Title: Statistical Physics Lecture - 120423

Speakers: Emilie Huffman

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Momentum Space Action:

$$S(\phi) = \frac{A^2\beta}{2}\frac{1}{V a^D}\sum_k B_k(1-\beta B_k)|\varphi_k|^2 - \beta A \frac{H}{a^D}\varphi_0$$

$$+ \frac{\beta^4 A^4}{12}\frac{1}{V^3 a^D}\sum_{\substack{k_1,k_2,\\k_3,k_4}} B_{k_1}B_{k_2}B_{k_3}B_{k_4}\,\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}$$

$$B_k = \lambda + 2J\sum_\mu \cos k_\mu a \approx \lambda + 2J\sum_\mu\left(1-\frac{k_\mu^2 a^2}{2}\right)$$

$$= \frac{1}{\beta_c} - Jk^2a^2 \longleftarrow T_c = \lambda + \frac{2JD}{k}$$

So we approximate

$$\frac{A^2}{a^D}\beta B_k (1-\beta B_k) \approx \frac{A^2}{a^D}\left(t+\beta_c J a^2 k^2\right) = \frac{t}{a^2 J \beta_c} + k^2$$

$$A = \frac{a^{(D-2)/2}}{\sqrt{\beta_c J}}$$

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some number

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$$T_c = \frac{\lambda + 2JD}{k}$$

(gaussian)

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$$r = \frac{A^2 t}{a^D} = \frac{t}{a^2 J\beta_c}$$

$$h = \frac{\beta A H}{a^D} = \frac{\beta H}{a^{(D+2)/2}\sqrt{\beta_c J}}$$

$$u = \frac{2\beta^4 A^4}{a^D \beta_c^4} = \frac{2\beta^4 a^{D-4}}{\beta_c^6 J^2}$$

$$\sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right)$$

expanding in powers of

$\beta A B_k$

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$$\sum_i \ln(\cosh(\beta A (B\phi)_i))$$

expanding in powers of

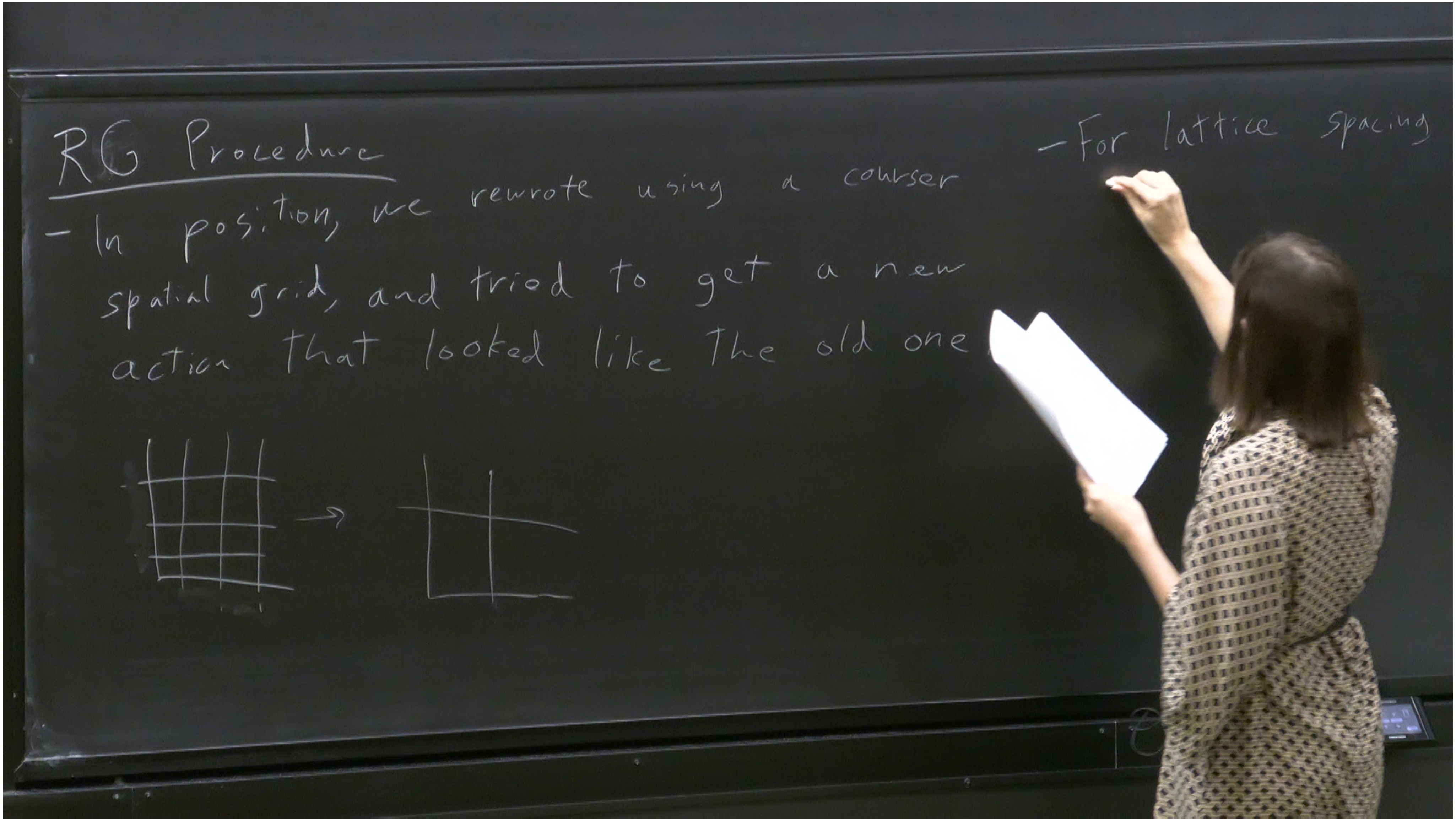
$$\beta A B_k \to \frac{\beta A}{\beta_c}$$

