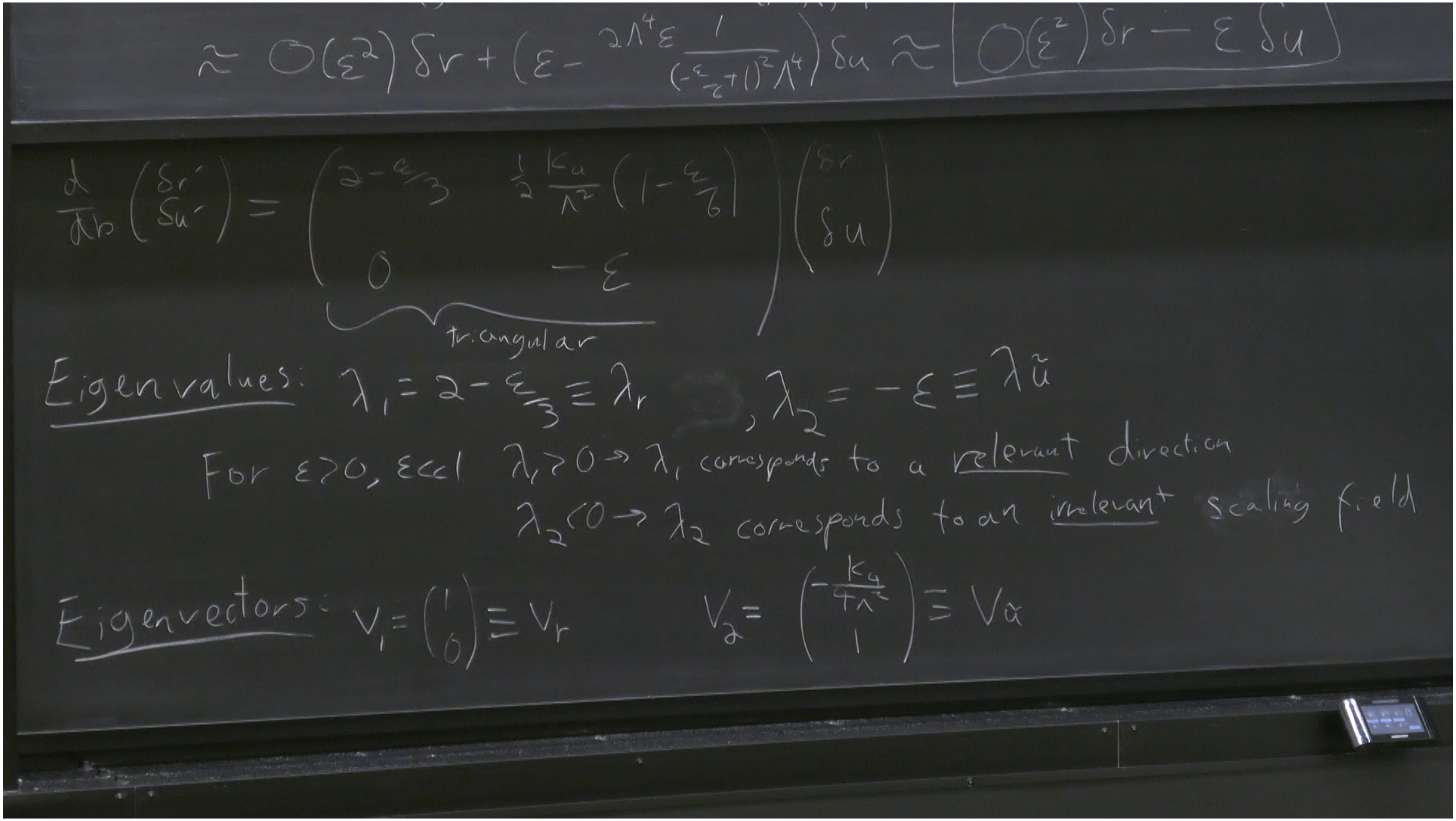
$$\approx \boxed{\left(2 - \frac{\varepsilon}{3}\right)\delta r + \frac{1}{2}\frac{K_4}{\Lambda^2}\left(1 - \frac{\varepsilon}{6}\right)\delta u}$$

$$\frac{d}{db}(\delta u') = \underbrace{\varepsilon u^* - \frac{3u^{*2}}{2}\frac{K_D}{(r^*+\Lambda^2)^2}}_{0} + \frac{\partial}{\partial r}\left(\varepsilon u - \frac{3u^2 K_D}{2(r+\Lambda^2)^2}\right)\Bigg|_{r^*, u^*} \delta r + \frac{\partial}{\partial u}\left(\varepsilon u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda^2)^2}\right)\Bigg|_{r^*, u^*} \delta u$$

$$\approx \frac{3u^{*2} K_D}{(r^*+\Lambda^2)^3}\delta r + \left(\varepsilon - 3u^*\frac{K_D}{(r^*+\Lambda^2)^2}\right)\delta u$$

$$\approx O(\varepsilon^2)\delta r + \left(\varepsilon - 2\Lambda^4\varepsilon\frac{1}{(-\frac{\varepsilon}{2}+1)^2\Lambda^4}\right)\delta u \approx \boxed{O(\varepsilon^2)\delta r - \varepsilon\,\delta u}$$



$$\approx O(\varepsilon^2)\,\delta r + \left(\varepsilon - 2\Lambda^4\varepsilon\frac{1}{(-\frac{\varepsilon}{6}+1)^2\Lambda^4}\right)\delta u \approx \boxed{O(\varepsilon^2)\,\delta r - \varepsilon\,\delta u}$$

