Title: Statistical Physics Lecture - 120123

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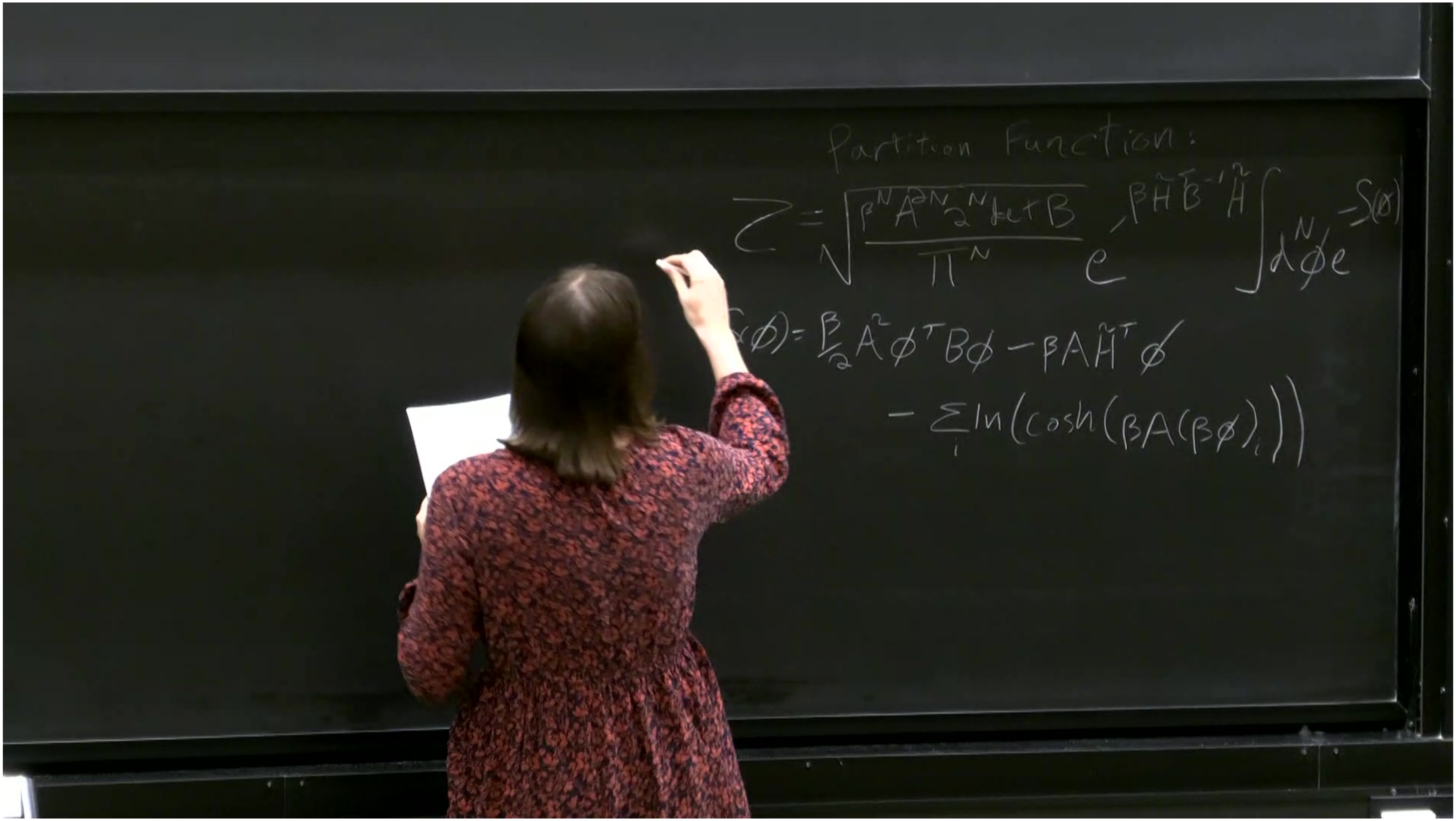
