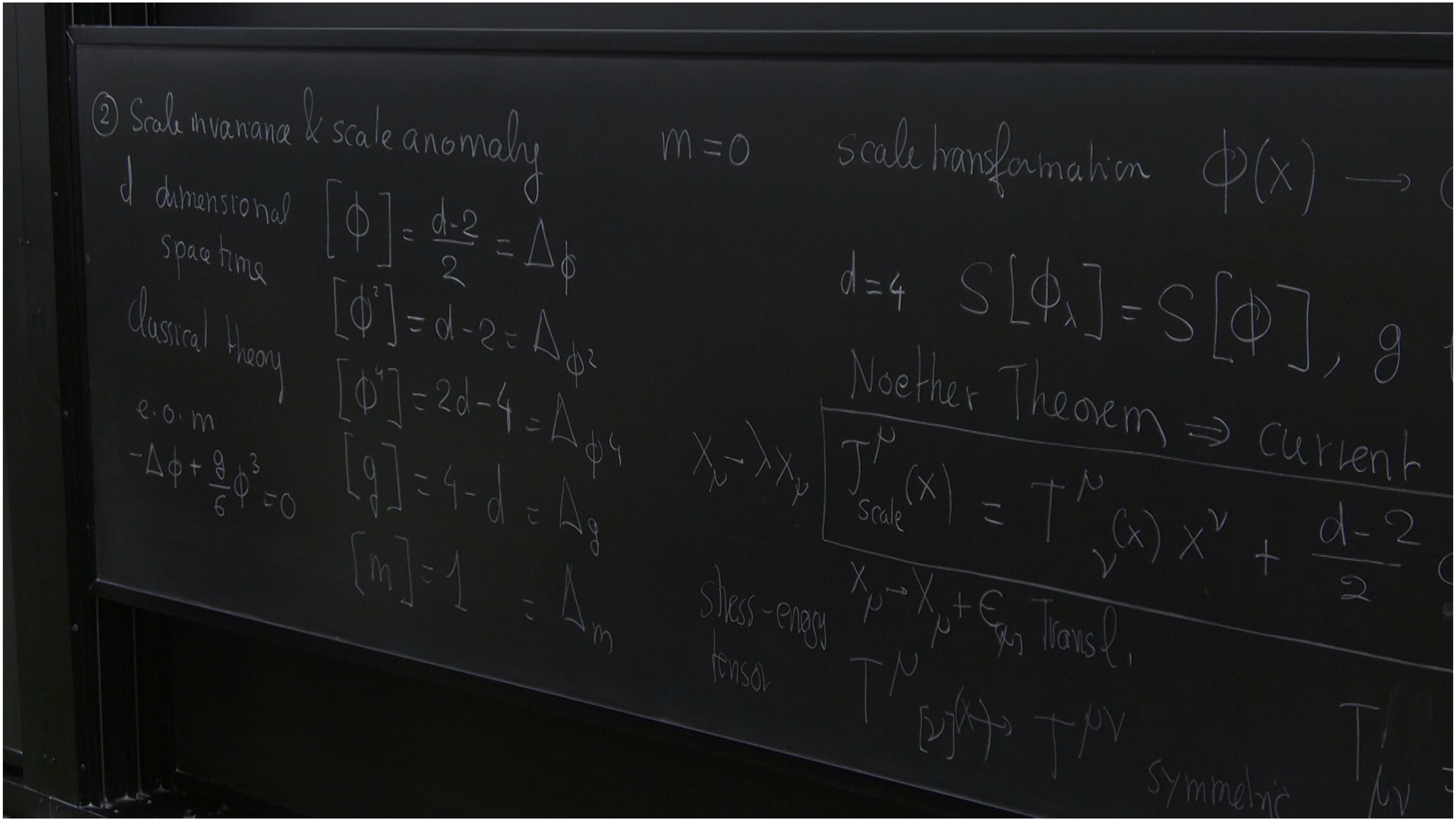
Title: QFT2 Lecture - 120623

Speakers: Francois David

Collection: Quantum Field Theory 2 2023/24

Date: December 06, 2023 - 9:00 AM

URL: https://pirsa.org/23120006



② Scale invariance & scale anomaly

d dimensional space time

$[\phi] = \frac{d-2}{2} = \Delta_\phi$

$[\phi^2] = d-2 = \Delta_{\phi^2}$

Classical theory

$[\phi^4] = 2d-4 = \Delta_{\phi^4}$

e.o.m

$-\Delta\phi + \frac{g}{6}\phi^3 = 0$

$[g] = 4-d = \Delta_g$

$[m] = 1 = \Delta_m$

$m = 0$

scale transformation $\phi(x) \longrightarrow$

$d=4$ $S[\phi_\lambda] = S[\phi]$, g

Noether Theorem ⇒ current

$x_\nu \to \lambda x_\nu$

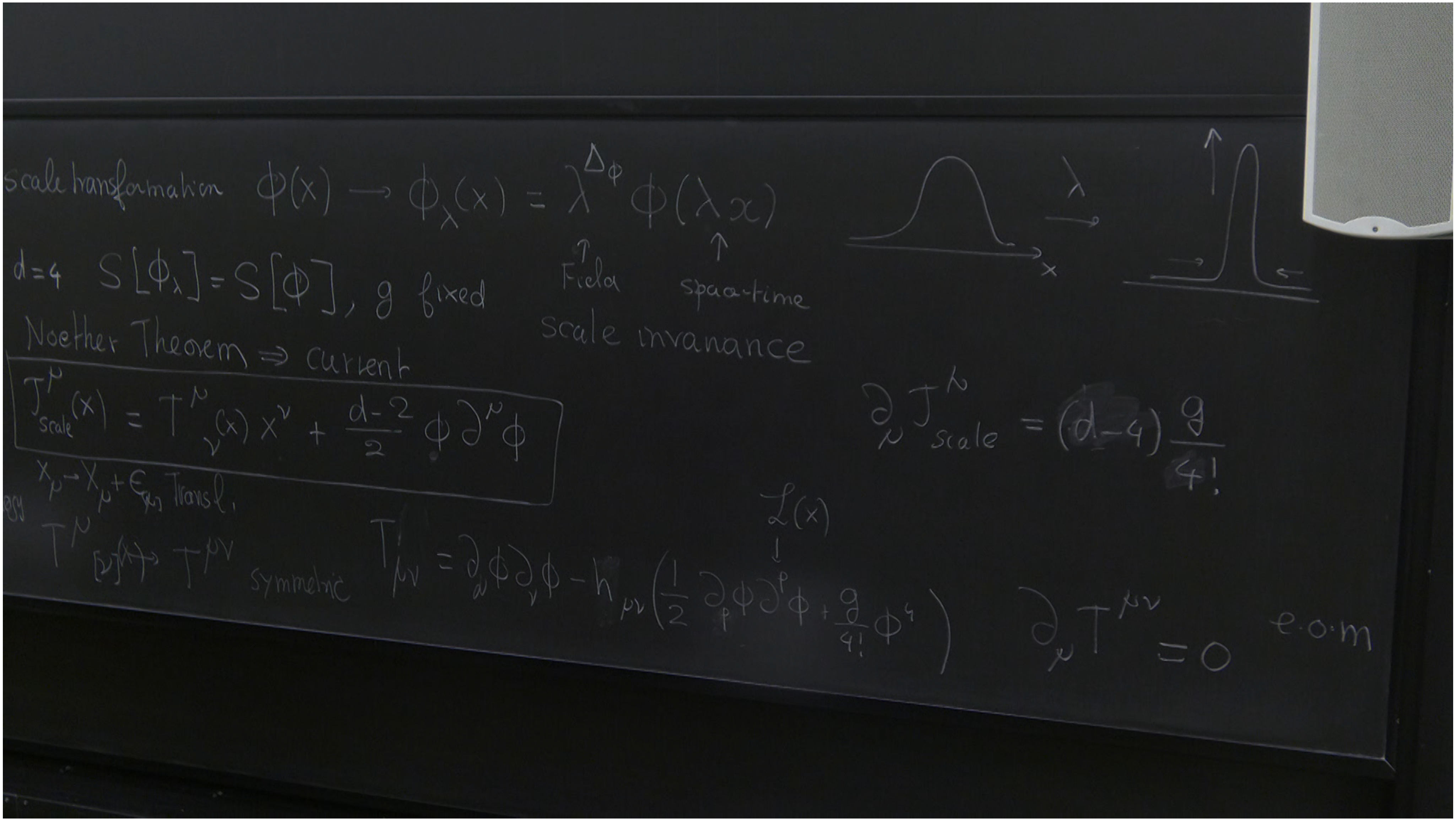
$$\boxed{J^\mu_{\text{scale}}(x) = T^\mu{}_\nu(x)\, x^\nu + \frac{d-2}{2}}$$

stress-energy tensor

$x_\mu \to x_\mu + \epsilon_{(x)}$ Transl.