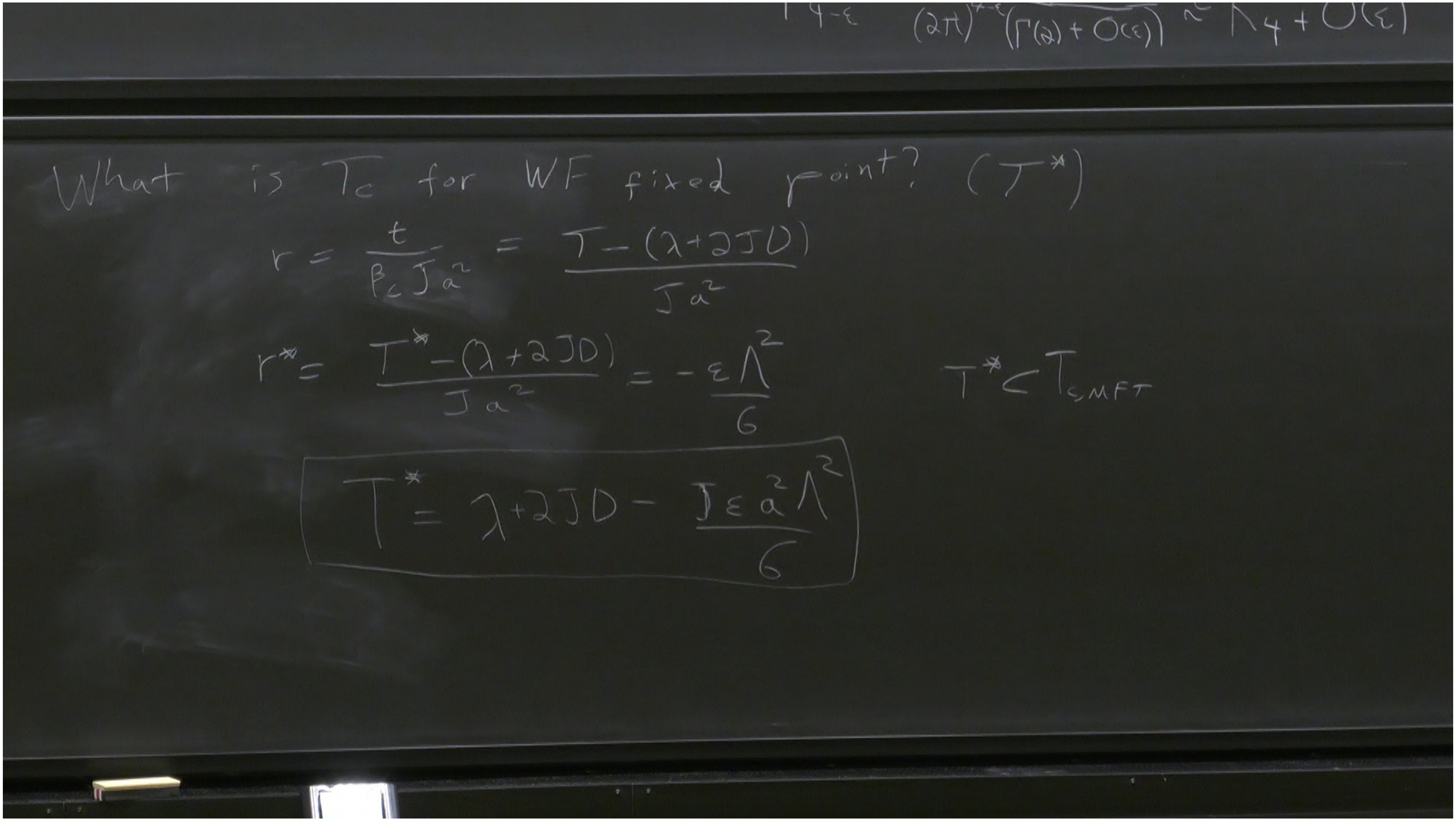
$$\frac{d}{db}\begin{pmatrix}\delta r'\\ \delta u'\end{pmatrix} = \underbrace{\begin{pmatrix}2-\varepsilon/3 & \frac{1}{2}\frac{K_u}{\Lambda^2}\left(1-\frac{\varepsilon}{6}\right)\\ 0 & -\varepsilon\end{pmatrix}}_{\text{triangular}}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2-\frac{\varepsilon}{3} \equiv \lambda_r$, $\lambda_2 = -\varepsilon \equiv \lambda_{\tilde{u}}$

For $\varepsilon > 0$, $\varepsilon \ll 1$ $\lambda_1 > 0 \rightarrow \lambda_1$ corresponds to a relevant direction

$\lambda_2 < 0 \rightarrow \lambda_2$ corresponds to an irrelevant scaling field

Eigenvectors: $V_1 = \begin{pmatrix}1\\0\end{pmatrix} \equiv V_r$ $\quad V_2 = \begin{pmatrix}-\frac{K_4}{4\Lambda^2}\\ 1\end{pmatrix} \equiv V_{\tilde{u}}$



$\Gamma_{4-\varepsilon} \quad (2\pi)^{4-\varepsilon}\left(\Gamma(2)+O(\varepsilon)\right) \sim \Lambda_4 + O(\varepsilon)$

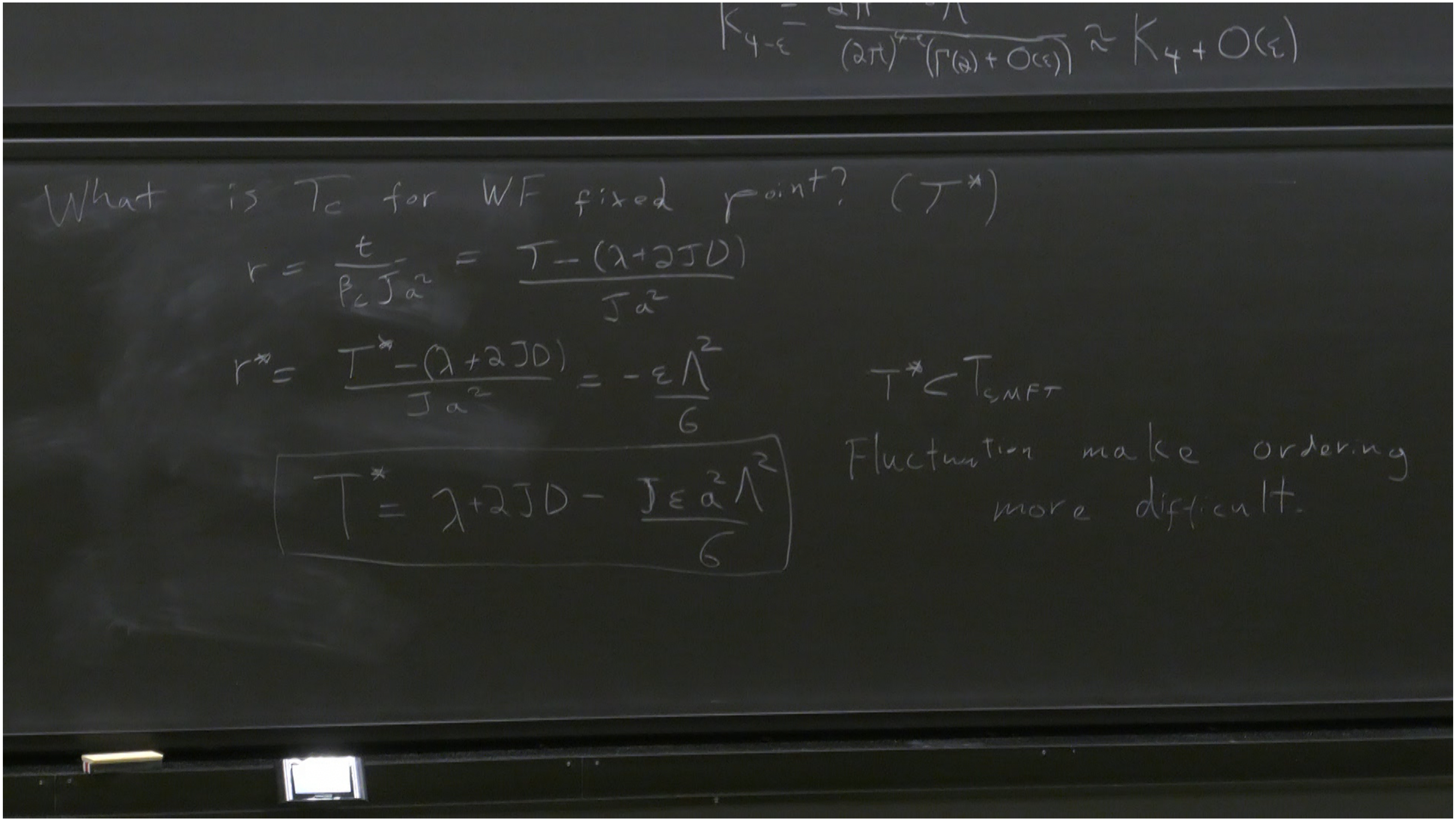
What is T_c for WF fixed point? (T^*)