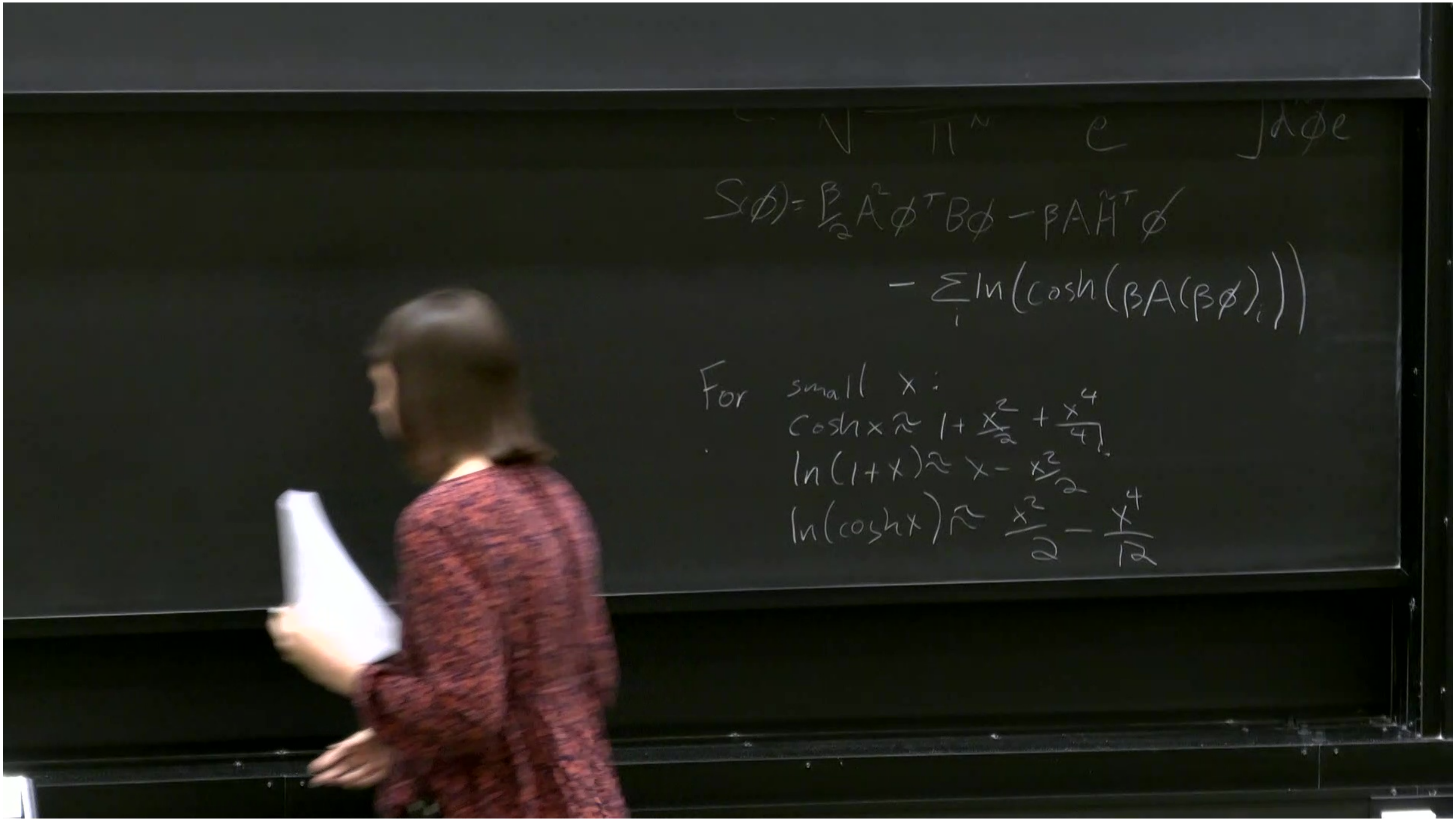
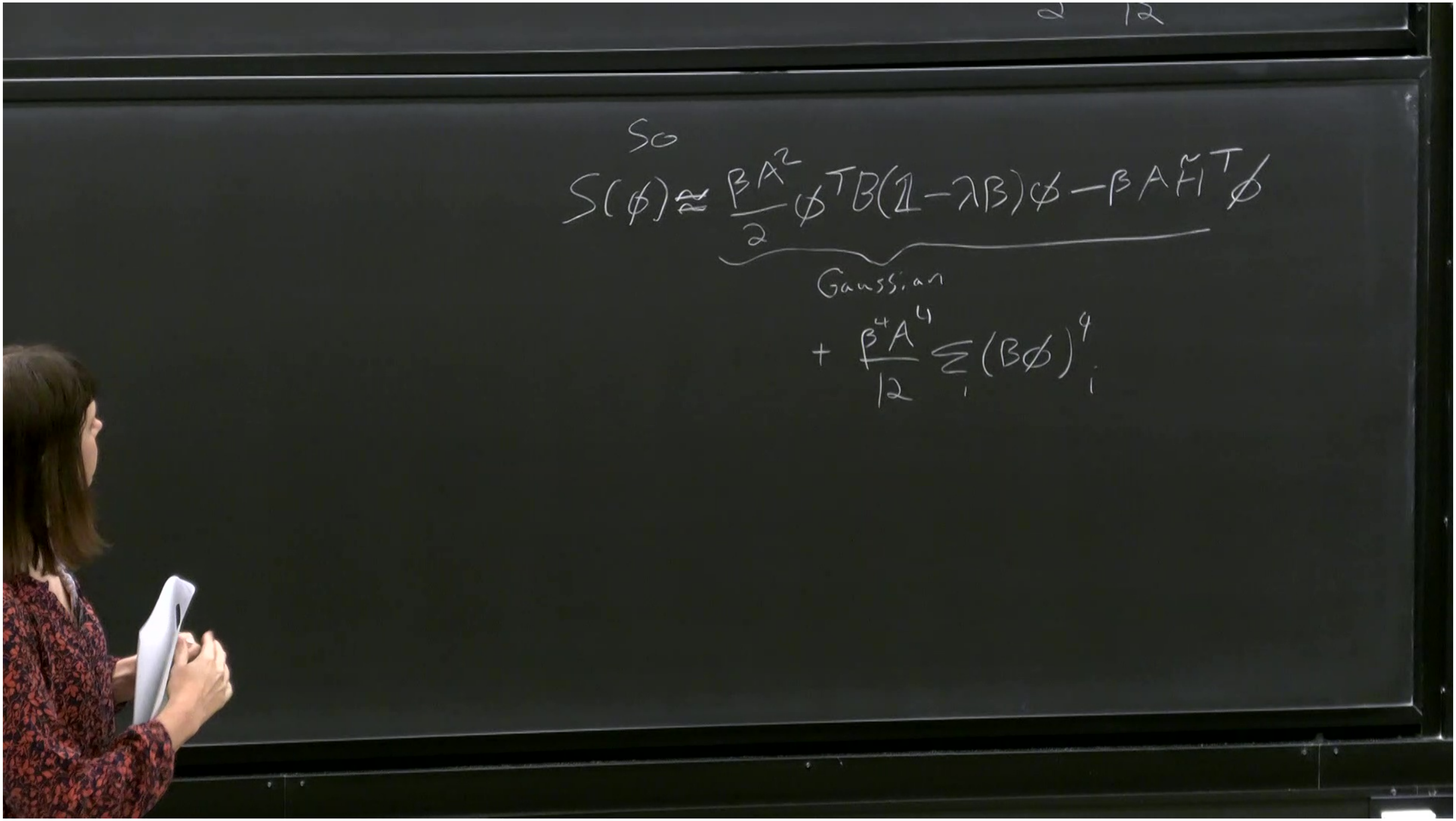
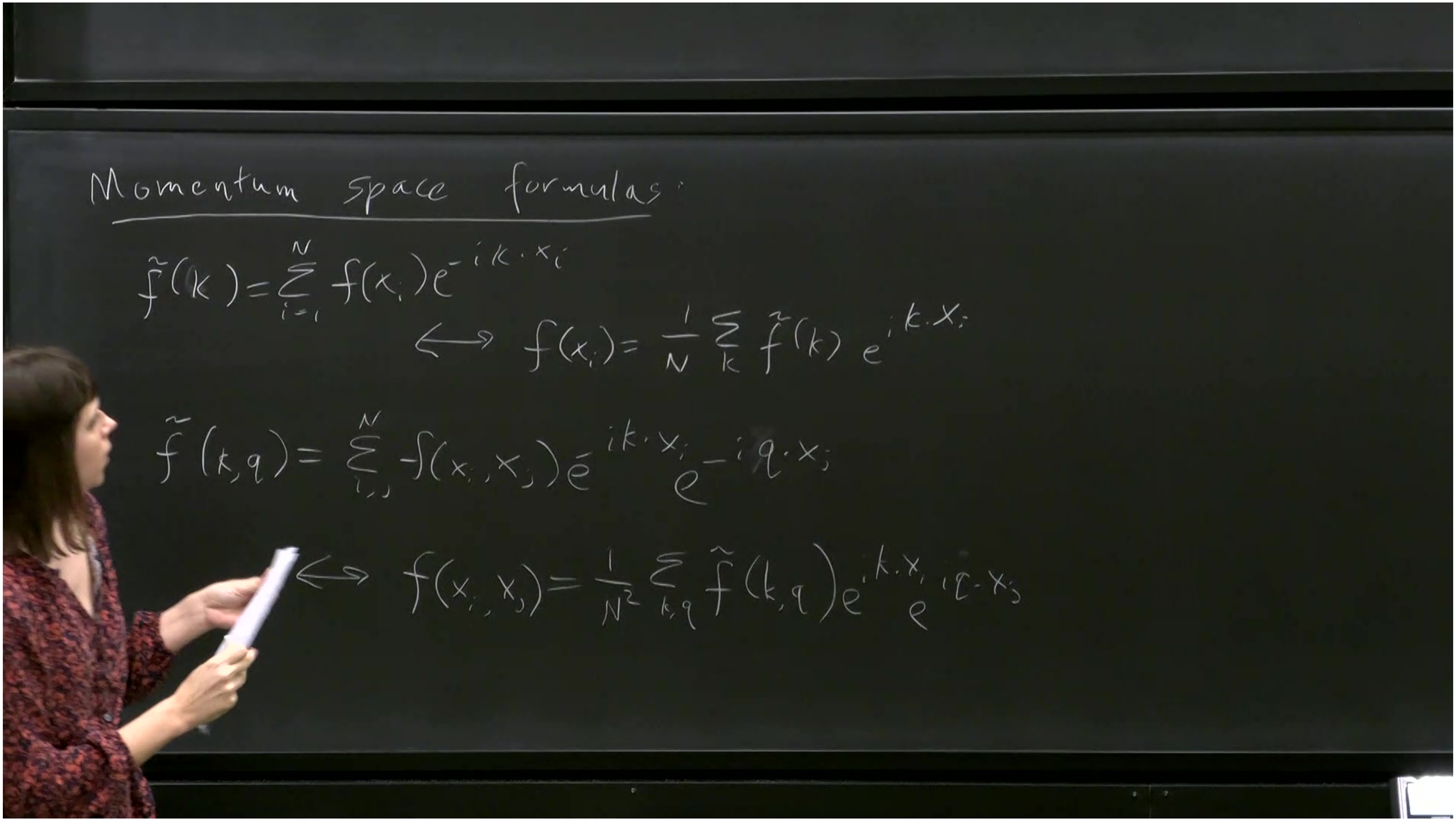