$$\frac{a^{(D-2)/2}}{\sqrt{\beta_c J}}$$

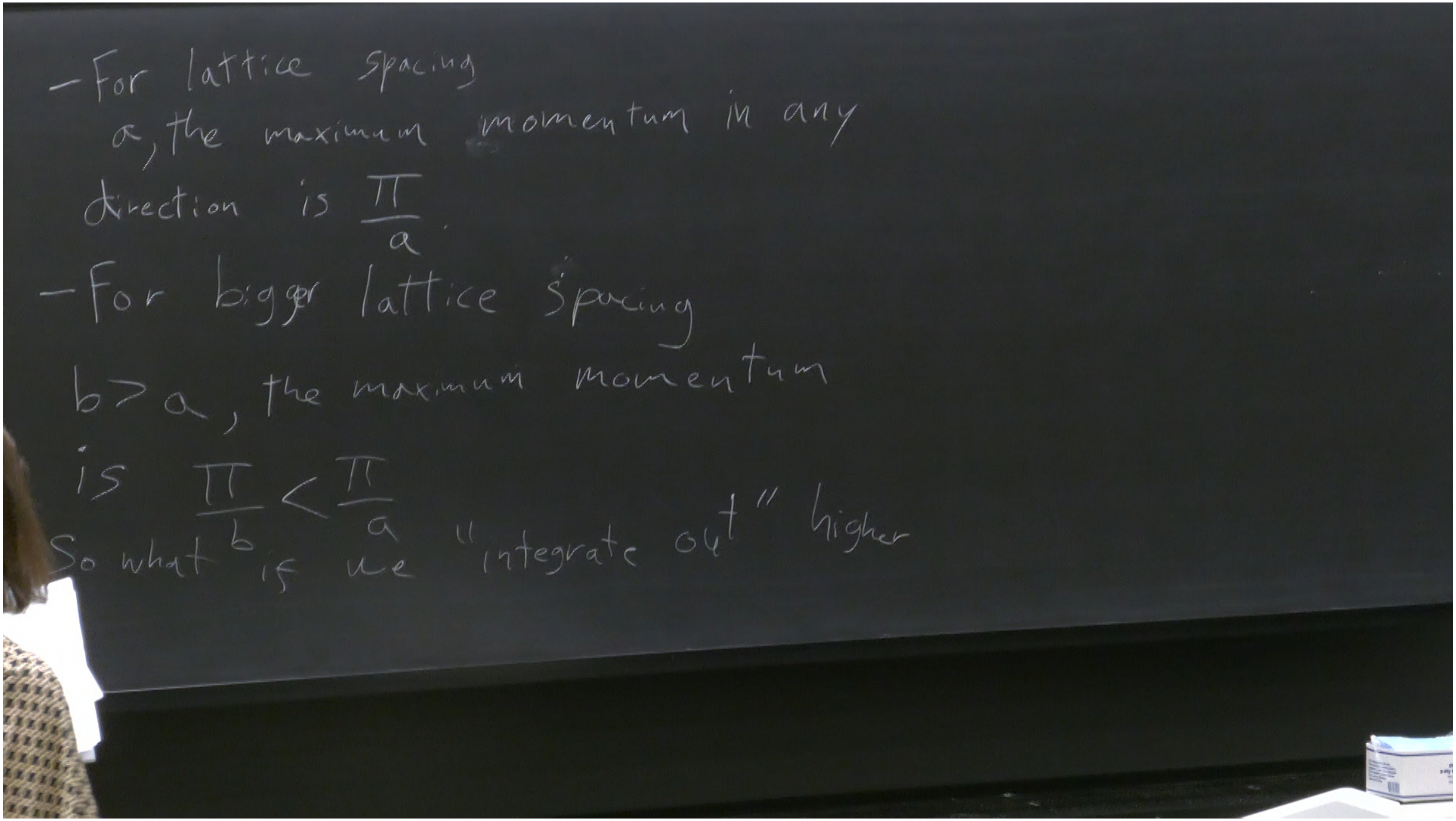
We have:

$$Z \simeq \frac{1}{(\pi V)^{N/2}} \int^{\Lambda} \mathcal{D}^N \varphi \, e^{-S(\varphi)}$$

$$S(\varphi) = \frac{1}{2V} \sum_k (r + k^2) |\varphi_k|^2 - h\varphi_0 + \frac{u}{4! \, V^3} \sum_{\substack{k_1, k_2, \\ k_3, k_4}} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4} \, \delta_{k_1+k_2+k_3+k_4, 0}$$


RG Procedure
- In position, we rewrote using a courser spatial grid, and tried to get a new action that looked like the old one
- For lattice spacing

- For lattice spacing a, the maximum momentum in any direction is $\frac{\pi}{a}$.