$$r = \frac{t}{\beta_c J a^2} = \frac{T-(\lambda+2JD)}{Ja^2}$$

$$r^* = \frac{T^*-(\lambda+2JD)}{Ja^2} = -\frac{\varepsilon\Lambda^2}{6}$$

$T^* < T_{SMFT}$

$$\boxed{T^* = \lambda+2JD - \frac{J\varepsilon a^2\Lambda^2}{6}}$$



$$K_{4-\varepsilon} = \frac{2\pi \quad \Lambda}{(2\pi)^{4-\varepsilon}(\Gamma(2) + O(\varepsilon))} \approx K_4 + O(\varepsilon)$$

What is T_c for WF fixed point? (T^*)

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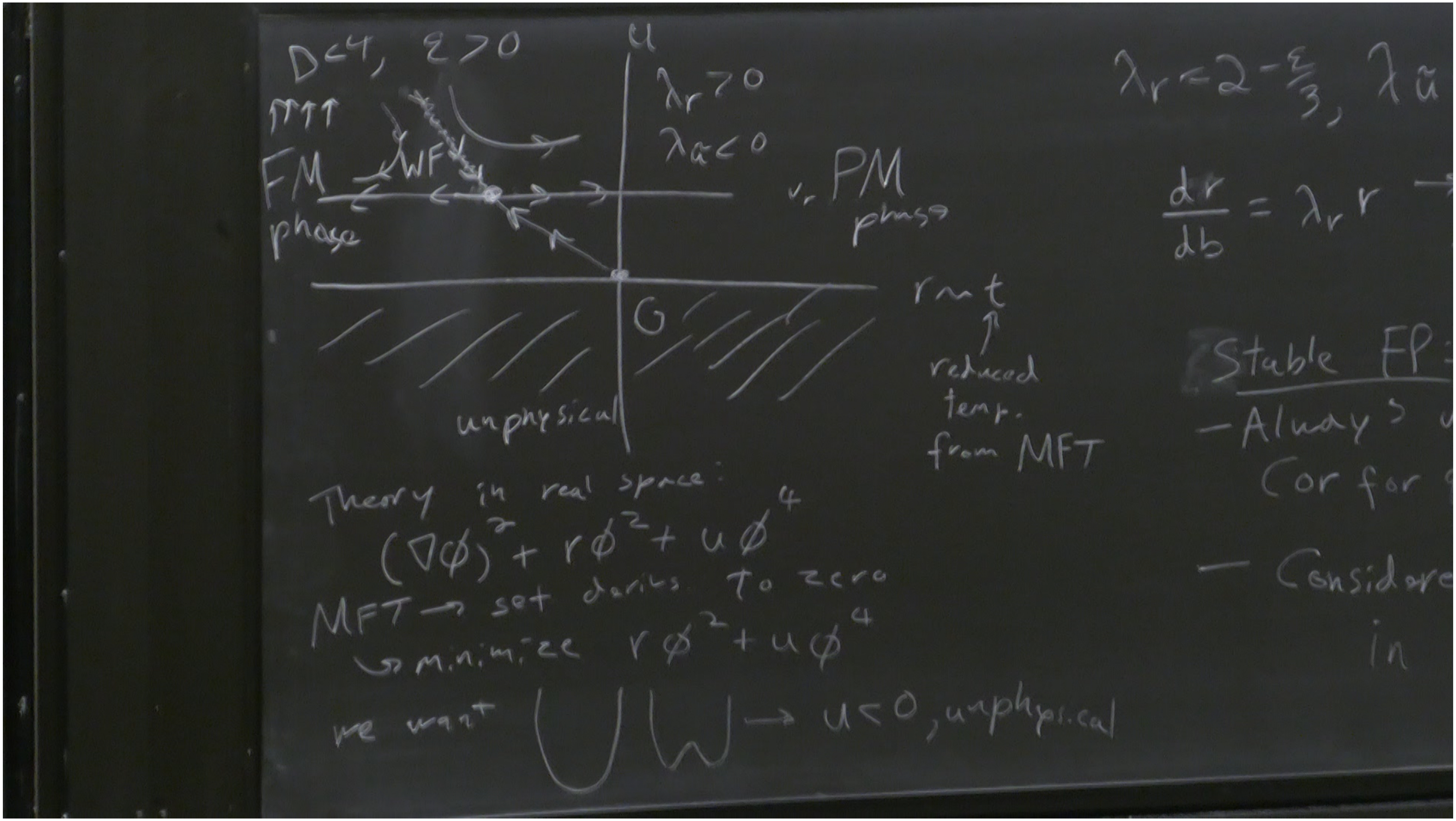
$T^* < T_{cMFT}$

$$\boxed{T^* = \lambda + 2JD - \frac{J \varepsilon a^2 \Lambda^2}{6}}$$

Fluctuation make ordering more difficult.