$T^\mu{}_{[\nu]}(x) \to T^{\mu\nu}$

symmetric $T_{\mu\nu}$



scale transformation $\phi(x) \longrightarrow \phi_\lambda(x) = \underset{\substack{\uparrow \\ \text{Field}}}{\lambda^{\Delta_\phi}} \phi(\underset{\substack{\uparrow \\ \text{space-time}}}{\lambda x})$

$d=4$ $S[\phi_\lambda] = S[\phi]$, g fixed

scale invariance

Noether Theorem ⇒ current

$$\boxed{J^\mu_{\text{scale}}(x) = T^\mu{}_\nu(x)\, x^\nu + \frac{d-2}{2}\phi\,\partial^\mu\phi}$$

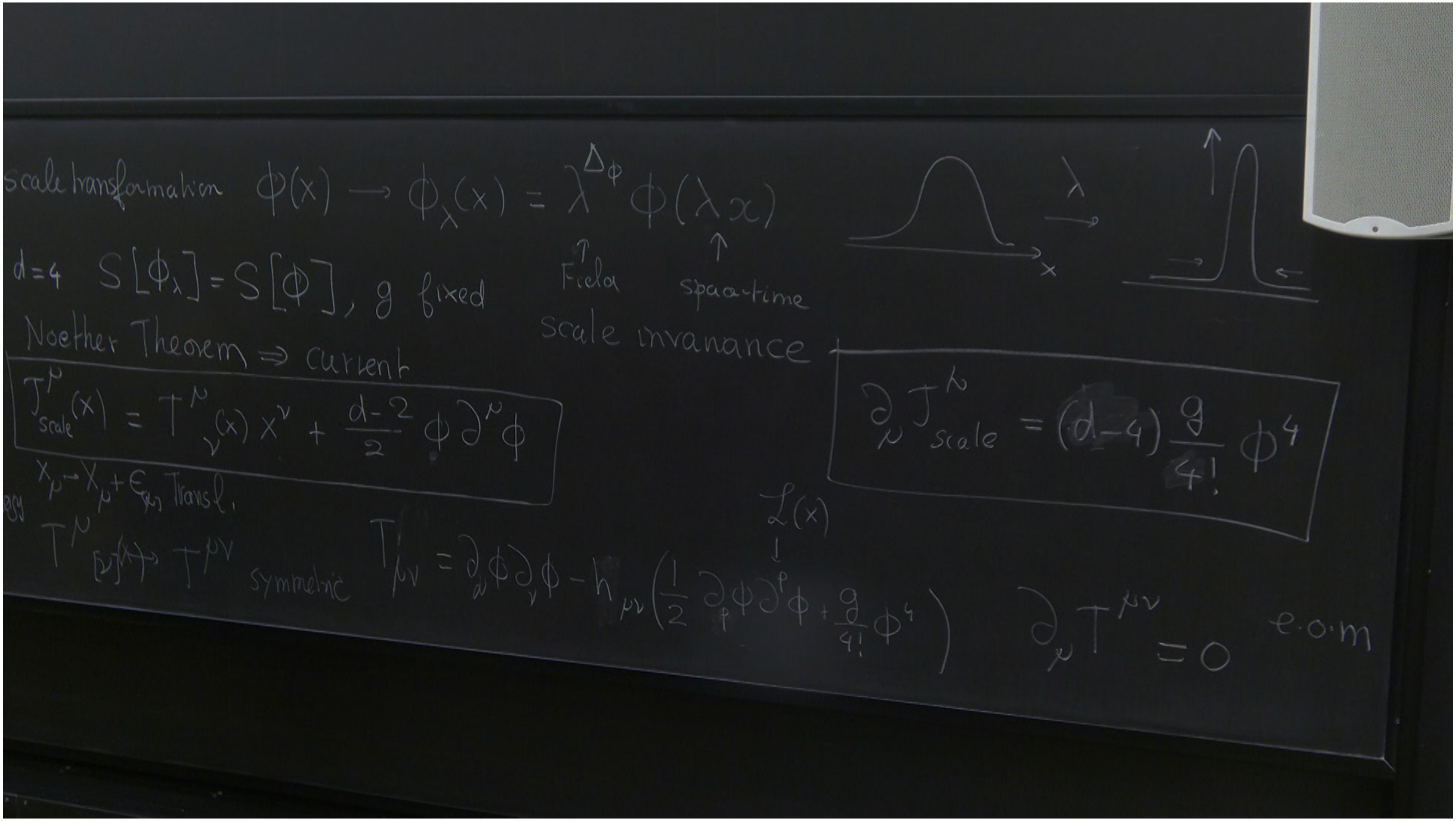
$$\partial_\mu J^\mu_{\text{scale}} = (d-4)\frac{g}{4!}$$

$x_\mu \to x_\mu + \epsilon_\mu$ Transl.

egy $T^\mu{}_{[\nu]}(x) \to T^{\mu\nu}$ symmetric

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \eta_{\mu\nu}\Big(\underset{\substack{\uparrow \\ \mathcal{L}(x)}}{\tfrac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + \tfrac{g}{4!}\phi^4}\Big)$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{e.o.m}$$



scale transformation $\phi(x) \longrightarrow \phi_\lambda(x) = \lambda^{\Delta_\phi} \phi(\lambda x)$

Field ↑ ↑ space-time

$d=4$ $S[\phi_\lambda] = S[\phi]$, g fixed

scale invariance

Noether Theorem ⇒ current

$$\boxed{J^\mu_{\text{scale}}(x) = T^\mu{}_\nu(x)\, x^\nu + \frac{d-2}{2} \phi \partial^\mu \phi}$$

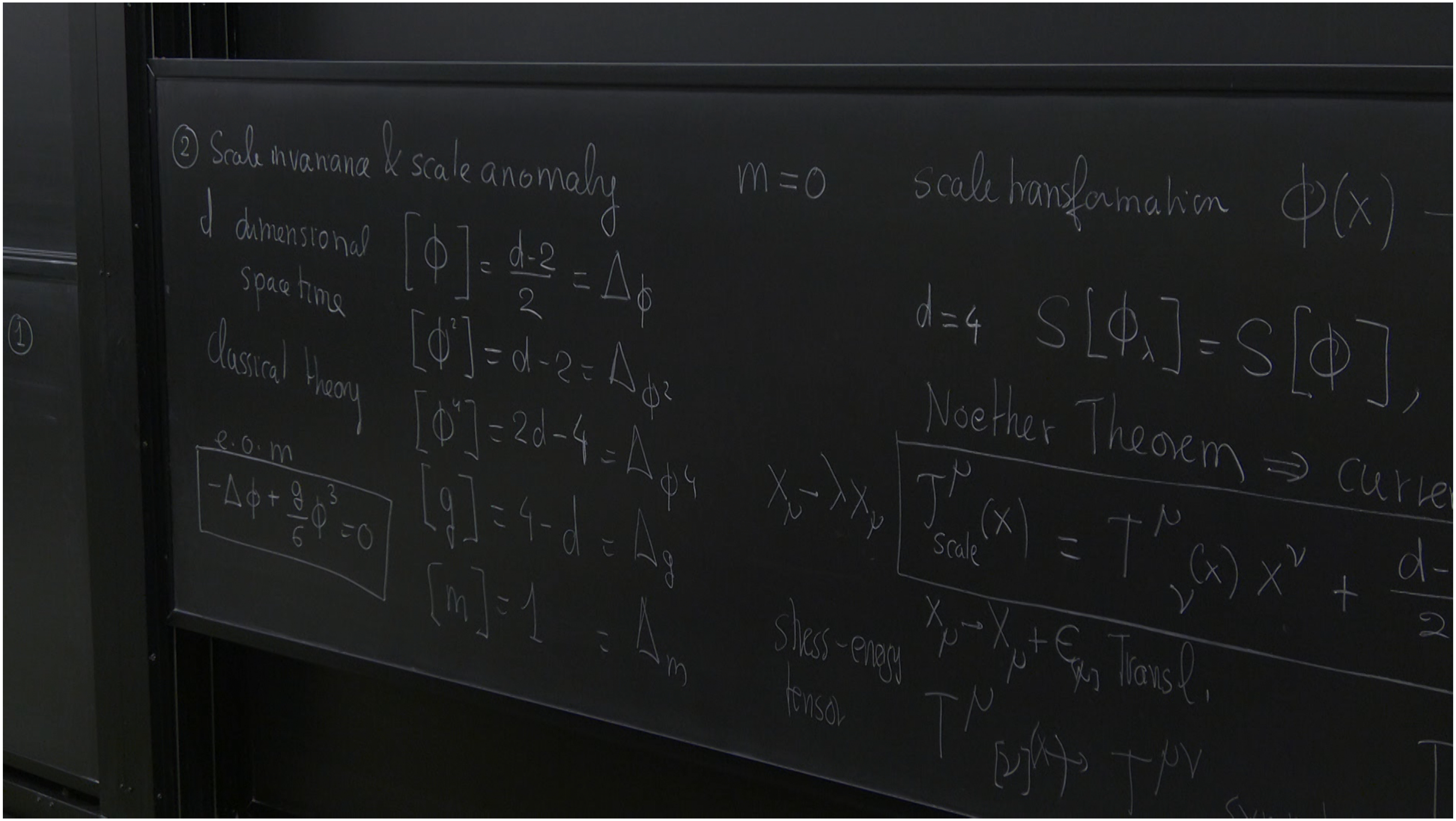
$$\boxed{\partial_\mu J^\mu_{\text{scale}} = (d-4) \frac{g}{4!} \phi^4}$$