- For lattice spacing
a, the maximum momentum in any
direction is $\frac{\pi}{a}$.
- For bigger lattice spacing
$b > a$, the maximum momentum
is $\frac{\pi}{b} < \frac{\pi}{a}$
So what if we "integrate out" higher

Make this precise:

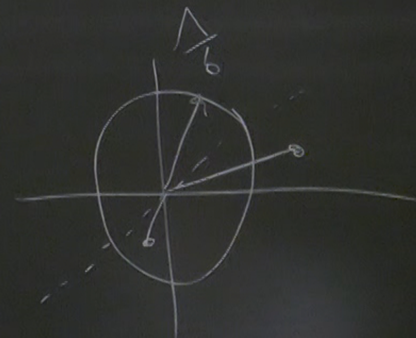
1. Split up φ into fast and slow modes:

$$\Lambda = \frac{\pi}{a}$$

$$\varphi_k^+ = \begin{cases} \varphi_k & \text{if } |k_\mu| > \frac{\Lambda}{b} \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_k = \varphi_k^+ + \varphi_k^-$$

$$\varphi_k^- = \begin{cases} \varphi_k & \text{if } |k_\mu| < \frac{\Lambda}{b} \\ 0 & \text{otherwise} \end{cases}$$



$$b > a = 1$$

Thus we have

$$\int^{\Lambda_L\,\pi/a} \mathcal{D}^N\varphi \, e^{-S(\varphi)} = \int^{\Lambda} \mathcal{D}\varphi^- \int^{\Lambda} \mathcal{D}\varphi^+ \, e^{-S(\varphi^- + \varphi^+)}$$

Now for the Gaussian model,
we have

Thus we have

$$\int^{\Lambda \approx \pi/a} \mathcal{D}^N\varphi \, e^{-S(\varphi)} = \int^{\Lambda} \mathcal{D}\varphi^- \int^{\Lambda} \mathcal{D}\varphi^+ \, e^{-S(\varphi^- + \varphi^+)} \overset{?}{\sim} \int^{\Lambda/b} \mathcal{D}^N\varphi^- \, e^{-S'(\varphi^-)}$$

Now for the Gaussian model, we have $S(\varphi^- + \varphi^+) = S(\varphi^-) + S(\varphi^+)$

$$= \int^{\Lambda/b} \mathcal{D}\varphi^- \, e^{-S(\varphi^-)} \int \mathcal{D}^N\varphi^+ \, e^{-S(\varphi^+)}$$

$$= \frac{1}{\beta_c} - Jk^2a^2 \longleftarrow T_c = \frac{\lambda + 2JD}{k}$$

$$S(\varphi^+) = \frac{1}{2V} \sum_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \left(r + \sum_\mu k_\mu^2\right) |\varphi_k^+|^2$$

$$\downarrow$$

$$e^{-S'(\varphi^-)} = e^{-S(\varphi^-)} \prod_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \sqrt{\frac{2V\pi}{r+k^2}}$$

$$S(\varphi^+) = \frac{1}{2V} \sum_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \left(r + \sum_\mu k_\mu^2\right) |\varphi_k^+|^2$$

$$\downarrow$$

$$e^{-S'(\varphi^-)} \sim e^{-S(\varphi^-)} \prod_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \sqrt{\frac{2V\pi}{r+k^2}}$$

$$S'(\varphi^-) = S(\varphi^-)$$

$$Z = \left(\prod_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \sqrt{\frac{2\pi V}{r^2+k^2}} \frac{1}{\sqrt{\pi V}} \right) \prod_{k \,|\, |k_\mu| < \frac{\Lambda}{b}} \frac{1}{\sqrt{\pi V}} \int^{\Lambda} \mathcal{D}^{\Lambda} \varphi^- \, e^{-S(\varphi^-)}$$

$= \frac{\lambda + 2JD}{k}$

$$h = \frac{\beta A H}{a^D} = \frac{\beta H}{a^{(D+2)/2}\sqrt{\beta_c J}}$$

$$u = \frac{2\beta^4 A^4}{a^D \beta^4} = 2\beta^4 a^{D-4}$$

$\sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right)$

expanding in powers of $\beta A B \to \beta A$

$\frac{a}{\sqrt{\beta_c J}}$

2. Now we are ready to <u>rescale</u>.
We are integrating only to Λ/b. Rescaling updates variables,
so we integrate to Λ again.

2. Now we are ready to rescale.

We are integrating only to Λ/b. Rescaling updates variables,

lattice sites

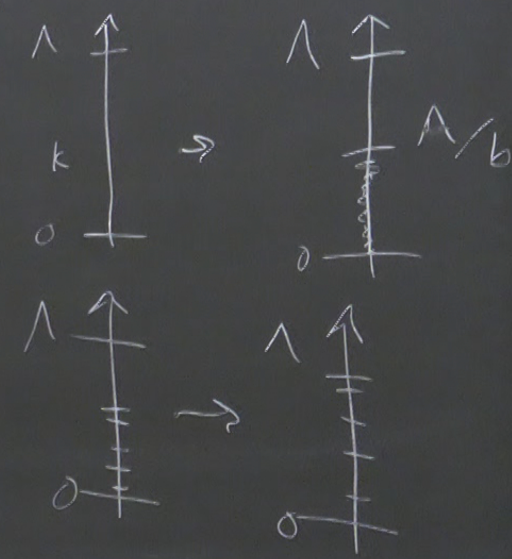
So we integrate to Λ again.

$$N' = \frac{N}{b^D} \qquad V' = \frac{V}{b^D}$$

$$k' = bk$$

take these smaller momenta (slow modes), and make them bigger to go up to Λ — no more fast modes in rescaled system

$$\varphi'_{k'} = z\,\varphi_k$$



$$S(\varphi^-) =$$

$$e \sim e \quad \prod_{k/|k_n|>\frac{\Lambda}{b}} \sqrt{r+k^2}$$

$$S'(\varphi^-) = S(\varphi^-)$$

$$Z = \left(\prod_{k/|k_n|>\frac{\Lambda}{b}} \sqrt{\frac{2\pi V}{r^2+k^2}} \frac{1}{\sqrt{\pi V}} \right) \prod_{k/|k_n|<\frac{\Lambda}{b}} \frac{1}{\sqrt{\pi V}} \int^{\Lambda} \mathcal{D}^N \varphi^- e^{-S(\varphi^-)}$$

$$k' = b k$$

$$\varphi'_{k'} = z \varphi_k$$

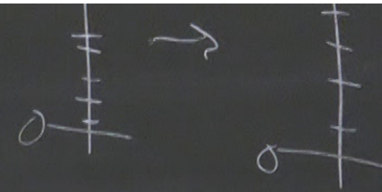
take the
and make
to Λ – ...
Sy...