$D<4,\ \varepsilon>0$

Theory in real space:

$$(\nabla\phi)^2 + r\phi^2 + u\phi^4$$

MFT → set derivs. to zero

↪ minimize $r\phi^2 + u\phi^4$

we want U W → $u<0$, unphysical

$\lambda_r < 2 - \frac{\varepsilon}{3},\ \lambda \tilde{u}$

$$\frac{dr}{db} = \lambda_r r \rightarrow$$

Stable FP

— Always

(or for

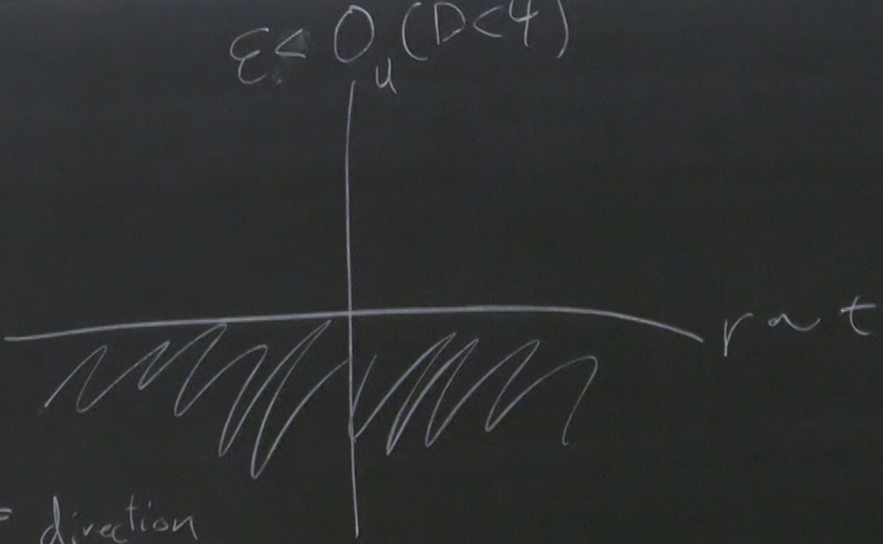
— Considere

in

$\lambda_r < 2 - \frac{\varepsilon}{3}$, $\lambda_{\tilde{u}} = -\varepsilon$ $\quad \lambda_r > 0$

$$\frac{dr}{db} = \lambda_r r \rightarrow r = r_0 e^{\lambda_r r}$$

$$\tilde{u} = \tilde{u}_0 e^{\lambda_{\tilde{u}} \tilde{u}}$$



Stable FP:

- Always unstable in the T direction (or for quantum phase transition, one coupling)
- Considered stable if it is attractive in the other parameters

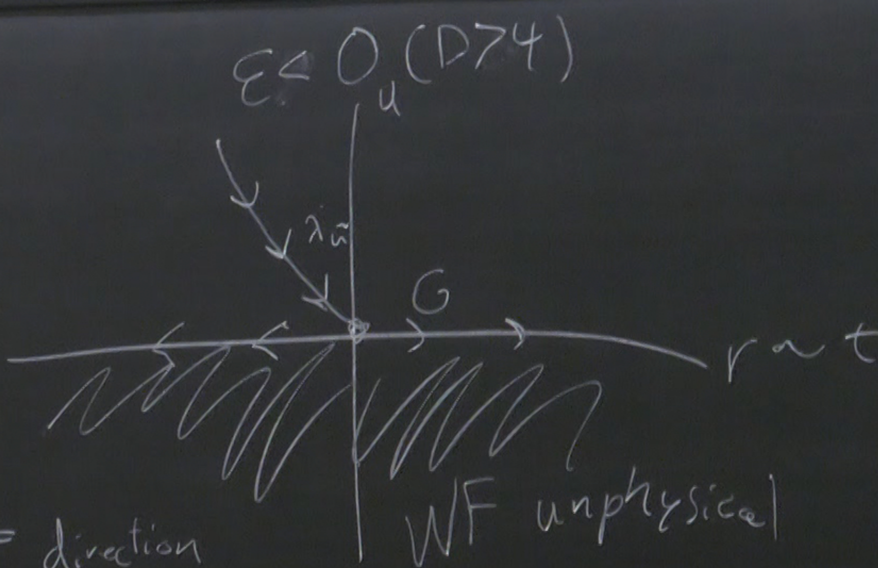
MFT

os.cal

$\lambda_r = 2 - \frac{\varepsilon}{3}$, $\lambda_{\tilde u} = -\varepsilon$

$$\frac{dr}{db} = \lambda_r r \rightarrow r = r_0 e^{\overset{\overset{\lambda_r > 0}{\uparrow}}{\lambda_r r}}$$

$$\tilde u = \tilde u_0 e^{\lambda_{\tilde u} \tilde u}$$



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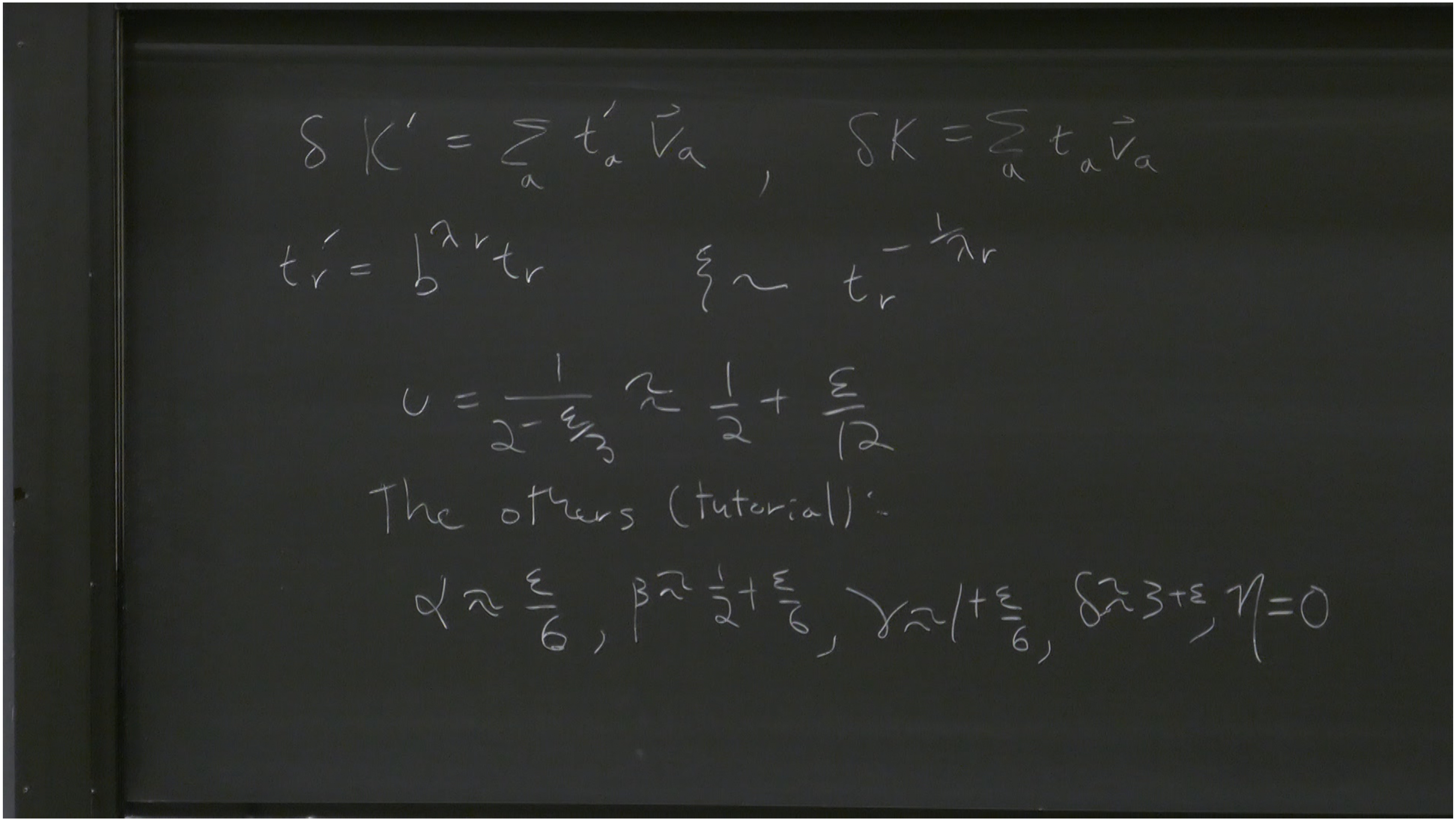