Partition Function:
$Z = \sqrt{\frac{\beta^N A^{2N} 2^N \det B}{\pi^N}} \, e^{-\beta \tilde{H}^T \tilde{B}^{-1} \tilde{H}} \int d^N\phi \, e^{-S(\phi)}$
$S(\phi) = \frac{\beta}{2} A^2 \phi^T B \phi - \beta A \tilde{H}^T \phi$
$- \sum_i \ln(\cosh(\beta A (B\phi)_i))$


$S(\phi) = \frac{\beta}{2} A^2 \phi^T B \phi - \beta A \tilde{H}^T \phi$
$- \sum_i \ln(\cosh(\beta A(\beta\phi)_i))$
For small x:
$\cosh x \approx 1 + \frac{x^2}{2} + \frac{x^4}{4!}$
$\ln(1+x) \approx x - \frac{x^2}{2}$
$\ln(\cosh x) \approx \frac{x^2}{2} - \frac{x^4}{12}$


So
$S(\phi) \approx \underbrace{\frac{\beta A^2}{2}\phi^T B(\mathbb{1}-\lambda B)\phi - \beta A \tilde{h}^T \phi}_{\text{Gaussian}}$
$+ \frac{\beta^4 A^4}{12}\sum_i (B\phi)_i^4$


Momentum space formulas:
$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i) e^{-ik\cdot x_i}$
$\longleftrightarrow \quad f(x_i) = \frac{1}{N} \sum_{k} \tilde{f}(k) e^{ik\cdot x_i}$
$\tilde{f}(k,q) = \sum_{i,j}^{N} f(x_i, x_j) e^{-ik\cdot x_i} e^{-iq\cdot x_j}$
$\longleftrightarrow \quad f(x_i, x_j) = \frac{1}{N^2} \sum_{k,q} \tilde{f}(k,q) e^{ik\cdot x_i} e^{iq\cdot x_j}$

For B_{ij}:

$$B_{ij} = J_{ij} + \lambda \mathbb{1}$$

nearest neighbors

$$B_{ij} = B(x_i, x_j)$$

$$B(x_i, x_i + \hat{e}_\mu)$$

$$\tilde{B}_{kq} = \sum_{i,j} B_{ij}\, e^{-ik\cdot x_i} e^{-iq\cdot x_j}$$

$$= \sum_i B_{ii} e^{-i(k+q)x_i} + \sum_{i,\mu} \left[B(x_i, x_i + \hat{e}_\mu)\, e^{-i(k+q)\cdot x_i} e^{-iq e_\mu a} + B(x_i - \hat{e}_\mu, x_i)\, e^{-i(k+q)\cdot x_i} e^{iq\cdot e_\mu a} \right]$$

$\hat{e}_\mu$

$$= \sum_i B_{ii} e^{-i(k+q)x_i} + \sum_{i,\mu} B(x_i, x_i + \hat{e}_\mu) e^{-i(k+q)\cdot x_i} \left(e^{-iq\cdot e_\mu a} + e^{iq\cdot e_\mu a} \right)$$

$B_{ij} = B(x_i, x_j)$

neighbors

$B(x_i, x_i + \hat{e}_\mu)$

$$\tilde{B}_{kq} = \sum_{i,j} B_{ij}\, e^{-ik\cdot x_i}\, e^{-iq\cdot x_j}$$

$$= \sum_i B_{ii}\, e^{-i(k+q)x_i} + \sum_{\substack{i,\mu \\ \uparrow \\ \hat{e}_\mu}} \left[B(x_i, x_i + \hat{e}_\mu)\, e^{-i(k+q)\cdot x_i}\, e^{-iq e_\mu a} + B(x_i - \hat{e}_\mu, x_i)\, e^{-i(k+q)\cdot x_i}\, e^{iq\cdot e_\mu a} \right]$$

$$= \sum_i B_{ii}\, e^{-i(k+q)x_i} + \sum_{i,\mu} B(x_i, x_i + \hat{e}_\mu)\, e^{-i(k+q)\cdot x_i} \overbrace{\left(e^{-iq\cdot e_\mu a} + e^{iq\cdot e_\mu a} \right)}^{2\cos(q_\mu a)}$$

$$= \sum_i \lambda e^{-i(k+q)\cdot x_i} + 2 \sum_{i,\mu} J e^{-i(k+q)\cdot x_i} \cos q_\mu a$$

$$\tilde{B}_{kq} = \lambda N \delta_{k,-q} + N 2 J \left(\sum_\mu \cos q_\mu a \right) \delta_{k,-q}$$

$$B_{ij} = \frac{1}{N^2} \sum_{k,q} \tilde{B}_{kq} e^{ik\cdot x_i} e^{iq\cdot x_j}$$

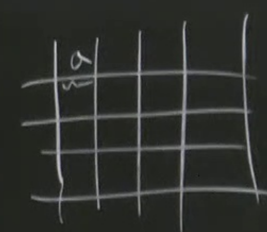
↑sum over q

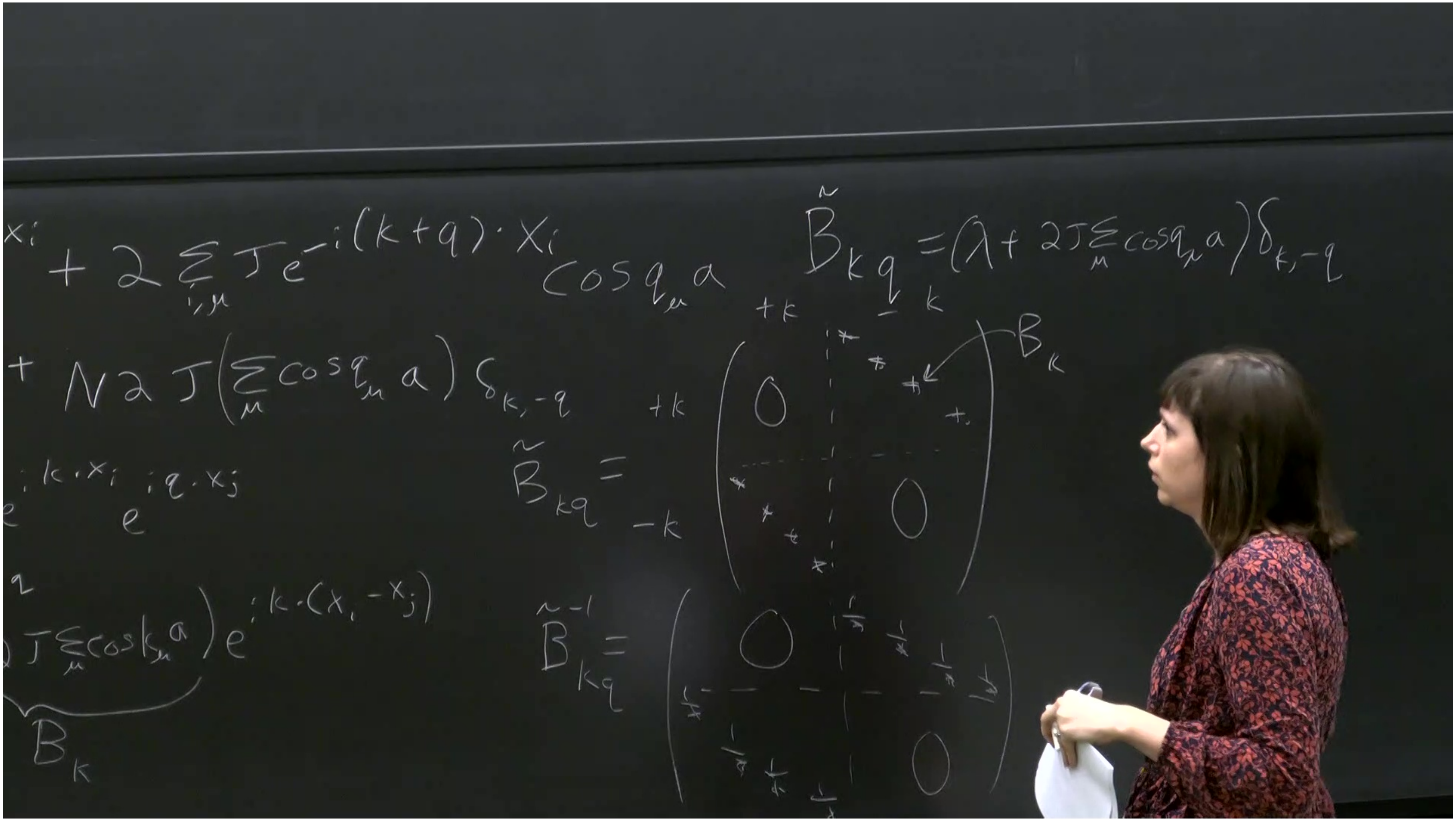
$$B_{ij} = \frac{1}{N} \sum_k \underbrace{\left(\lambda + 2J \sum_\mu \cos k_\mu a \right)}_{B_k} e^{ik\cdot(x_i - x_j)}$$