We have:

$$\boxed{\begin{aligned} Z &\simeq \frac{1}{(\pi V)^{N/2}} \int^{\Lambda} \mathcal{D}^N \varphi \, e^{-S(\varphi)} \\ S(\varphi) &= \frac{1}{2V} \sum_k (r+k^2)|\varphi_k|^2 - h\varphi_0 + \frac{u}{4!V^3} \sum_{\substack{k_1,k_2,\\k_3,k_4}} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\, \delta_{k_1+k_2+k_3+k_4,0} \end{aligned}}$$

S(

$= \varphi_k$ to Λ — no more fast modes in rescaled system

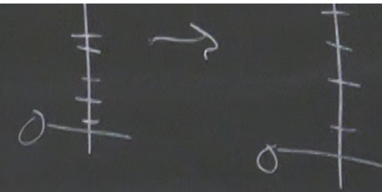


$$S(\varphi^-) = \frac{b^D}{2V'} \sum_{k'} (r + b^{-2}k'^2) z^{-2} |\varphi^-_{k'}|^2 - h z^{-1} \varphi_0$$

$$b^D b^{-2} z^{-2} = 1$$

$\int \ldots k_1 + k_2 + k_3 + k_4, 0$

φ_k to Λ — no more fast modes in rescaled system



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$$b^{-D} b^{-2} z^{-2} = 1$$

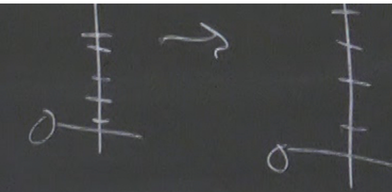
$$z^2 = b^{-D-2}$$

$$z = b^{-\frac{D+2}{2}}$$

$$\delta_{k_1+k_2+k_3+k_4,\,0}$$

$$Z(N,r,h) \simeq \left(\prod_{|k_\mu| > \Lambda/b} \frac{\sqrt{2}}{\sqrt{r+k^2}} \right) z^{-N'} b^{-DN'/2} \qquad b^{N'}$$

$\Sigma\, \varphi_k$ to Λ — no more fast modes in rescaled System



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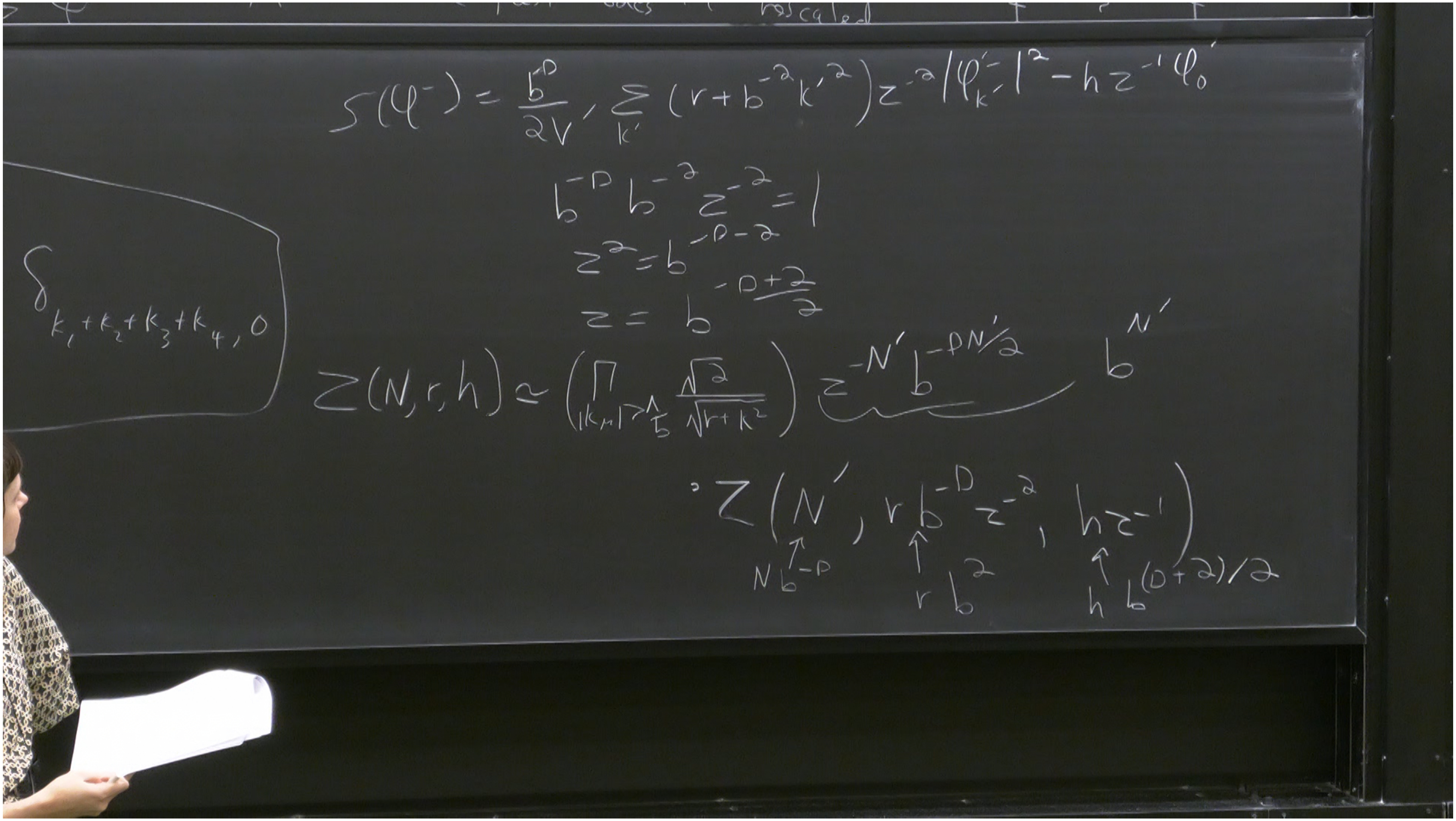
$$\cdot Z(N', r b^{-D} z^{-2},$$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$\cdot Z(N', r b^{-D} z^{-2}, h z^{-1})$
$N b^{-D}$ $r b^2$ $h b^{(D+2)/2}$