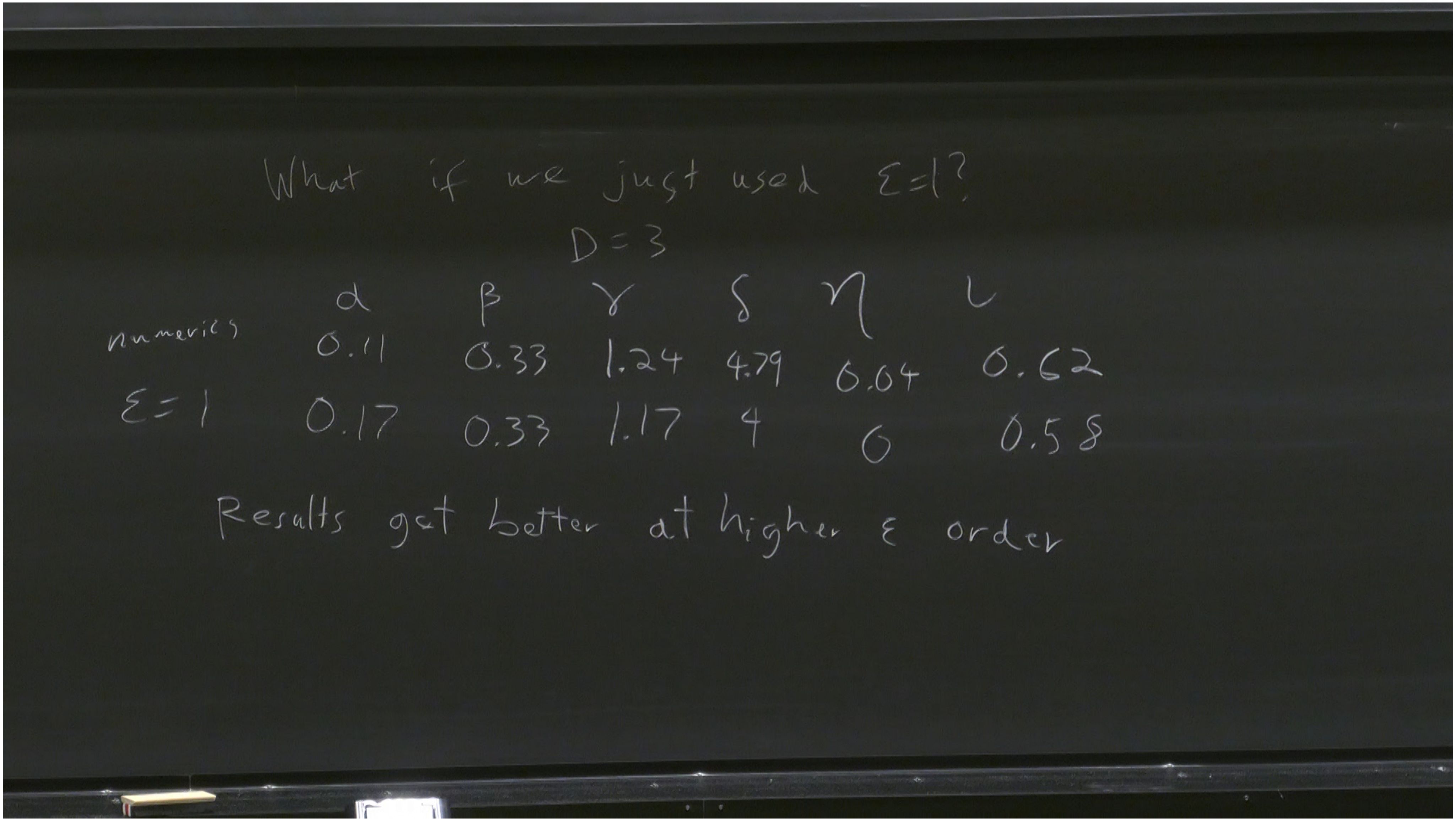
$$\delta K' = \sum_a t'_a \vec{V}_a \quad , \quad \delta K = \sum_a t_a \vec{V}_a$$

$$t'_r = b^{\lambda_r} t_r \qquad \xi \sim t_r^{-1/\lambda_r}$$

$$\nu = \frac{1}{2-\varepsilon/3} \approx \frac{1}{2} + \frac{\varepsilon}{12}$$

The others (tutorial):

$$\alpha \approx \frac{\varepsilon}{6}, \; \beta \approx \frac{1}{2} + \frac{\varepsilon}{6}, \; \gamma \approx 1 + \frac{\varepsilon}{6}, \; \delta \approx 3 + \varepsilon, \; \eta = 0$$