So

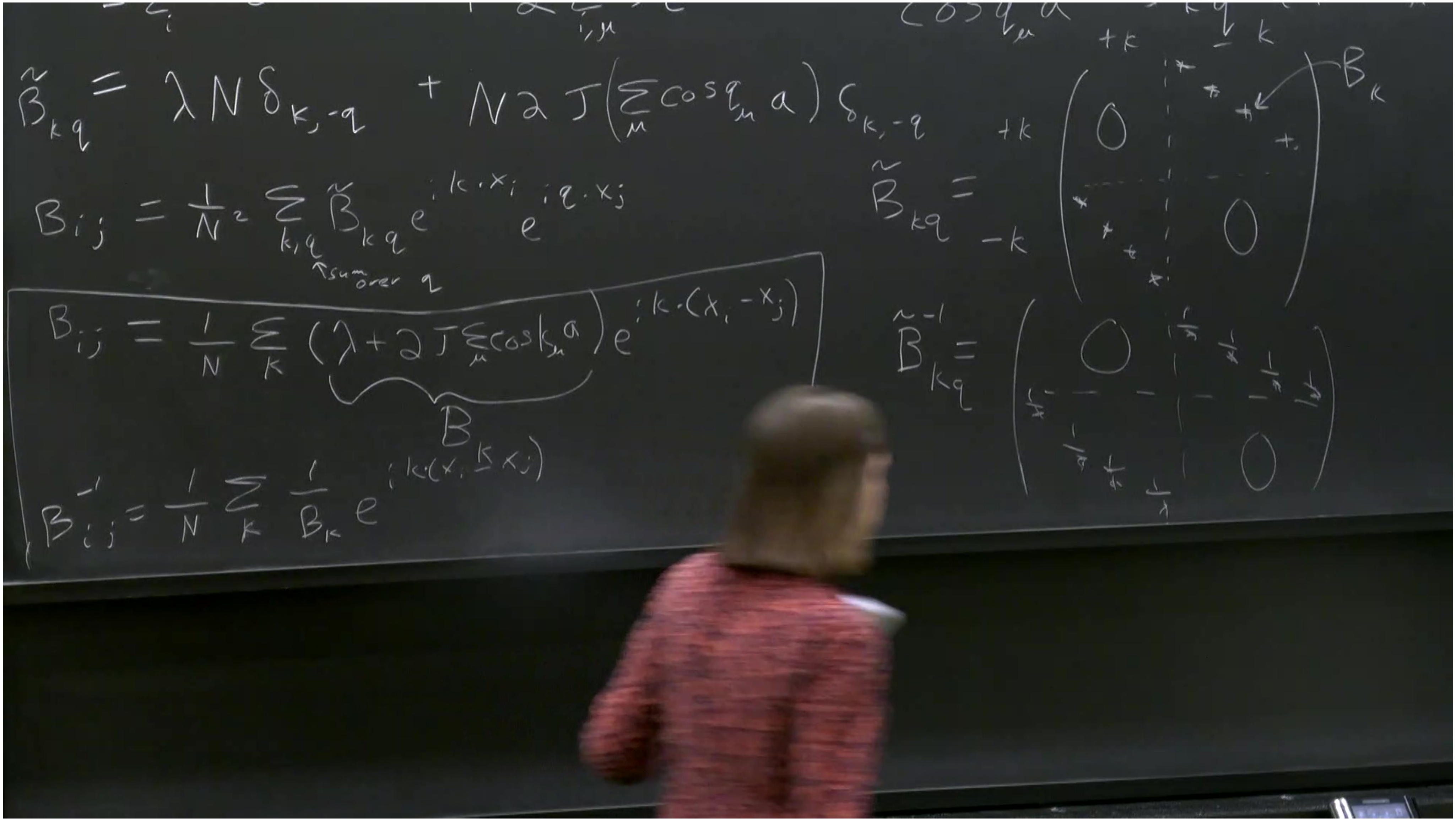
$$S(\phi) \simeq \underbrace{\frac{\beta A^2}{2} \phi^T B(\mathbb{1} - \lambda B)\phi - \beta A \tilde{h}^T \phi}_{\text{Gaussian}}$$

$$+ \frac{\beta^4 A^4}{12} \sum_i (B\phi)_i^4$$





$x_i + 2 \sum_{i,\mu} J e^{-i(k+q)\cdot x_i} \cos q_\mu a$
$+ N 2 J \left(\sum_\mu \cos q_\mu a\right) \delta_{k,-q}$
$\tilde{B}_{kq} = (\lambda + 2J \sum_\mu \cos q_\mu a)\, \delta_{k,-q}$
B_k
$+k$
$-k$
$\tilde{B}_{kq} =$
$e^{ik\cdot x_i} \; e^{iq\cdot x_j}$
$J \sum_\mu \cos k_\mu a) e^{ik\cdot(x_i - x_j)}$
B_k
$\tilde{B}^{-1}_{kq} =$


$\tilde{B}_{kq} = \lambda N \delta_{k,-q} + N 2 J \left(\sum_{\mu} \cos q_{\mu} a\right) \delta_{k,-q}$
$B_{ij} = \frac{1}{N^2} \sum_{k,q} \tilde{B}_{kq} e^{ik\cdot x_i} e^{iq\cdot x_j}$
sum over q
$B_{ij} = \frac{1}{N} \sum_{k} \underbrace{\left(\lambda + 2J \sum_{\mu} \cos k_{\mu} a\right)}_{B} e^{ik\cdot(x_i - x_j)}$
$B^{-1}_{ij} = \frac{1}{N} \sum_{k} \frac{1}{B_k} e^{ik\cdot(x_i - x_j)}$
$\tilde{B}_{kq} =$
B_k
+k
-k
$\tilde{B}^{-1}_{kq} =$

$$B$$

$$B^{-1}_{ij} = \frac{1}{N}\sum_k \frac{1}{B_k} e^{ik\cdot(x_i - x_j)}$$

$$\boxed{\det(B) = \prod_k B_k}$$

$$B^2_{il} = \frac{1}{N}\sum_k B_k^2\, e^{ikx_i} e^{-ikx_l}$$

$$\phi^T B \phi = \sum_{ij} \phi_i B_{ij} \phi_j = \frac{1}{aN}\frac{1}{aN}\frac{1}{N} \sum_{\substack{i,j \\ k,q,k'}} \varphi_k e^{ik\cdot x_i} B_q e^{iq(x_i - x_j)} \varphi_{k'} e^{ik'\cdot x_j}$$

$$= \frac{1}{V^2 N} \sum_{kqk'} \varphi_k B_q \varphi_{k'} \sum_{i,j} e^{i(k+q)\cdot x_i} e^{i(k'-q)\cdot x_j}$$