$$RG(r) = rb^2$$

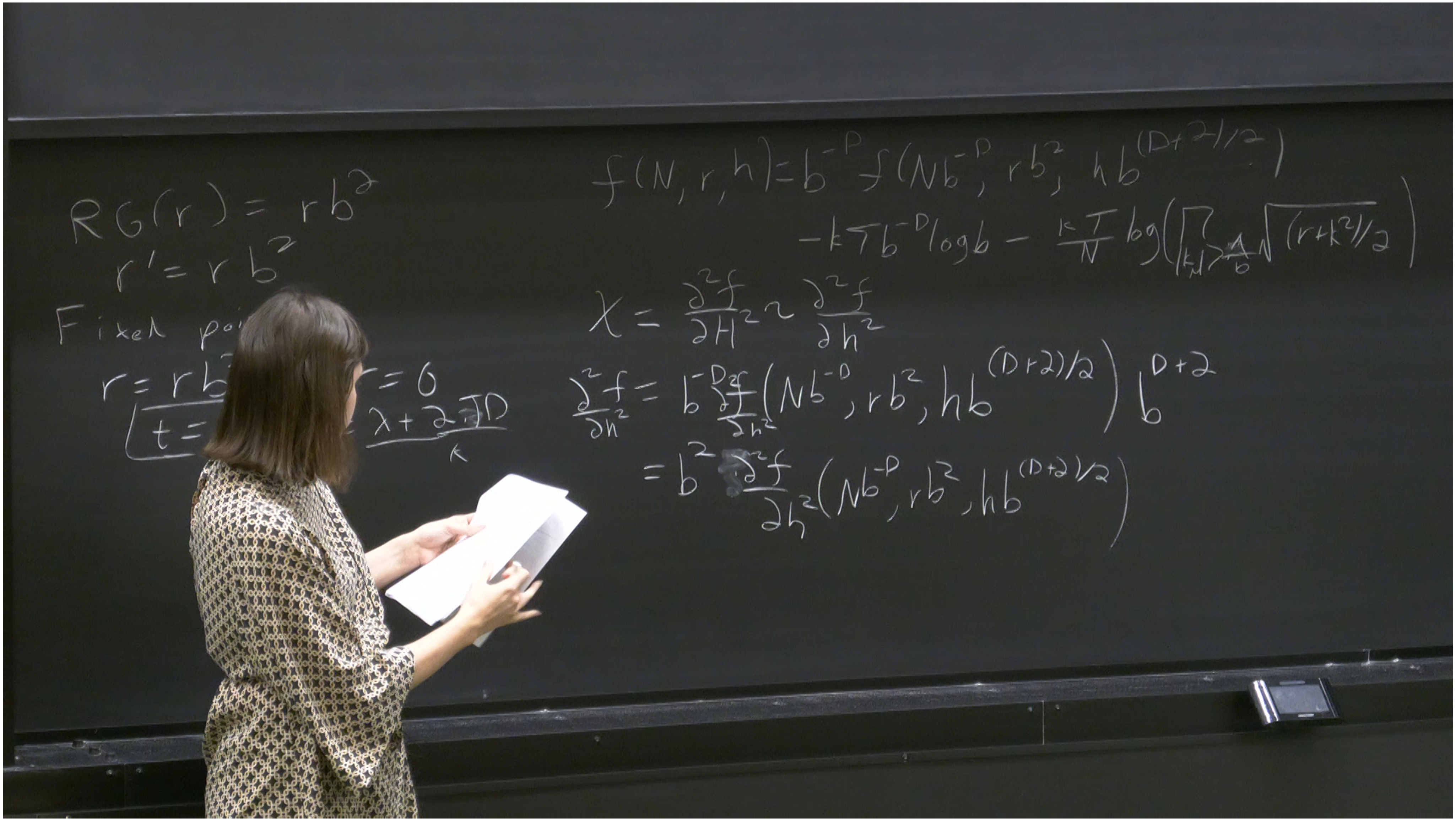
$$r' = rb^2$$

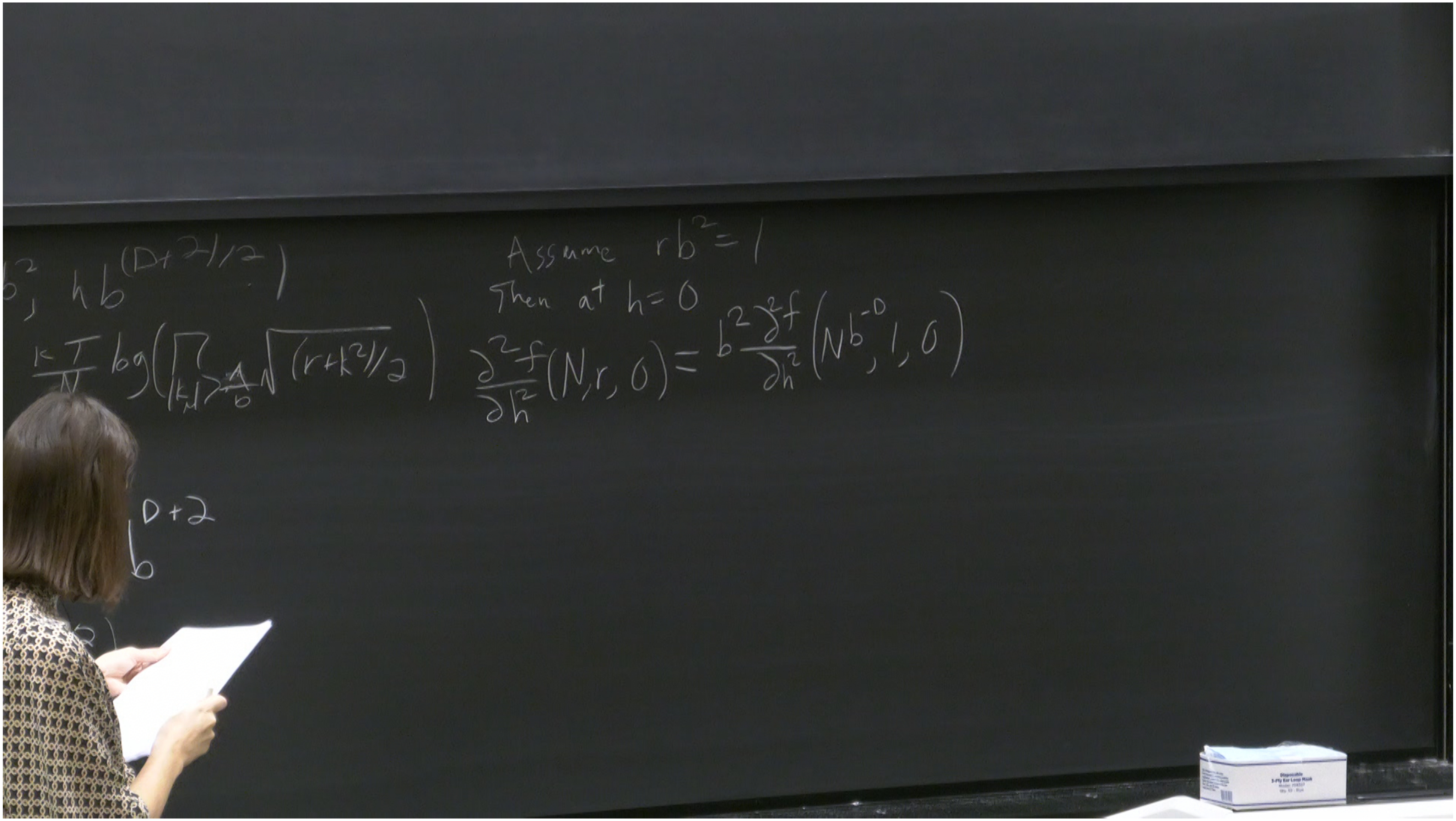
Fixed point:

$$r = rb^2 \rightarrow r = 0$$

$$\boxed{t = 0} \qquad T_c = \frac{\lambda + 2JD}{k}$$


RG(r) = rb²
r' = rb²
Fixed point
r = rb²
t = 0
f(N, r, h) = b^{-D} f(Nb^{-D}, rb², hb^{(D+2)/2})
−kT b^{-D} log b − (kT/N) log(∏_{|k|>Λ/b} √(r+k²/2))
χ = ∂²f/∂H² ∼ ∂²f/∂h²


$RG(r) = rb^2$
$r' = rb^2$
Fixed poi
$r = rb^2$
$= 0$
$t =$
$\frac{\lambda + 2 - D}{\kappa}$
$f(N, r, h) = b^{-D} f(Nb^{-D}, rb^2, hb^{(D+2)/2})$
$- kTb^{-D} \log b - \frac{kT}{N} \log\left(\prod_{|k|>\Lambda/b} \sqrt{(r+k^2)/2}\right)$
$\chi = \frac{\partial^2 f}{\partial H^2} \sim \frac{\partial^2 f}{\partial h^2}$