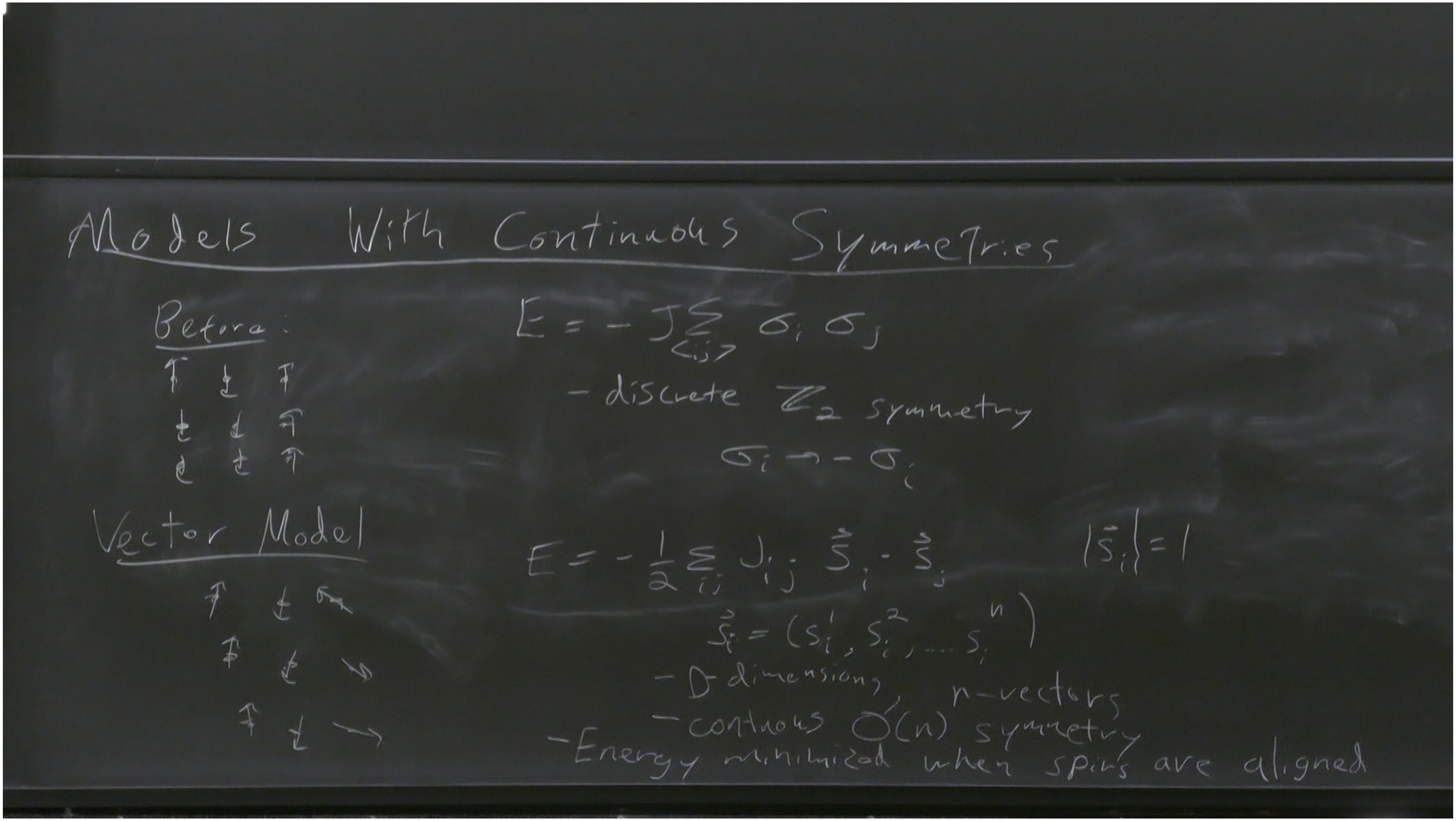


What if we just used $\varepsilon=1$?

$D=3$

	α	β	γ	δ	η	ν
numerics	0.11	0.33	1.24	4.79	0.04	0.62
$\varepsilon=1$	0.17	0.33	1.17	4	0	0.58

Results get better at higher ε order



Models With Continuous Symmetries

Before:

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

- discrete $\mathbb{Z}_2$ symmetry

$$\sigma_i \to -\sigma_i$$

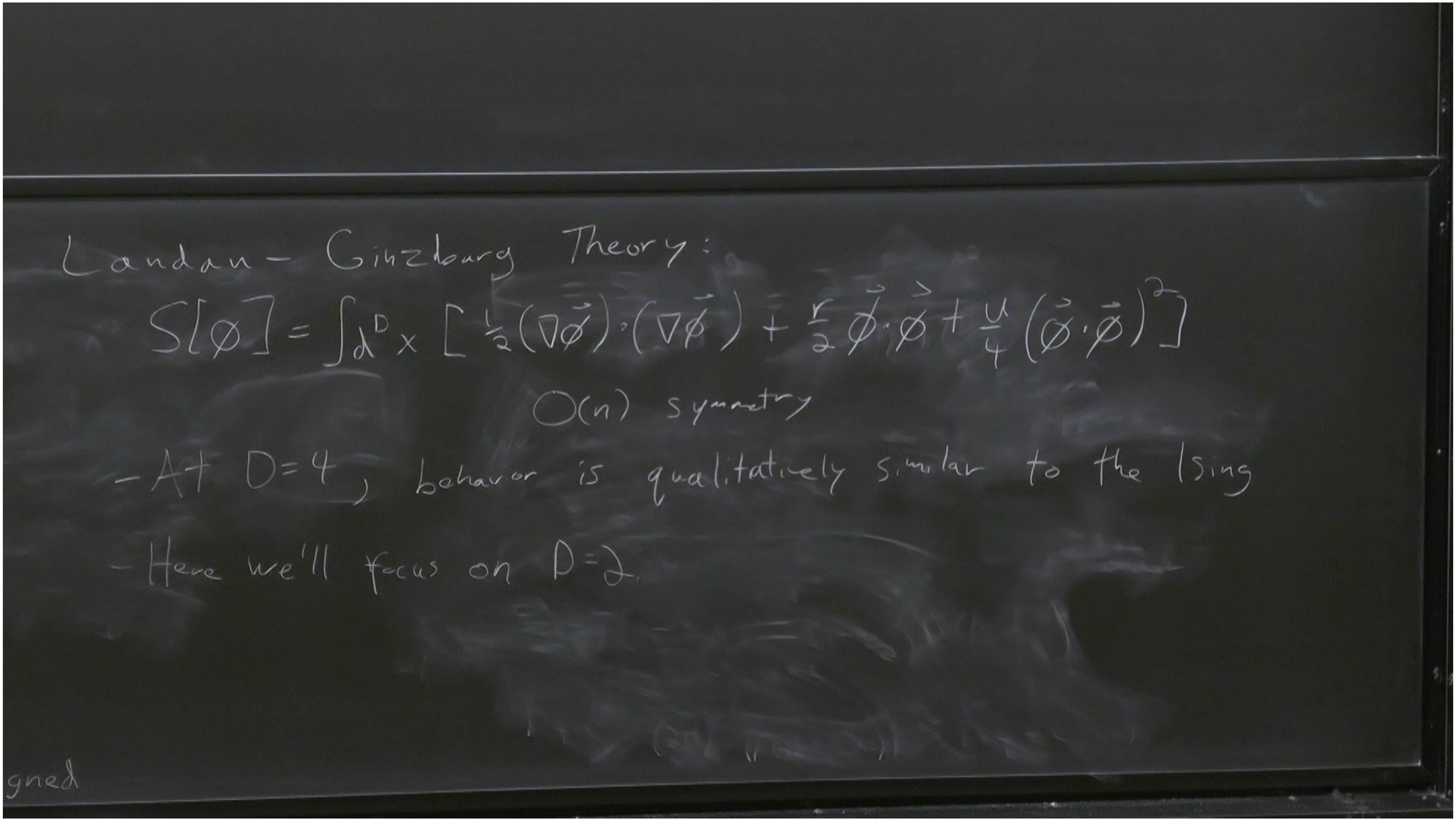
Vector Model

$$E = -\frac{1}{2} \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j \qquad |\vec{S}_i| = 1$$

$$\vec{S}_i = (S_i^1, S_i^2, \dots S_i^n)$$

- D dimensions, n-vectors
- contnous O(n) symmetry

- Energy minimized when spins are aligned



Landau-Ginzburg Theory:

$$S[\vec\phi] = \int d^Dx \left[\frac{1}{2}(\nabla\vec\phi)\cdot(\nabla\vec\phi) + \frac{r}{2}\vec\phi\cdot\vec\phi + \frac{u}{4}(\vec\phi\cdot\vec\phi)^2 \right]$$