$x_\mu \to x_\mu + \epsilon_{(x)}$ Transl.

ergy $T^\mu{}_{[\nu]}(x) \to T^{\mu\nu}$ symmetric

$\mathcal{L}(x)$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \left(\frac{1}{2} \partial_\rho \phi \partial^\rho \phi + \frac{g}{4!} \phi^4 \right) \qquad \partial_\mu T^{\mu\nu} = 0 \quad \text{e.o.m}$$



①

② Scale invariance & scale anomaly

d dimensional spacetime

$[\phi] = \frac{d-2}{2} = \Delta_\phi$

$[\phi^2] = d-2 = \Delta_{\phi^2}$

classical theory

$[\phi^4] = 2d-4 = \Delta_{\phi^4}$

e.o.m

$$\boxed{-\Delta\phi + \frac{g}{6}\phi^3 = 0}$$

$[g] = 4-d = \Delta_g$

$[m] = 1 = \Delta_m$

$m = 0$

scale transformation $\phi(x)$

$d=4$ $S[\phi_\lambda] = S[\phi]$,

Noether Theorem ⇒ curre

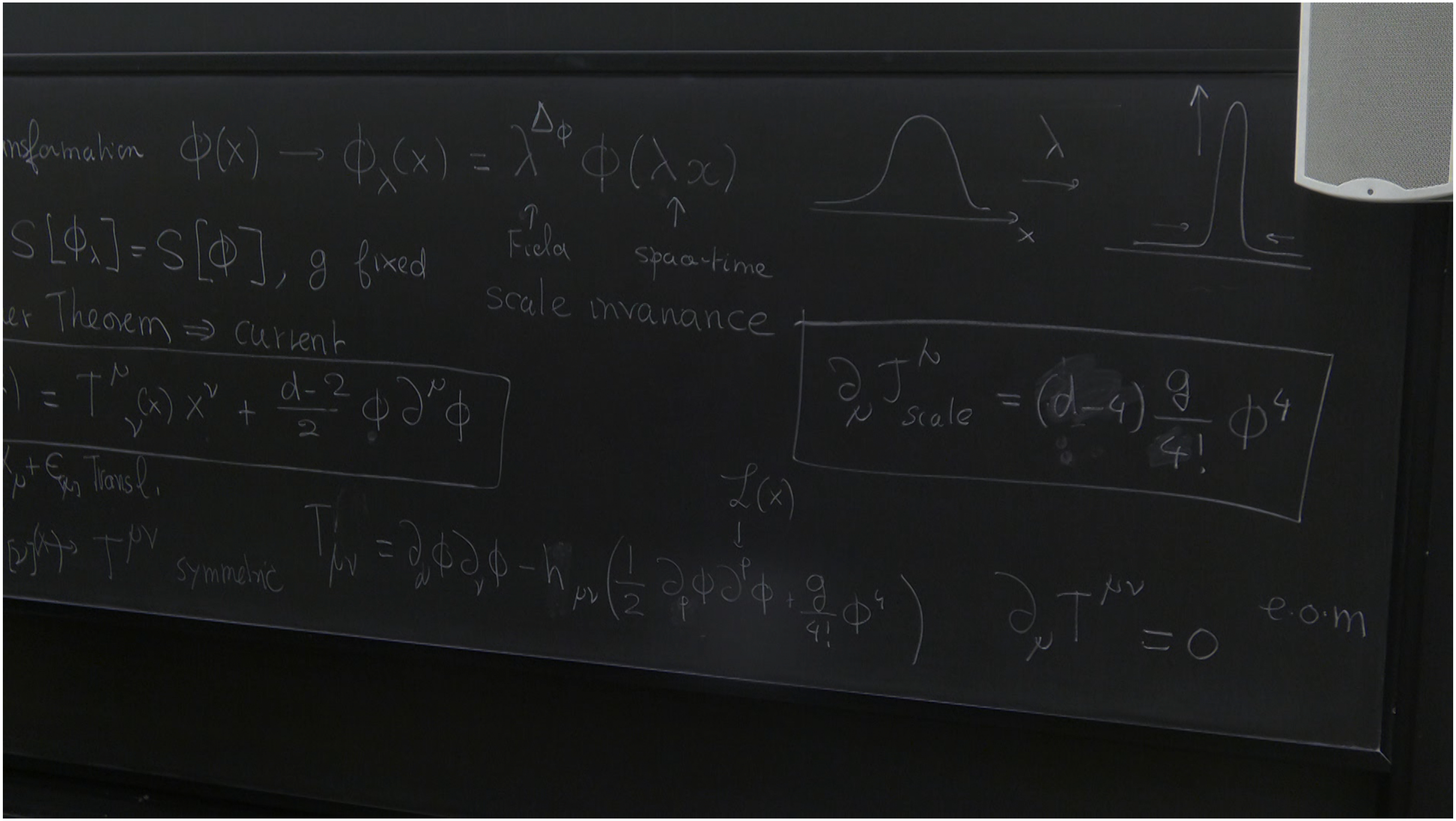
$x_\nu \to \lambda x_\nu$

$$\boxed{J^{\mu}_{\text{scale}}(x) = T^{\mu}{}_{\nu}(x)\, x^{\nu} + \frac{d-}{2}}$$

$x_\mu \to x_\mu + \epsilon_{(x)}$ Transl.

stress-energy tensor

$T^{\mu}{}_{[\nu]}(x) \to T^{\mu\nu}$



nsformation $\phi(x) \longrightarrow \phi_\lambda(x) = \lambda^{\Delta_\phi}\phi(\lambda x)$