$\frac{\partial^2 f}{\partial h^2} = b^{-D} \frac{\partial^2 f}{\partial h^2}\left(Nb^{-D}, rb^2, hb^{(D+2)/2}\right) b^{D+2}$
$= b^2 \frac{\partial^2 f}{\partial h^2}\left(Nb^{-D}, rb^2, hb^{(D+2)/2}\right)$


Assume $rb^2=1$
Then at $h=0$
$\frac{\partial^2 f}{\partial h^2}(N,r,0) = b^2 \frac{\partial^2 f}{\partial h^2}(Nb^{-D},1,0)$

$b^2, \; h b^{(D+2)/2})$

$\frac{k T}{N} \log\Big(\prod_{|k_\mu| > \frac{A}{b}} \sqrt{(r + k^2)/2} \Big)$

$+2)/2 \Big) b^{D+2}$

$D+2)/2 \Big)$

Assume $r b^2 = 1$

Then at $h = 0$

$$\frac{\partial^2 f}{\partial h^2}(N, r, 0) = \frac{1}{r} \frac{\partial^2 f}{\partial h^2}(N b^{-D}, 1, 0)$$

const.

$$\boxed{\chi \sim \frac{1}{r}}$$

Fixed Points and Critical Exponents

- Better understanding of how to get RG fixed points and critical exponents.