$O(n)$ symmetry

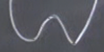
- At $D=4$, behavior is qualitatively similar to the Ising
- Here we'll focus on $D=2$.

gned

Write $\vec{\phi}(x) = \overset{\text{number}}{\rho(x)}\,\underset{\text{unit vector}}{\vec{n}(x)}$

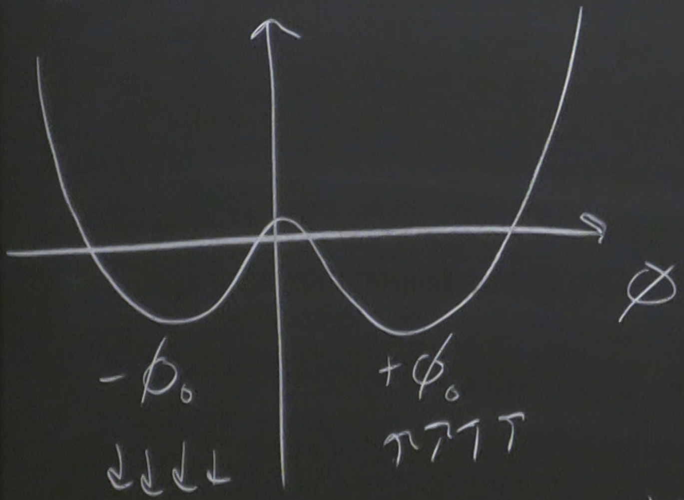
$$S[\rho,\vec{n}] = \int d^Dx \left[\frac{1}{2}(\nabla\rho)^2 + \frac{1}{2}\rho^2(\nabla n^\alpha)(\nabla n^\alpha) + \underbrace{\frac{r}{2}\rho^2 + \frac{u}{4}\rho^4}\right]$$

MFT: derivatives are zero, mimize



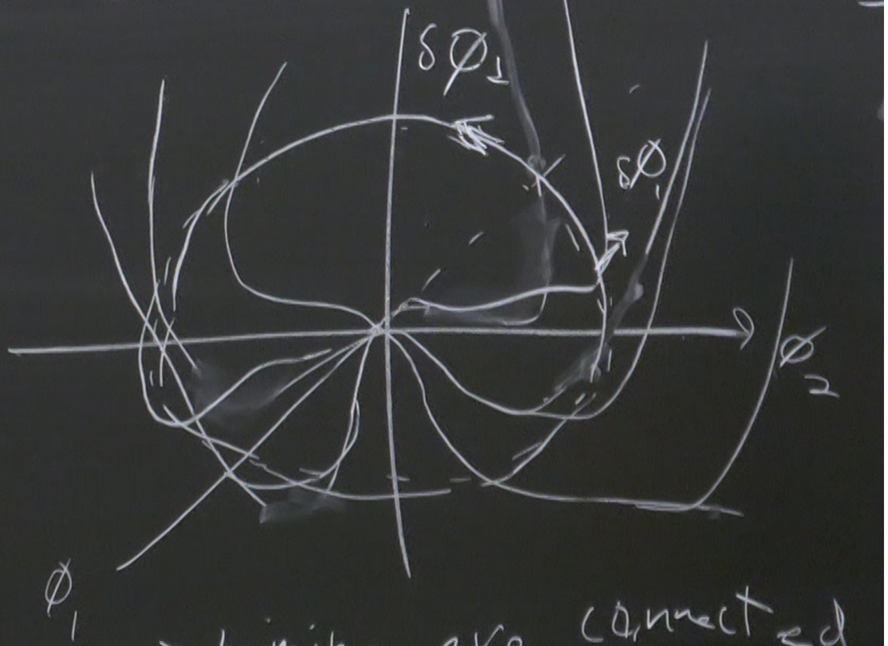
$$\underset{\text{minimum}}{\rho_0} = \begin{cases} \sqrt{\frac{-r}{u}}, & r<0 \\ 0, & r>0 \end{cases}$$