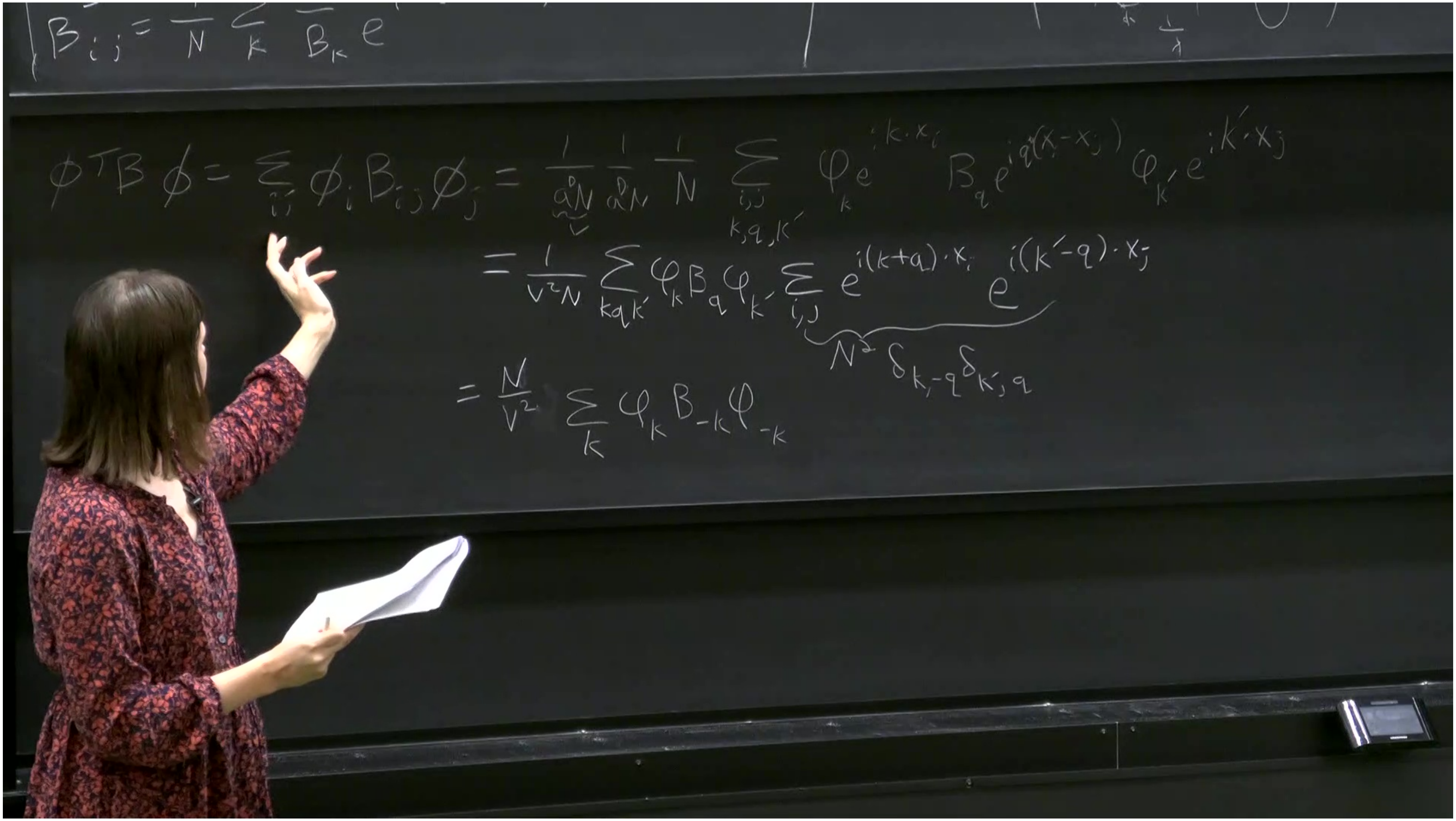
Momentum space formulas:

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot x_i}$$

$$\Longleftrightarrow \quad f(x_i) = \frac{1}{N} \sum_{k} \tilde{f}(k)\, e^{ik\cdot x_i}$$

$$\tilde{f}(k,q) = \sum_{i,j}^{N} f(x_i, x_j)\, e^{-ik\cdot x_i}\, e^{-iq\cdot x_j}$$

$$\Longleftrightarrow \quad f(x_i, x_j) = \frac{1}{N^2} \sum_{k,q} \tilde{f}(k,q)\, e^{ik\cdot x_i}\, e^{iq\cdot x_j}$$



$B_{ij} = \frac{1}{N}\sum_k B_k e$
$\phi^T B \phi = \sum_{ij} \phi_i B_{ij} \phi_j = \frac{1}{a^D N}\frac{1}{a^D N}\frac{1}{N}\sum_{i,j,k,q,k'} \varphi_k e^{ik\cdot x_i} B_q e^{iq(x_i-x_j)} \varphi_{k'} e^{ik'\cdot x_j}$
V
$= \frac{1}{V^2 N}\sum_{kqk'} \varphi_k B_q \varphi_{k'} \underbrace{\sum_{i,j} e^{i(k+q)\cdot x_i} e^{i(k'-q)\cdot x_j}}_{N^2 \delta_{k,-q}\delta_{k',q}}$
$= \frac{N}{V^2}\sum_k \varphi_k B_{-k}\varphi_{-k}$

$$B_{ij} = \frac{1}{N}\sum_k B_k e$$

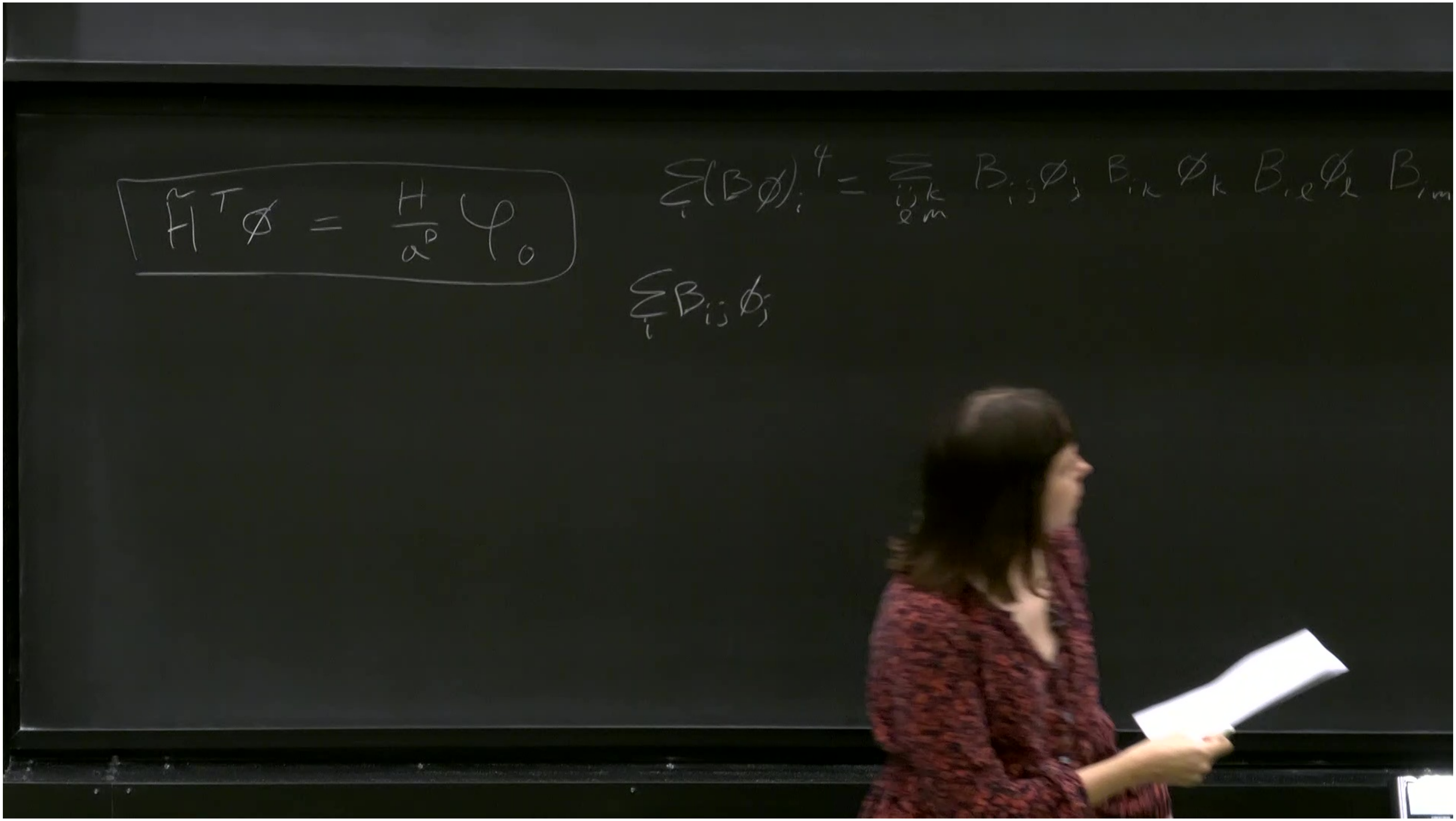
$$\phi^T B \phi = \sum_{ij} \phi_i B_{ij} \phi_j = \frac{1}{\underbrace{a^D N}_{V}} \frac{1}{a^D N} \frac{1}{N} \sum_{\substack{i,j \\ k,q,k'}} \varphi_k e^{ik\cdot x_i} B_q e^{iq(x_i - x_j)} \varphi_{k'} e^{ik'\cdot x_j}$$

$$= \frac{1}{V^2 N} \sum_{kqk'} \varphi_k B_q \varphi_{k'} \underbrace{\sum_{i,j} e^{i(k+q)\cdot x_i} e^{i(k'-q)\cdot x_j}}_{N^2 \delta_{k,-q}\delta_{k',q}}$$

$$B_k = B_{-k}$$

$$= \frac{N}{V^2} \sum_k \varphi_k B_{-k} \varphi_{-k}$$

$$\varphi_{-k} = \varphi_k^*$$


$\boxed{\tilde{H}^T \phi = \frac{H}{a^D} \varphi_0}$
$\sum_i (B\phi)_i^4 = \sum_{\substack{ijk \\ lm}} B_{ij}\phi_j \, B_{ik} \phi_k \, B_{il}\phi_l \, B_{im}$
$\sum_i B_{ij} \phi_j$