Field

space-time

$S[\phi_\lambda] = S[\phi]$, g fixed

scale invariance

er Theorem ⇒ current

$$\boxed{) = T^\mu{}_\nu(x)\, x^\nu + \frac{d-2}{2}\phi\,\partial^\mu\phi}$$

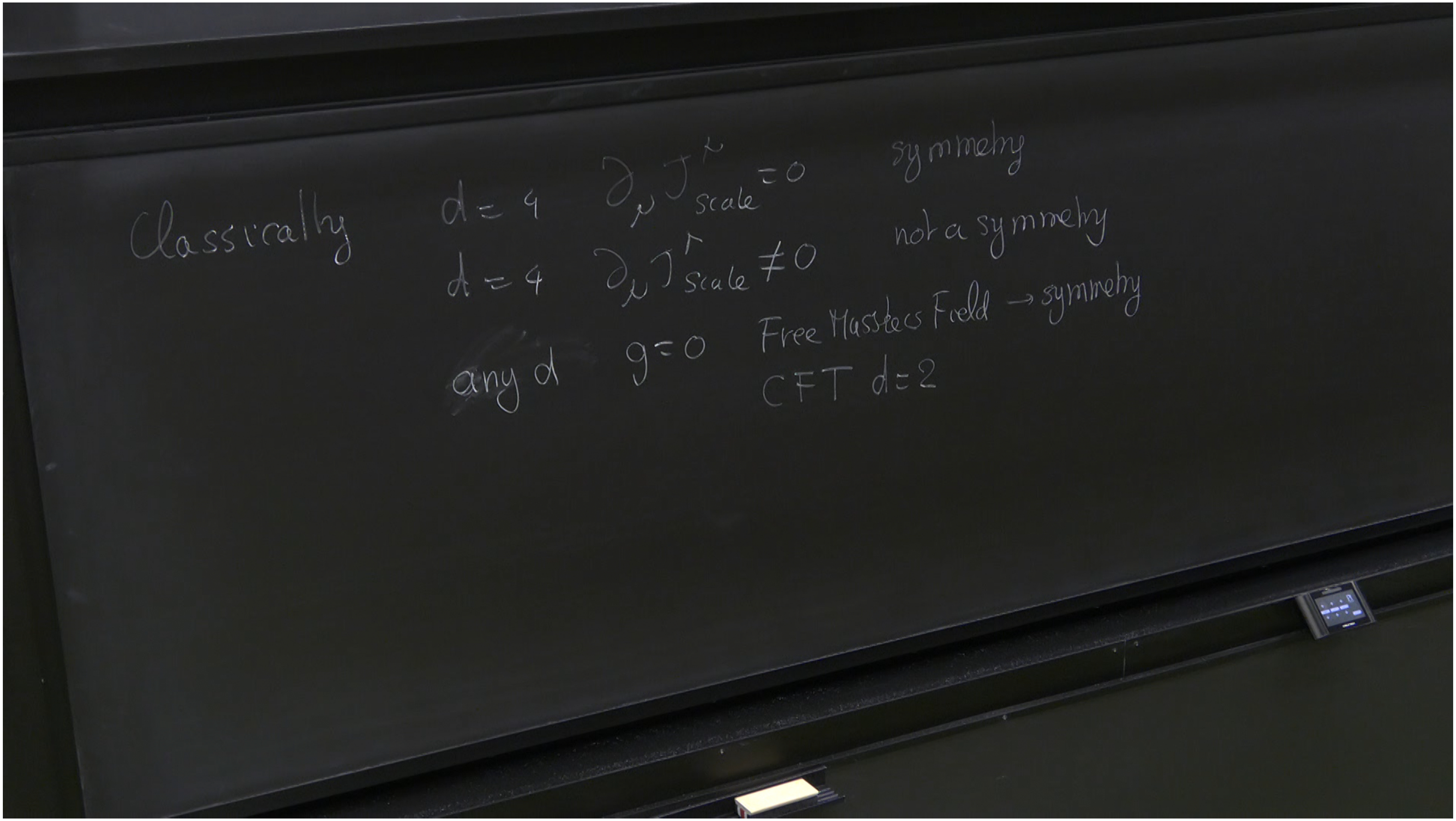
$$\boxed{\partial_\mu J^\mu_{scale} = (d-4)\frac{g}{4!}\phi^4}$$

$x_\mu + \epsilon_{(x)}$ Transl.

$[\nu](x) \to T^{\mu\nu}$ symmetric

$$T'_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - h_{\mu\nu}\Big(\underbrace{\tfrac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + \tfrac{g}{4!}\phi^4}_{\mathcal{L}(x)}\Big)$$

$\partial_\mu T^{\mu\nu} = 0$ e.o.m

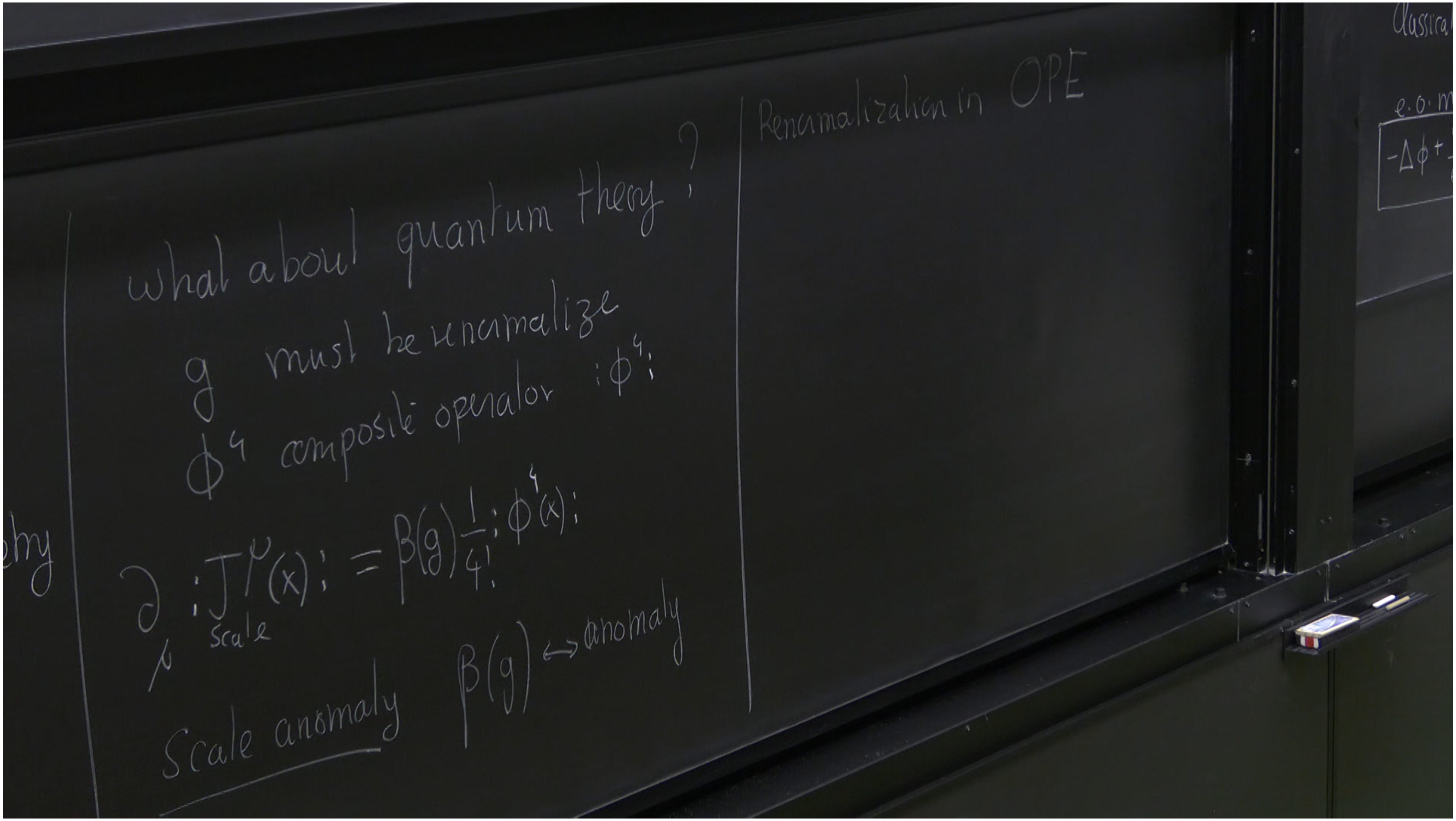


Classically $d = 4$ $\partial_\mu J^\mu_{scale} = 0$ symmetry

$d = 4$ $\partial_\mu J^\mu_{scale} \neq 0$ not a symmetry

any d $g = 0$ Free Massless Field → symmetry

CFT $d = 2$



what about quantum theory ?