Ising : $f(\phi)$

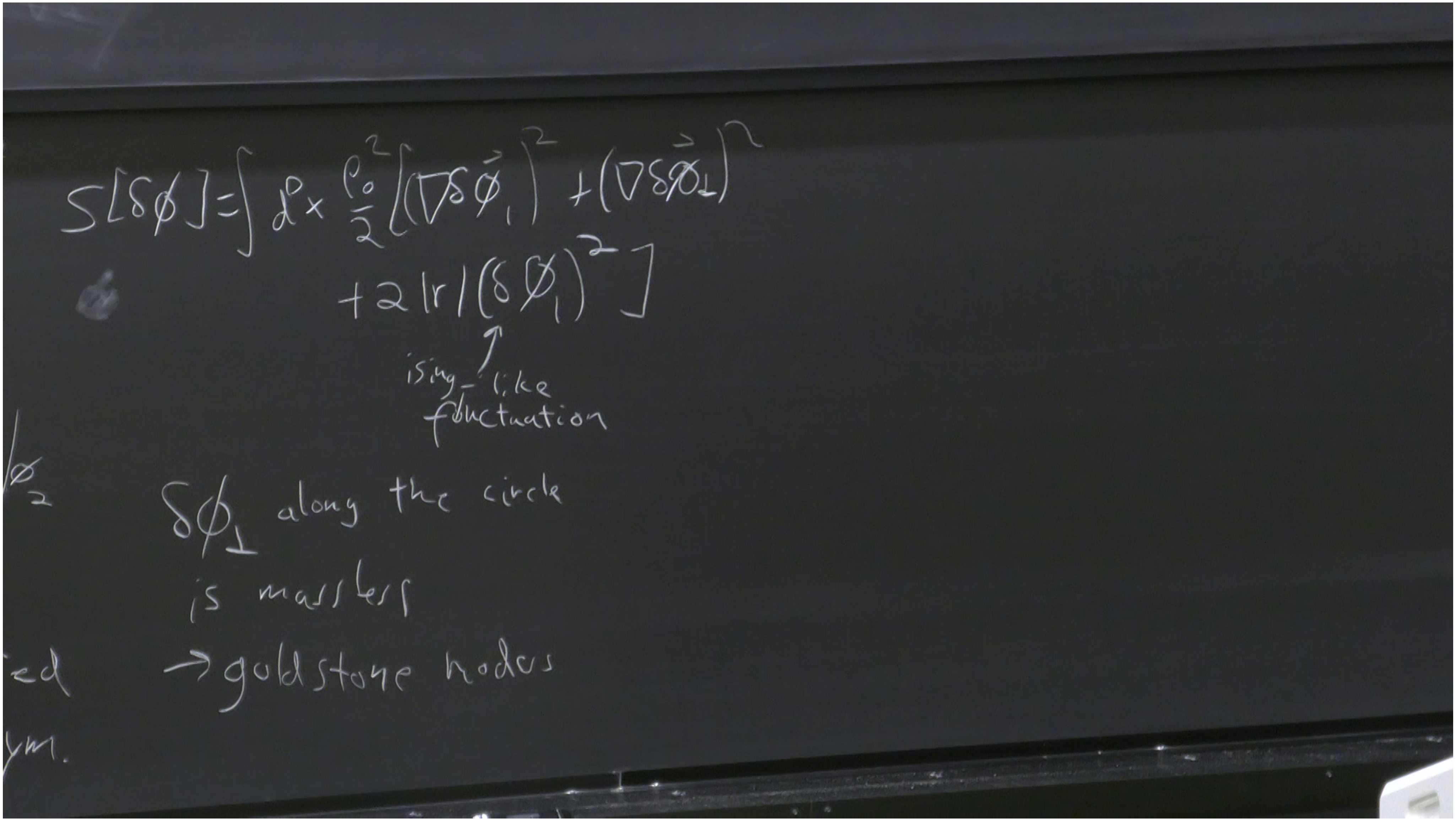


- Barrier between the two minima

Vector Model: $D=2$, $n=2$



- minima are connected by continuous sym.



$$S[\delta\phi] = \int d^Dx \, \frac{\rho_0^2}{2}\Big[(\nabla\delta\vec{\phi}_1)^2 + (\nabla\delta\vec{\phi}_\perp)^2 + 2|r|(\delta\phi_1)^2\Big]$$

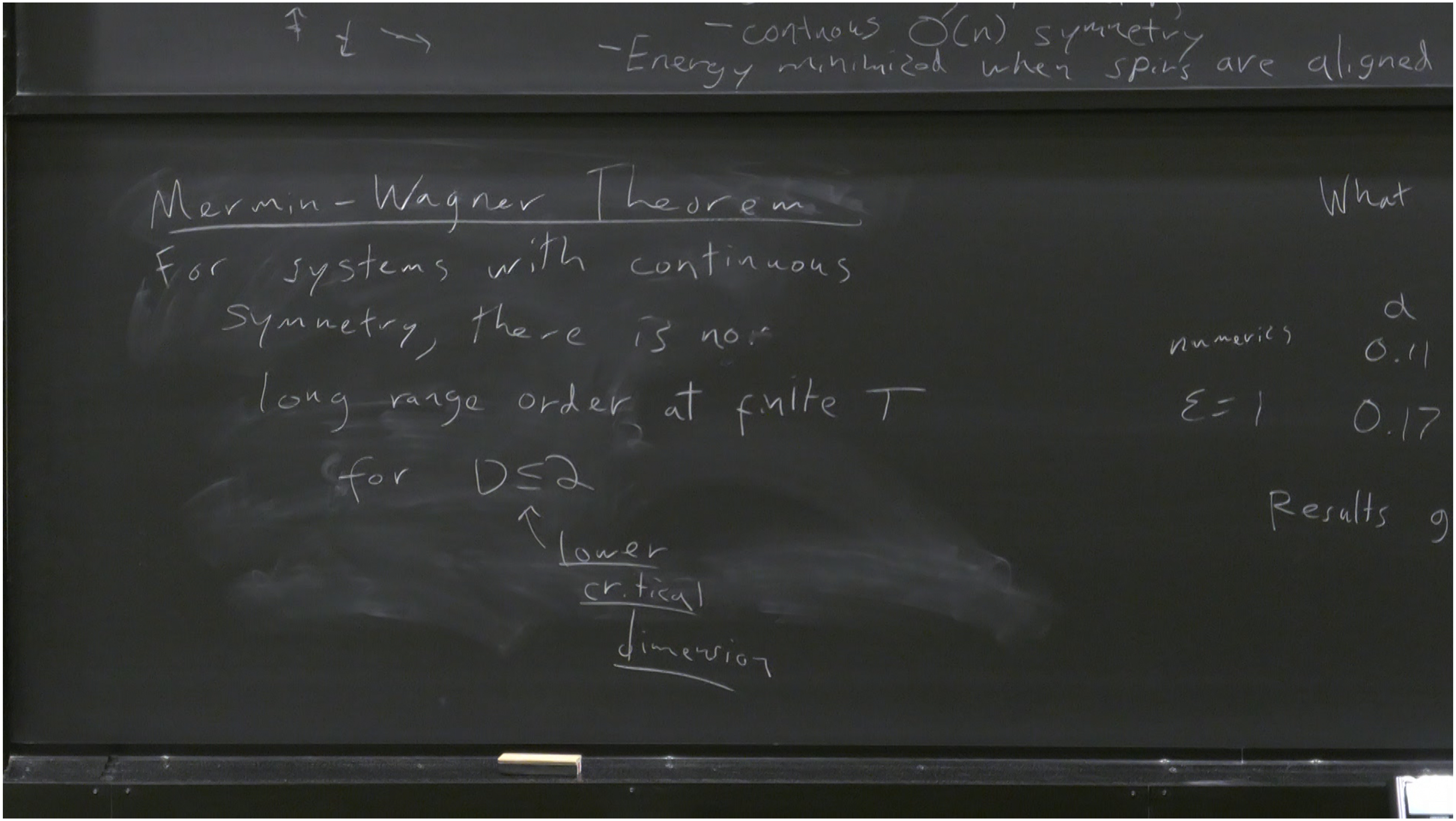
ising-like fluctuation

ϕ_2

$\delta\phi_\perp$ along the circle is massless

ed → goldstone modes

ym.



- contnuous $O(n)$ symmetry
- Energy minimized when spins are aligned

Mermin-Wagner Theorem

For systems with continuous symmetry, there is no long range order at finite T for $D \leq 2$

↑ lower critical dimension

What

	α
numerics	0.11
$\varepsilon = 1$	0.17

Results g