$\boxed{\varphi_0}$

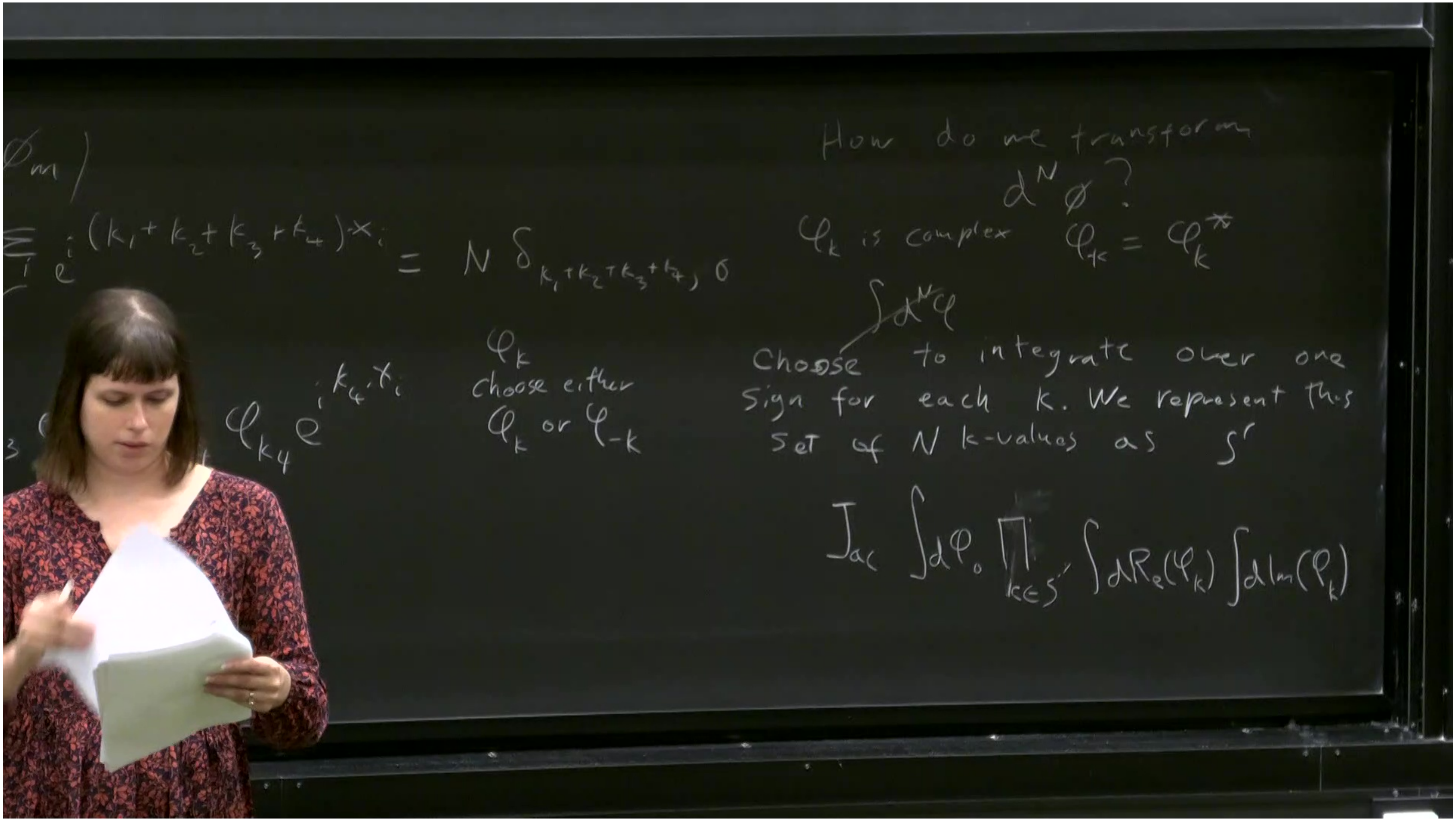
$$\sum_i (B\phi)_i^4 = \sum_{\substack{ijk\\lm}} (B_{ij}\phi_j)(B_{ik}\phi_k)(B_{il}\phi_l)(B_{im}\phi_m)$$

$$\sum_j B_{ij}\phi_j = \frac{1}{V}\sum_k B_k \varphi_k e^{ik\cdot x_i}$$

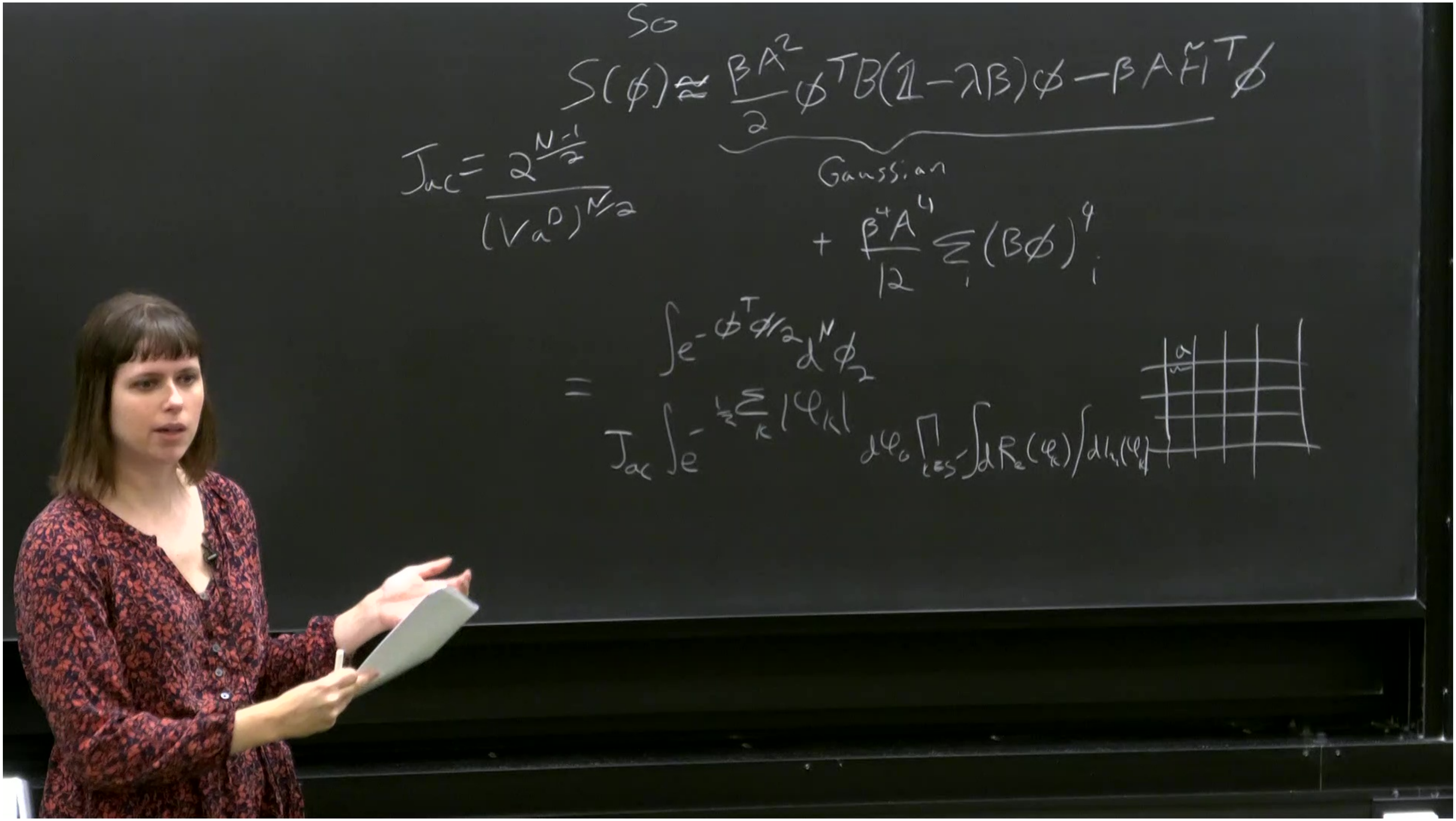
$$\sum_i e^{i(k_1+k_2+k_3+k_4)\cdot x_i} =$$

$$\sum_i (B\phi)_i^4 = \frac{1}{V^4}\sum_{\substack{i,k_1,\\k_2,k_3,k_4}} B_{k_1}\varphi_{k_1} e^{ik_1\cdot x_i} B_{k_2}\varphi_{k_2} e^{ik_2\cdot x_i} B_{k_3}\varphi_{k_3} e^{ik_3\cdot x_i} B_{k_4}\varphi_{k_4} e^{ik_4\cdot x_i}$$

$$\boxed{\sum_i (B\phi)_i^4 = \frac{1}{V^3 a^D}\sum_{\substack{k_1,k_2,\\k_3,k_4}} B_{k_1}B_{k_2}B_{k_3}B_{k_4}\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}}$$