g must be renormalize

ϕ^4 composite operator $:\phi^4:$

$$\partial_\mu :J^\mu_{\text{scale}}(x): = \beta(g)\frac{1}{4!}:\phi^4(x):$$

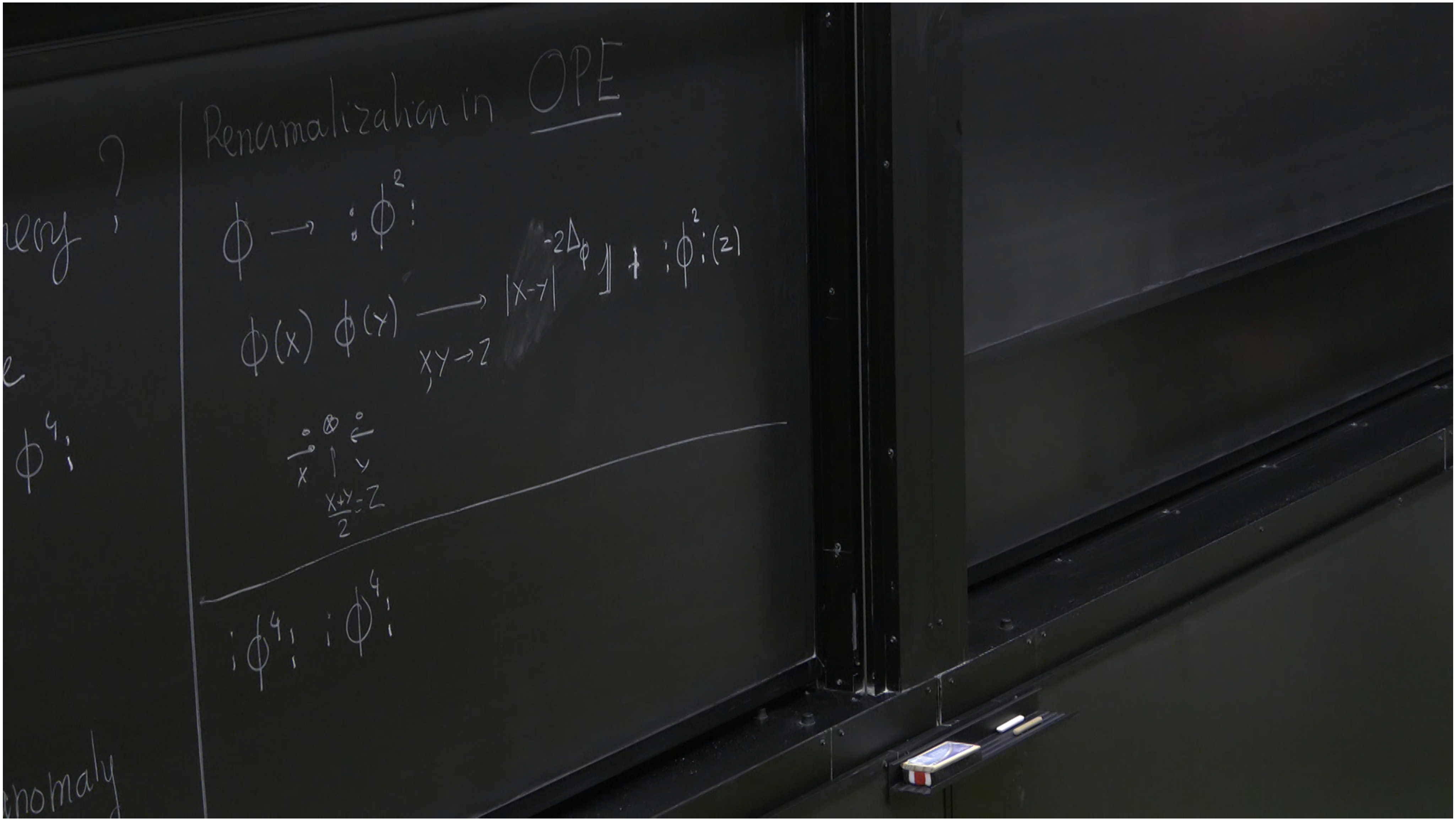
Scale anomaly $\beta(g) \leftrightarrow$ anomaly

Renormalization in OPE

Classica

e.o.m

$-\Delta\phi +$



nery ?

e

$:\phi^4:$

nomaly

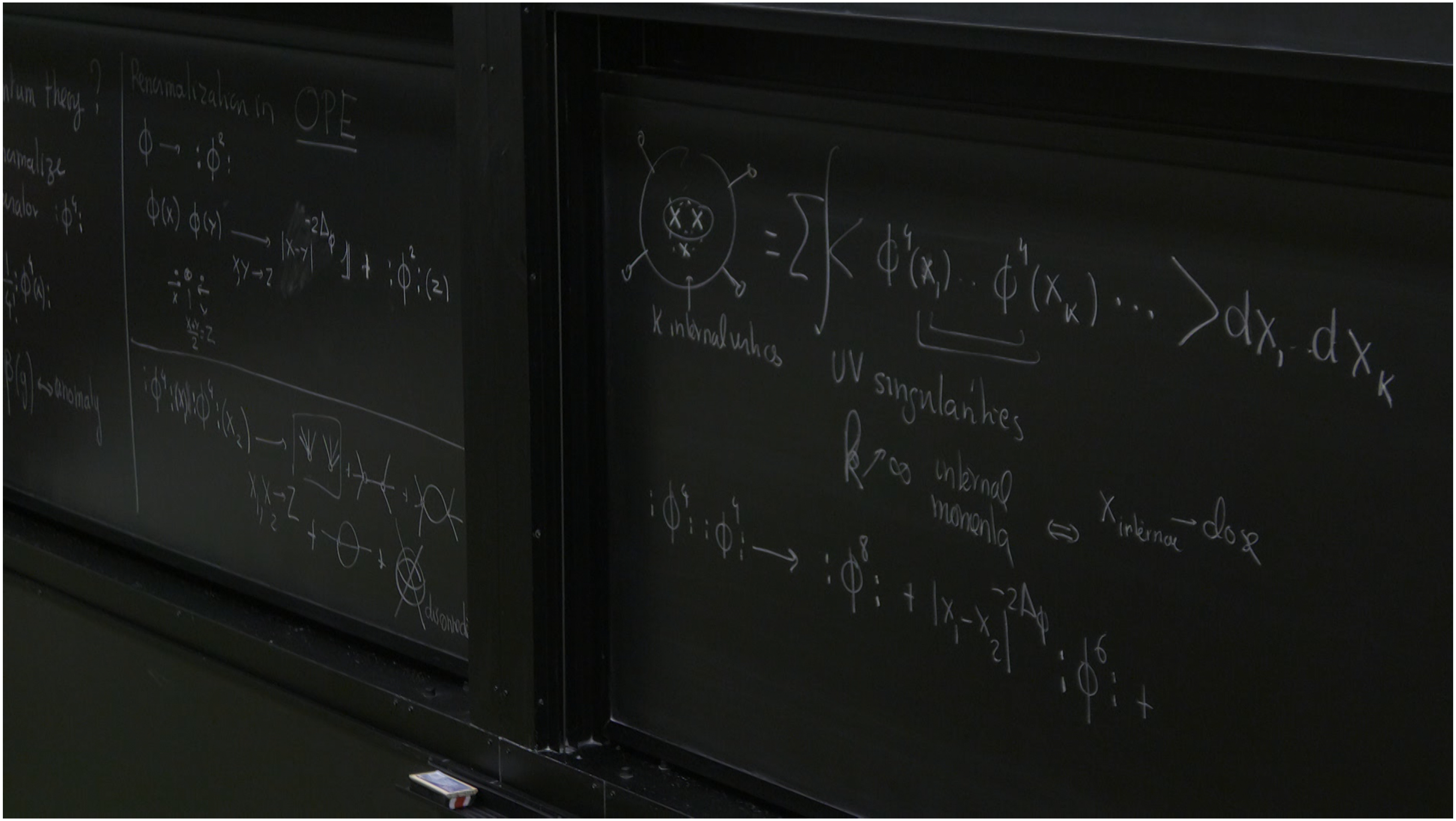
Renormalization in OPE

$$\phi \longrightarrow :\phi^2:$$

$$\phi(x)\,\phi(y) \xrightarrow[x,y\to z]{} |x-y|^{-2\Delta_\phi}\,\mathbb{1} + :\phi^2:(z)$$

$$\frac{x+y}{2}=z$$

$$:\phi^4: \; :\phi^4:$$



ntum theory ?

rmalize

rator $:\phi^4:$

$\frac{1}{4!} :\phi^4(x):$

$\beta(g) \leftrightarrow$ anomaly

Renormalization in OPE

$\phi \to :\phi^2:$

$$\phi(x)\,\phi(y) \xrightarrow[x,y\to z]{} |x-y|^{-2\Delta_\phi}\,\mathbb{1} + :\phi^2:(z)$$