Assume these parameters:

$$K = (K_1, K_2, \ldots)$$

(gaussian model: r, h
ϕ^4 : r, h, u)

$N b^{-D}$ $r b^2$ $h b^{(D+2)/2}$

- After mom. space RGT

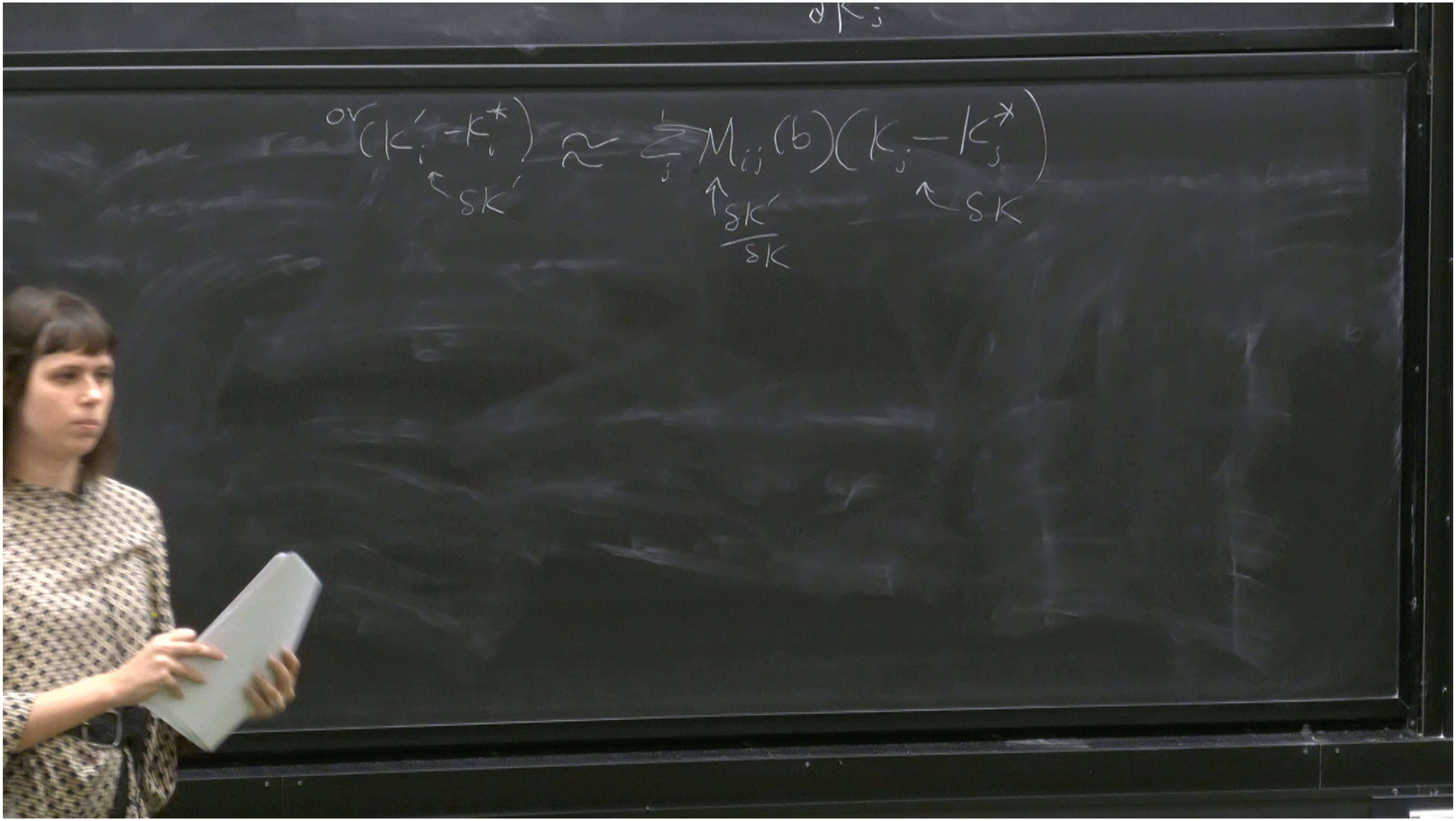
$$Z(N,K) \propto Z(N',K') \propto \int \hat{D}^{N'} \varphi' \, e^{-S(\varphi';K')}$$

$$K' = RG_b(K), \quad N' = N b^{-D}, \quad \varphi'_{k'} = z\, \varphi_{k'/b}$$

Fixed points: $K^* = RG_b(K^*)$

Linearize near the fixed points:

$$K'_i \approx K^*_i + \sum_j M_{ij}(b)(K_j - K^*_j), \quad M_{ij} = \frac{\partial K'_i}{\partial K_j}$$



∂K_j
or (K'_i − K*_i) ≈ Σ_j M_ij(b)(K_j − K*_j)
δK'
δK'/δK
δK

∂K_j

or $(K'_i - K^*_i) \approx \sum_j M_{ij}(b)(K_j - K^*_j)$

$\delta K'$ $\quad$ $\frac{\delta K'}{\delta K}$ $\quad$ δK

$$\underbrace{\delta K'}_{\text{vector}} = \underbrace{M(b)}_{\text{matrix}} \underbrace{\delta K}_{\text{vector}}$$

If M is diagonalizable,

$$M(b)\vec{v}_a = \lambda_a(b)\vec{v}_a$$

$\uparrow$ eigenvalues

RG transformations form a semigroup.