$\phi_m)$
$\sum_i e^{i(k_1+k_2+k_3+k_4)\cdot x_i} = N\delta_{k_1+k_2+k_3+k_4,0}$
$\varphi_{k_4} e^{ik_4\cdot x_i}$
φ_k
choose either
φ_k or φ_{-k}
How do we transform
$d^N\phi$?
φ_k is complex $\varphi_{-k} = \varphi_k^*$
$\int d^N\varphi$
Choose to integrate over one
sign for each k. We represent this
set of N k-values as S'
$J_{ac}\int d\varphi_0 \prod_{k\in S'} \int d\mathrm{Re}(\varphi_k)\int d\mathrm{Im}(\varphi_k)$


So
$S(\phi) \simeq \underbrace{\frac{\beta A^2}{2}\phi^T B(\mathbb{1}-\lambda B)\phi - \beta A \tilde{H}^T \phi}_{\text{Gaussian}}$
$+ \frac{\beta^4 A^4}{12}\sum_i (B\phi)_i^4$
$J_{ac} = \frac{2^{\frac{N-1}{2}}}{(Va^D)^{N/2}}$
$= \int e^{-\phi^T\phi/2} d^N\phi_2$
$J_{ac}\int e^{-\frac{1}{2}\sum_k |\varphi_k|} d\varphi_0 \prod_{k\in S'} \int dR_e(\varphi_k)\int dI_m(\varphi_k)$
a

Final Result:

$$Z = \frac{1}{\sqrt{2}} \prod_k \sqrt{\frac{\beta A^2 4 B_k}{\pi a^D}} \int D^N \varphi \, e^{-S(\varphi)}$$

where

$$S(\varphi) \simeq \frac{A^2 \beta}{2} \frac{1}{V a^D} \sum_k B_k (1-\beta B_k) |\varphi_k|^2 - \beta A \frac{H}{a^D} \varphi_0$$

$$+ \frac{\beta^4 A^4}{12} \frac{1}{V^3 a^D} \sum_{\substack{k_1 k_2 \\ k_3 k_4}} B_{k_1} B_{k_2} B_{k_3} B_{k_4} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4} \delta_{k_1+k_2+k_3+k_4, 0}$$

The Gaussian Model: Exact Solution

$$\int d^n x \, e^{-\frac{1}{2} x^T A x + x^T w} = \sqrt{\det(2\pi A^{-1})}\, e^{\frac{1}{2} w^T A^{-1} w}$$

$$A = B(1-\beta B) \qquad w = \tilde{H}$$

$$Z = \sqrt{\det\left(\frac{2\beta A^{-2} B}{\pi}\right)} \sqrt{\det\left(2\pi B^{-1}\right.}$$

$$B^{-1}_{ij} = \frac{1}{N}\sum_{k} \frac{1}{B_k} e^{ik(x_i - x_j)}$$

The Gaussian Model: Exact Solution

$$\int d^n x\, e^{-\frac{1}{2} x^T A x + x^T w} = \sqrt{\det(2\pi A^{-1})}\, e^{\frac{1}{2} w^T A^{-1} w}$$

$$A = \beta \tilde{A}^2 B (1 - \beta B) \qquad w = \beta A \tilde{H}$$

$$Z = \sqrt{\det\left(\frac{2\beta \tilde{A}^2 B}{\pi}\right)} \sqrt{\det\left(\frac{2\pi}{\beta \tilde{A}^2} B^{-1}(1-\beta B)^{-1}\right)}\, e^{\frac{1}{2}\frac{\beta^2 \tilde{A}^2}{\beta \tilde{A}^2} \tilde{H} N B^{-1}(1-\beta B)^{-1}}$$

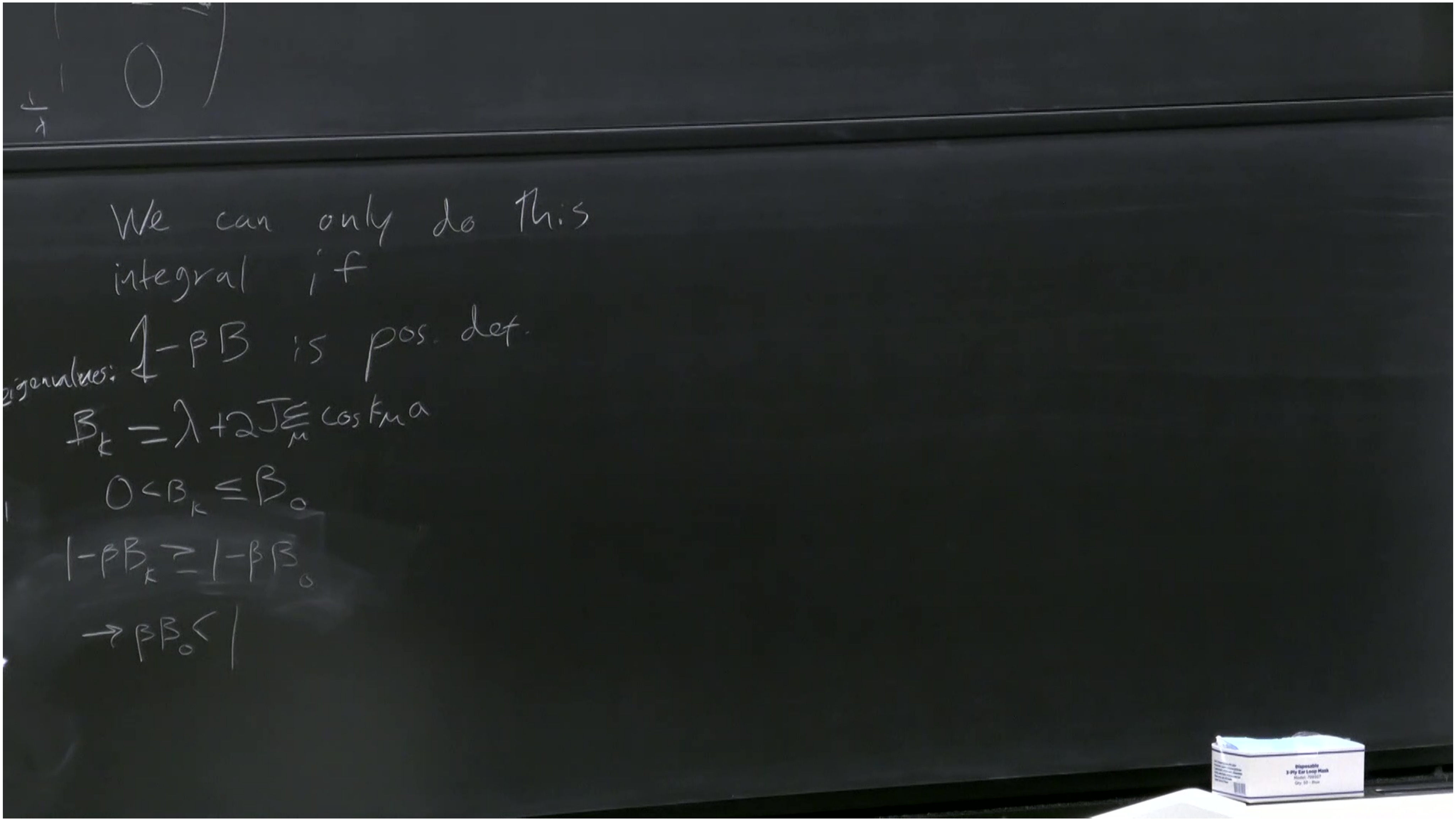
The Gaussian Model: Exact Solution

$$\int d^n x\, e^{-\frac{1}{2}x^T \underset{\text{pos. def.}}{A} x + x^T w} = \sqrt{\det(2\pi A^{-1})}\, e^{\frac{1}{2} w^T A^{-1} w}$$

$$A = \beta A^2 B (1 - \beta B) \qquad w = \beta A \tilde{H}$$

$$Z = \sqrt{\det\left(\frac{2\beta A^2 B}{\pi}\right)} \sqrt{\det\left(\frac{2\pi}{\beta A^2} B^{-1} (1-\beta B)^{-1}\right)}\, e^{\frac{1}{2}\beta^2 \frac{A^2}{\beta A^2} \tilde{H} N B^{-1} (1-\beta B)^{-1}}$$