$\frac{x+y}{2} = z$

$$:\phi^4:(x_1)\,:\phi^4:(x_2) \xrightarrow[x_1,x_2\to z]{}$$

disconnd

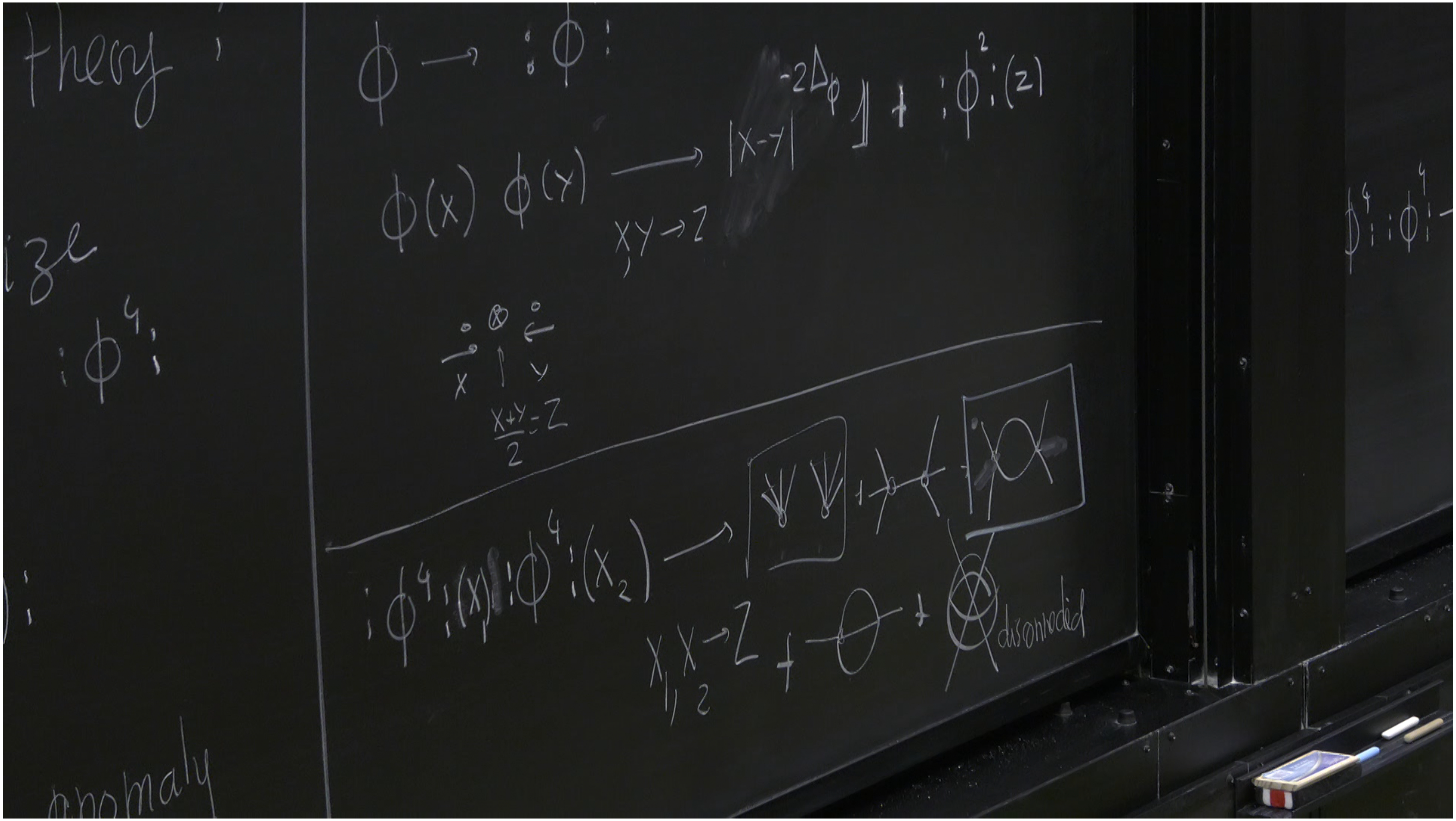
$$= \sum \int \langle \phi^4(x_1) \cdots \underbrace{\phi^4(x_k)}_{} \cdots \rangle\, dx_1 \cdots dx_k$$

k internal vertices

UV singularities

$k \nearrow \infty$ internal momenta $\Leftrightarrow$ $x_{\text{internal}} \to$ close

$$:\phi^4:\,:\phi^4: \longrightarrow\; :\phi^8: + |x_1-x_2|^{-2\Delta_\phi} :\phi^6: +$$



theory ;

ize

$:\phi^4:$

$):$

anomaly

$\phi \longrightarrow :\phi:$

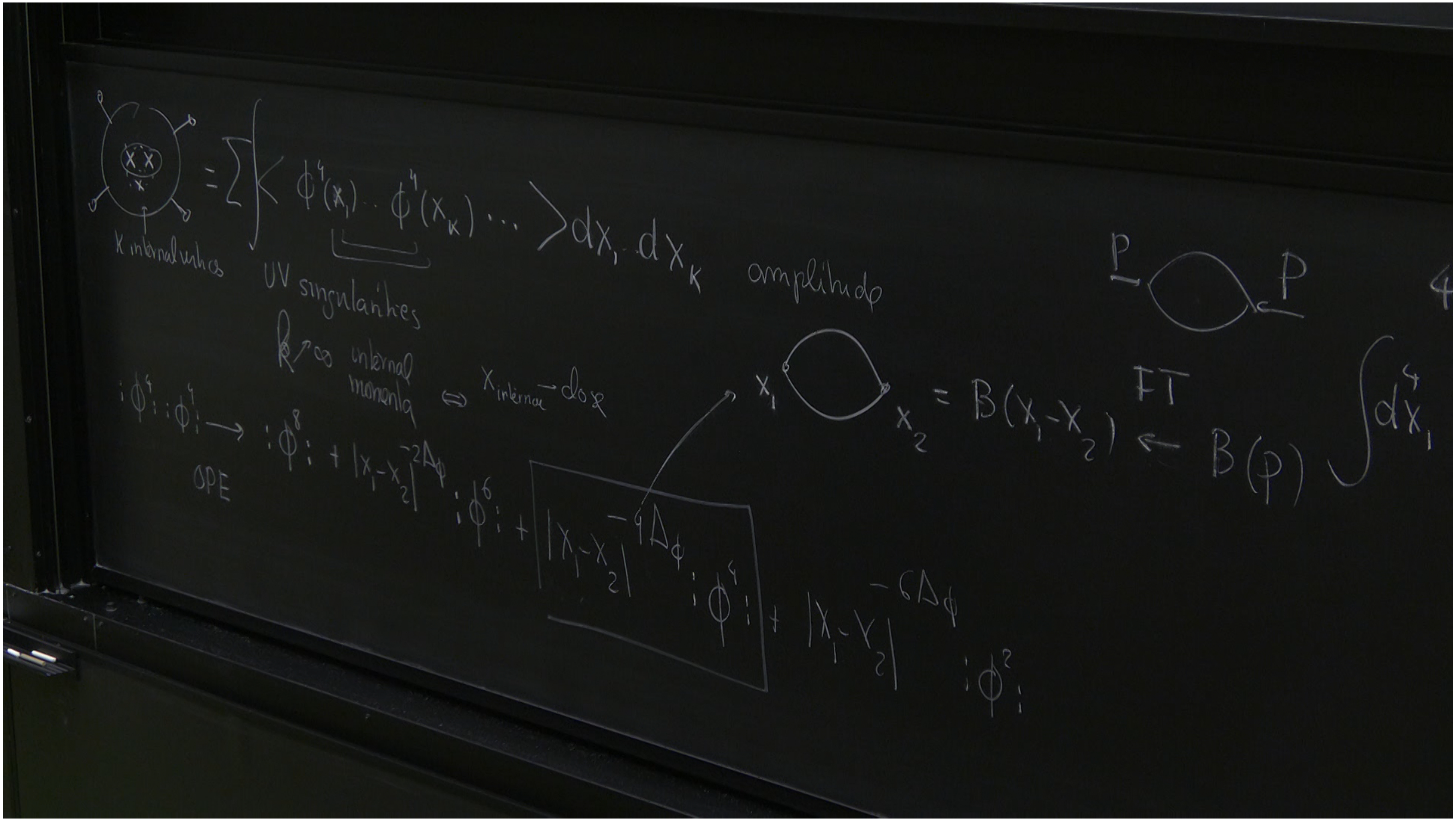
$\phi(x)\,\phi(y) \xrightarrow[x,y\to z]{} |x-y|^{-2\Delta_\phi}\,\mathbb{1} + :\phi^2:(z)$

$\frac{x+y}{2} = z$

$:\phi^4:(x_1)\,:\phi^4:(x_2) \xrightarrow[x_1,x_2\to z]{}$

disconnected

$:\phi^4: \; :\phi^4: -$



k internal vertices

$$= \sum \int \langle \underbrace{\phi^4(x_1) \cdots \phi^4(x_k)}_{} \cdots \rangle dx_1 \cdots dx_k$$

amplitude

UV singularities

$k \nearrow \infty$ internal momenta $\Leftrightarrow$ $x_{\text{internal}} \to$ close