$$M(b_1)M(b_2) = M(b_1 b_2) = M(b_2 b_1) = M(b_2)M(b_1)$$

- So we can simultaneously diagonalize $M(b)$ matrices

$$\lambda_a(b_1)\lambda_a(b_2) = \lambda_a(b_1 b_2)$$

$$\frac{\partial K_i}{\partial K_j}$$

or $\underbrace{(K_i' - K_i^*)}_{\delta K'} \approx \sum_j \underbrace{M_{ij}(b)}_{\frac{\delta K'}{\delta K}} \underbrace{(K_j - K_j^*)}_{\delta K}$

$$\underbrace{\delta K'}_{\text{vector}} = \underbrace{M(b)}_{\text{matrix}} \underbrace{\delta K}_{\text{vector}}$$

$$\lambda_a(b_1)\lambda_a(b_2) = \lambda_a(b_1 b_2)$$

works for

This $\rightarrow$ $b_1^{y_a} \, b_2^{y_a} = (b_1 b_2)^{y_a}$

Because $\vec{v}_a$ are a basis, we have:

$$\delta K = \sum_a t_a v_a$$

$$\delta K' = \sum_a t'_a v_a$$

t_a and t'_a are <u>scaling fields</u> in this basis, the RGT is simpler.

$(\cancel{A})$

We have: $t'_a = b^{y_a} t_a$

$$RG_b(t_1, t_2, \ldots) = (b^{y_1} t_1, b^{y_2} t_2, \ldots)$$

– We see that the sign of y_a is very important.

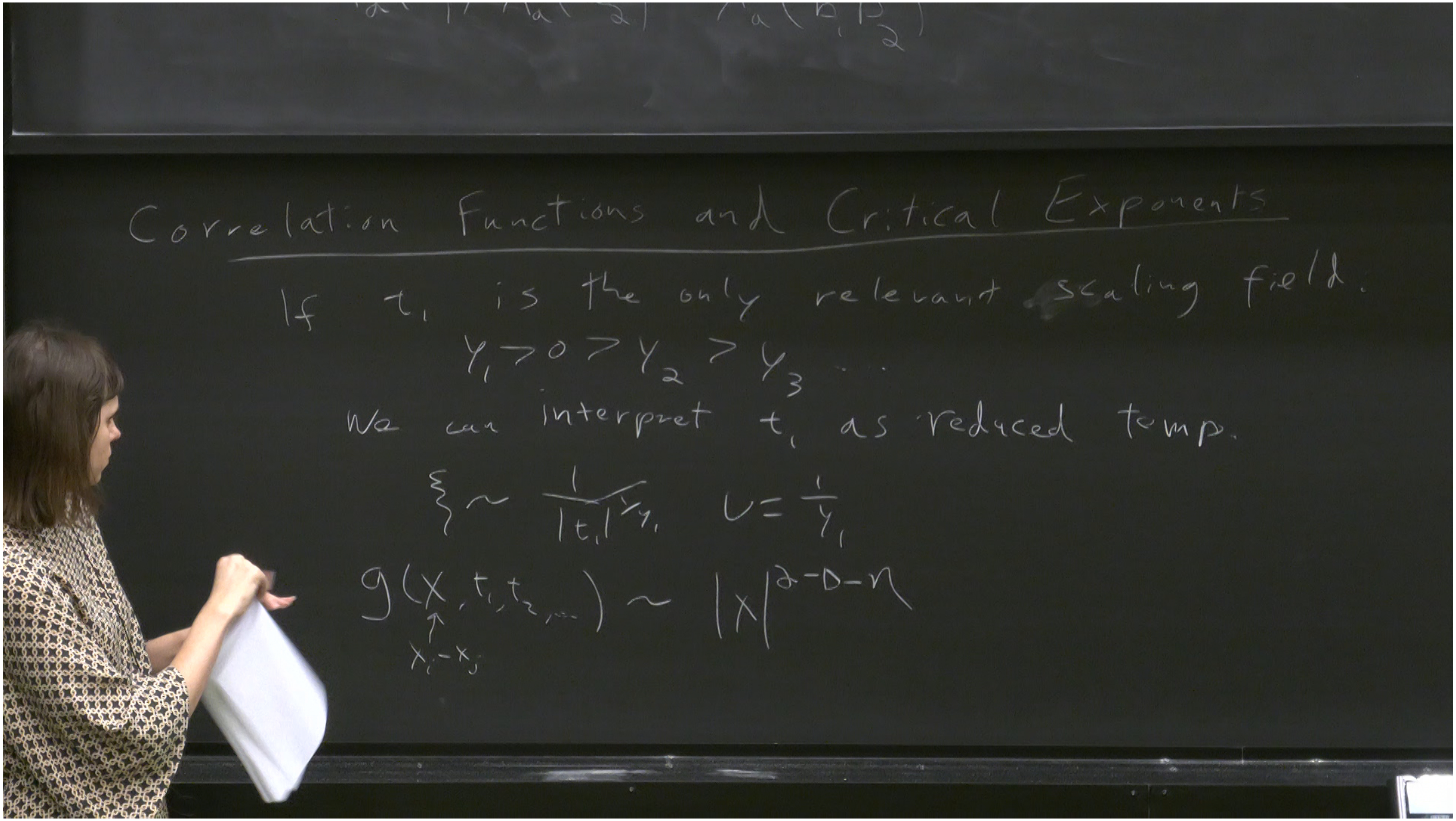
A scaling field is:

asis, <u>relevant</u>: if $y_a > 0$

<u>irrelevant</u>: if $y_a < 0$

<u>marginal</u>: if $y_a = 0$

$\rightarrow$ we need to go beyond linear to understand


Correlation Functions and Critical Exponents
If t_1 is the only relevant scaling field:
$y_1 > 0 > y_2 > y_3 \ldots$
We can interpret t_1 as reduced temp.
$\xi \sim \frac{1}{|t_1|^{1/y_1}}$ $\nu = \frac{1}{y_1}$
$g(x, t_1, t_2, \ldots) \sim |x|^{2-D-\eta}$
$x_i - x_j$