We can o


$\frac{1}{\lambda}$ 0
We can only do this
integral if
$\mathbb{1} - \beta B$ is pos. def.
eigenvalues: $B_k = \lambda + 2J \sum_{\mu} \cos k_\mu a$
$0 < B_k \leq B_0$
$1 - \beta B_k \geq 1 - \beta B_0$
$\rightarrow \beta B_0 < 1$

We can only do this integral if

eigenvalues: $\mathbb{1} - \beta B$ is pos. def.

$$B_k = \lambda + 2J \sum_\mu \cos k_\mu a$$

$$0 < B_k \leq B_0$$

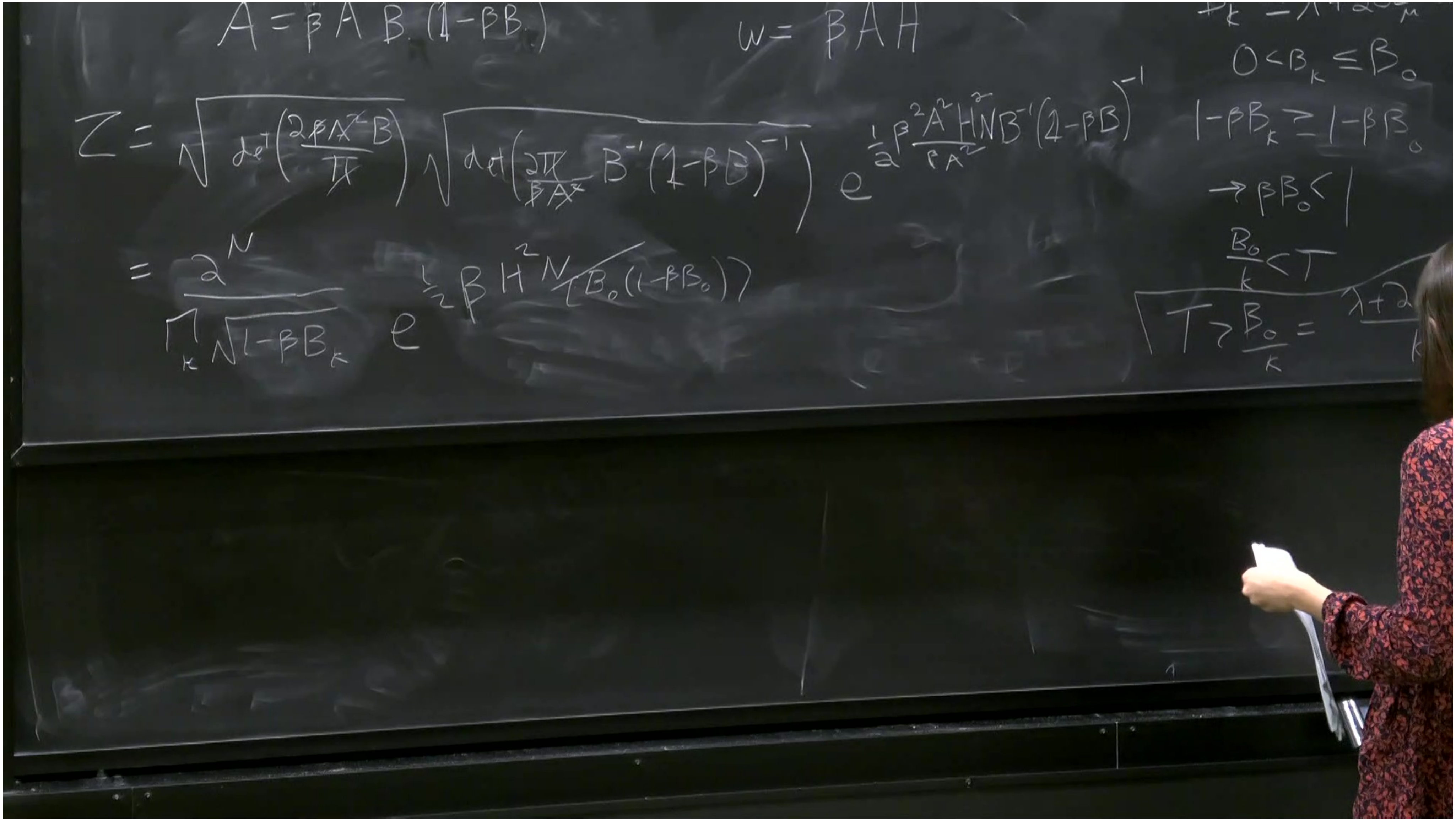
$$(\ldots - \beta B)^{-1}$$

$$1 - \beta B_k \geq 1 - \beta B_0$$

$$\rightarrow \beta B_0 < 1$$

$$\frac{B_0}{k} < T$$

$$\boxed{T > \frac{B_0}{k} = \frac{\lambda + 2JD}{k} = T_c}$$


A = βAB(1 − βB)
w = βAH
0 < B_k ≤ B_0
1 − βB_k ≥ 1 − βB_0
→ βB_0 < 1
B_0/k < T
T > B_0/k =

Critical Exponents:

$$t = \frac{T - T_c}{T_c} \longrightarrow \beta = \frac{\beta_c}{(1+t)}$$

$$f = -\frac{1}{N} kT \ln Z = -kT \ln 2 + \frac{\beta N H^2}{2\beta_0(1-\beta\beta_0)} + \frac{kT}{2N}\sum_k \ln(1-\beta B_k)$$

$$M = -\frac{\partial f}{\partial H} = \frac{\partial}{\partial H}\left(\frac{\beta_c^2 N H^2}{2t}\right) = \frac{\beta_c^2 N H}{t}$$

$$\uparrow \frac{\beta_c^2 N H^2}{2t}$$

$$\chi = \frac{\partial M}{\partial H} = \frac{\beta_c^2 N}{t}$$

$$\chi \sim t^{-1} \qquad \boxed{\gamma = 1}$$

$$\alpha = \begin{cases} 2 - D/2 & \text{if } D < 4 \\ 0 & \text{if } D \geq 4 \end{cases}$$

$$S_0$$

$$\boxed{S(\phi) \approx \frac{\beta A^2}{2} \phi^T B(\mathbb{1} - }$$

Gaussian

$$Z = \sum_{\{\sigma\}} e^{\sum_{ij} \beta \sigma_i J_{ij} \sigma_j + \sum_i H \sigma_i}$$

$$\frac{1}{N} \frac{\partial \ln Z}{\partial H} = \frac{1}{N} \frac{1}{Z} \sum_{\{\sigma\}} \left(\sum_i \sigma_i \right) e^{\left(\sum_{ij} \beta \sigma_i J_{ij} \sigma_j + \sum_i H \sigma_i \right)}$$

$$J_{ac} = \frac{2^{\frac{N-1}{2}}}{(V a^D)^{N/2}}$$

$$+ \frac{\beta^4 A^4}{12}$$

$$\chi = \sum_{ij} \sigma_i \sigma_j$$

$$\int e^{-\phi^T \phi / 2} d^N \phi_2$$

$$= J_{ac} \int e^{-\frac{1}{2} \sum_k |\varphi_k|} d\varphi_0$$