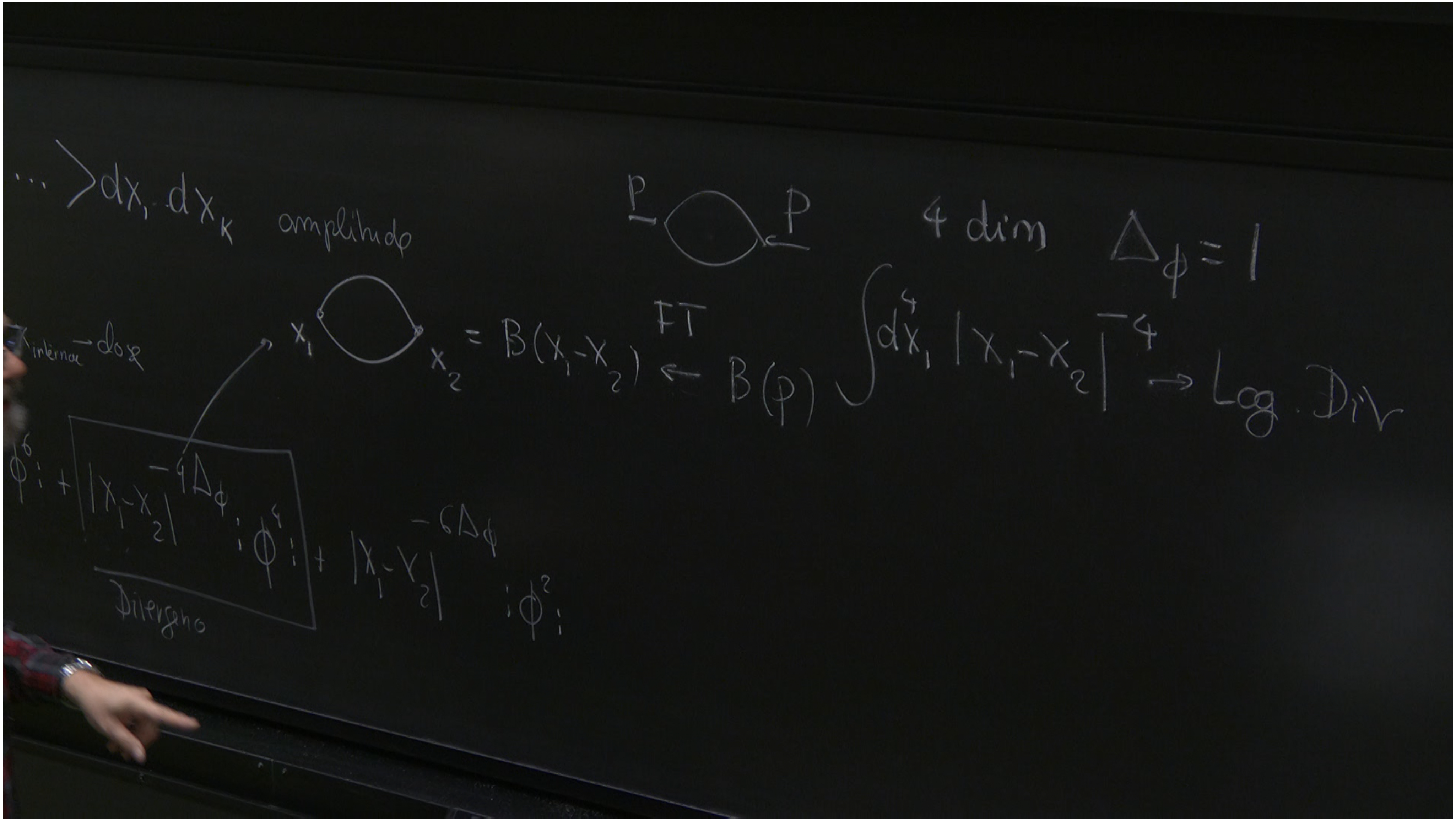
$$:\phi^4::\phi^4: \longrightarrow :\phi^8: + |x_1-x_2|^{-2\Delta_\phi} :\phi^6: + \boxed{|x_1-x_2|^{-4\Delta_\phi}} :\phi^4: + |x_1-x_2|^{-6\Delta_\phi} :\phi^2:$$

OPE

x_1 x_2 $= B(x_1-x_2) \xleftarrow{FT} B(p) \int d^4x_1$

P P

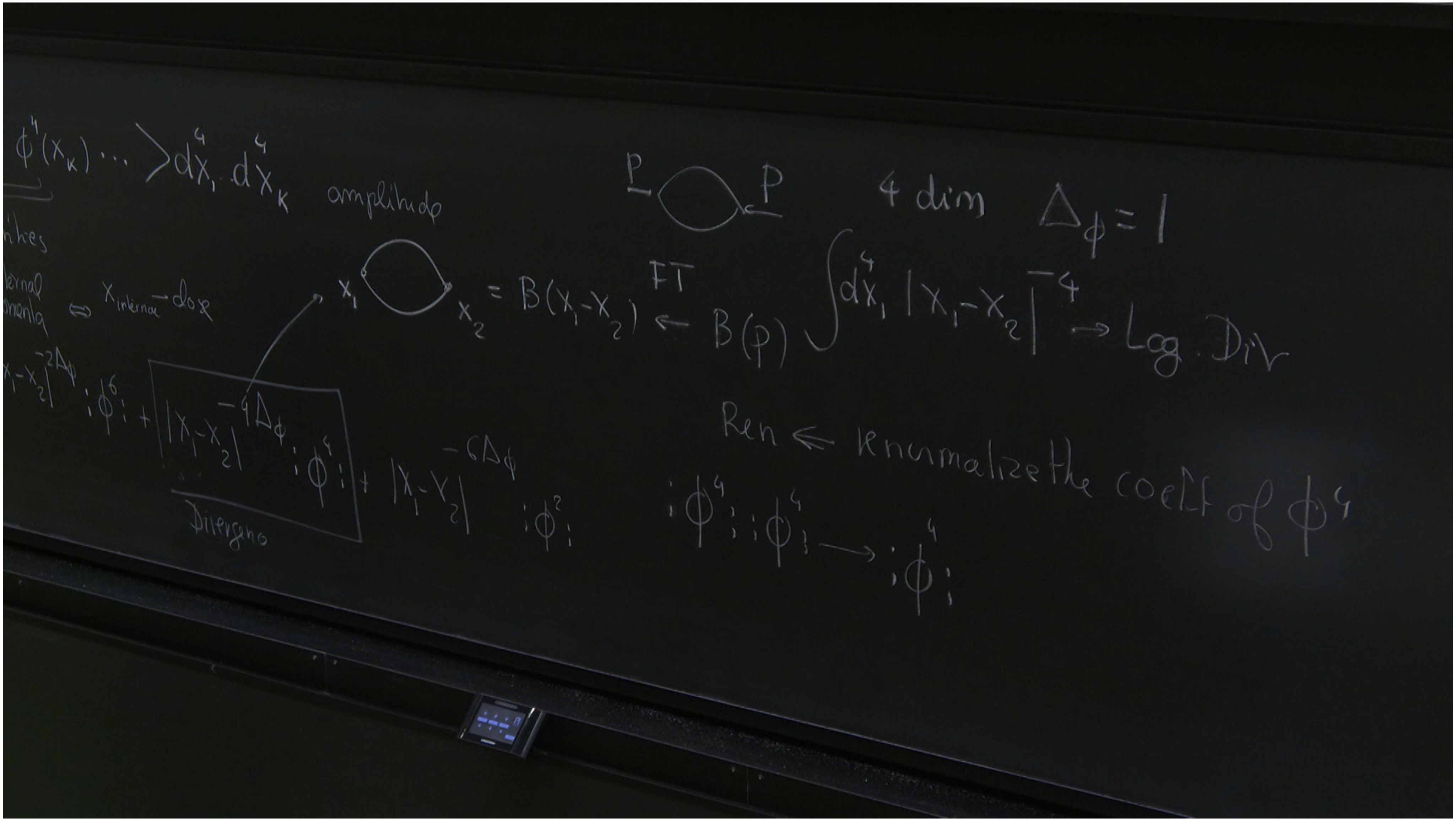
4



4 dim $\Delta_\phi = 1$

$$x_1 \bigcirc x_2 = B(x_1-x_2) \xleftarrow{FT} B(p) \quad \int d^4x_1 \, |x_1-x_2|^{-4} \to \text{Log. Div}$$

$$:\phi^6: + \boxed{|x_1-x_2|^{-4\Delta_\phi} :\phi^4:} + |x_1-x_2|^{-6\Delta_\phi} :\phi^2:$$



$\phi^4(x_k) \cdots \rangle d^4x_1 \cdots d^4x_k$ amplitude

ihes

ternal
menta ⇔ $x_{\text{internal}} \to$ dox

$|x_1-x_2|^{-2\Delta_\phi} :\phi^6: + \boxed{|x_1-x_2|^{-4\Delta_\phi} :\phi^4:} + |x_1-x_2|^{-6\Delta_\phi} :\phi^2:$

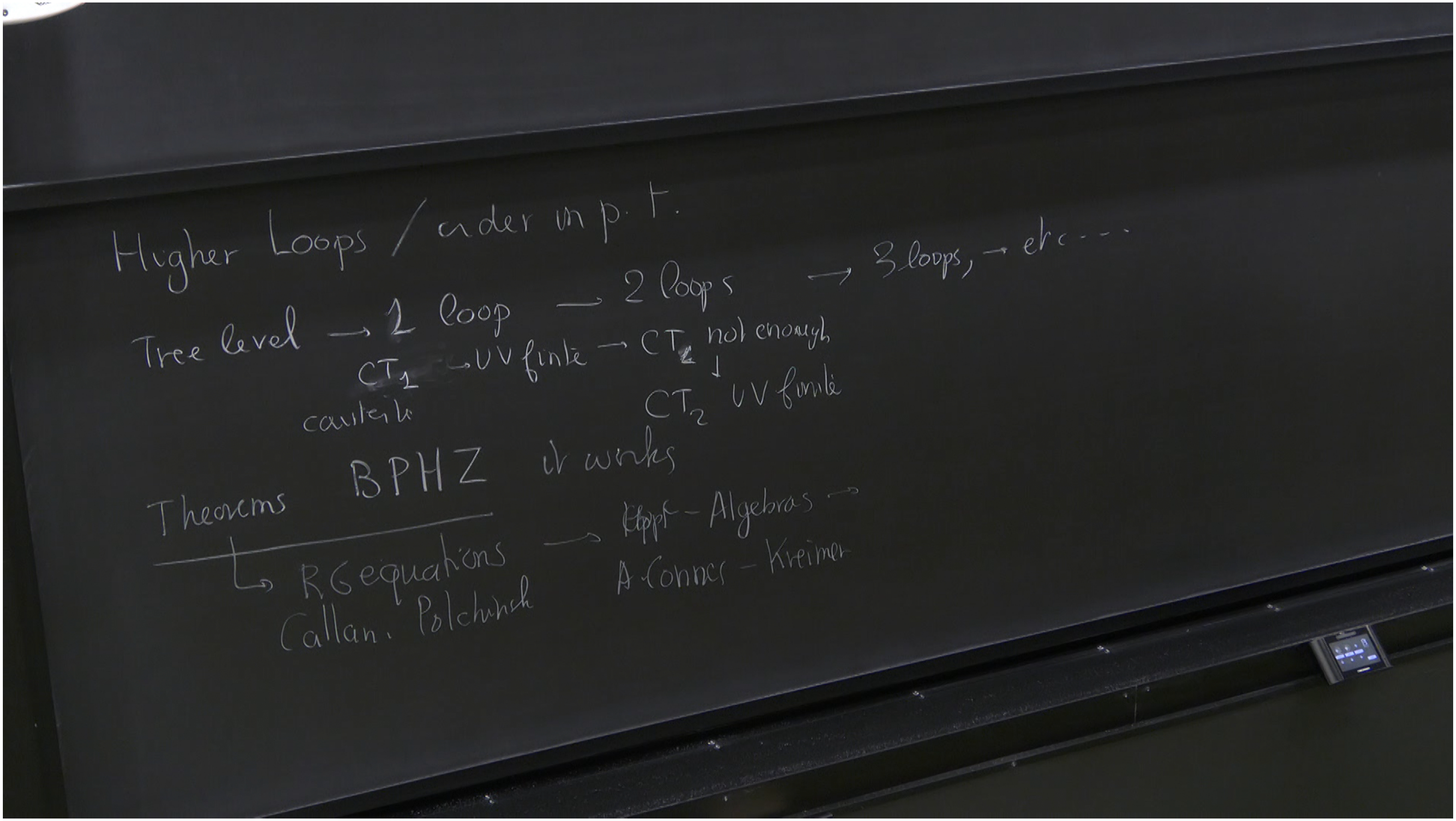
Divergence

4 dim $\Delta_\phi = 1$

$= B(x_1-x_2) \xleftarrow{FT} B(p) \int d^4x_1 |x_1-x_2|^{-4} \to$ Log. Div

Ren ⇐ renormalize the coeff of ϕ^4

$:\phi^4: :\phi^4: \longrightarrow :\phi^4:$



Higher Loops / order in p. T.

Tree level → 1 loop → 2 loops → 3 loops, → etc . . .

CT_1 → UV finite → CT_1 not enough

counterlr. ↓

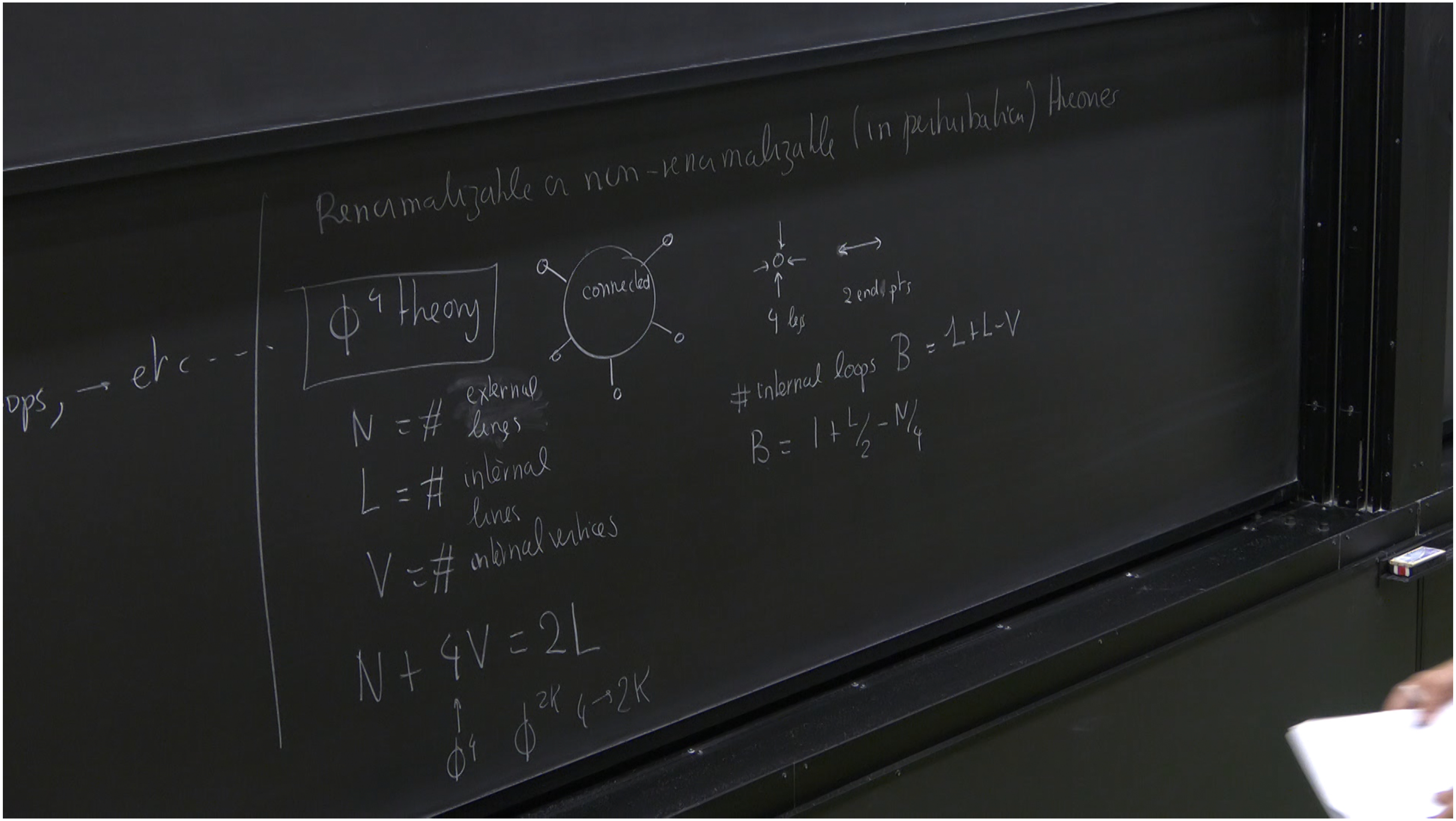
CT_2 UV finite

BPHZ it works

Theorems

↳ RG equations → Hopf - Algebras →

Callan. Polchinsk A. Connes - Kreimer



ops, → etc - - -

Renormalizable or non-renormalizable (in perturbation) theories

$\boxed{\phi^4 \text{ theory}}$

4 legs

2 end pts

N = # external lines

L = # internal lines

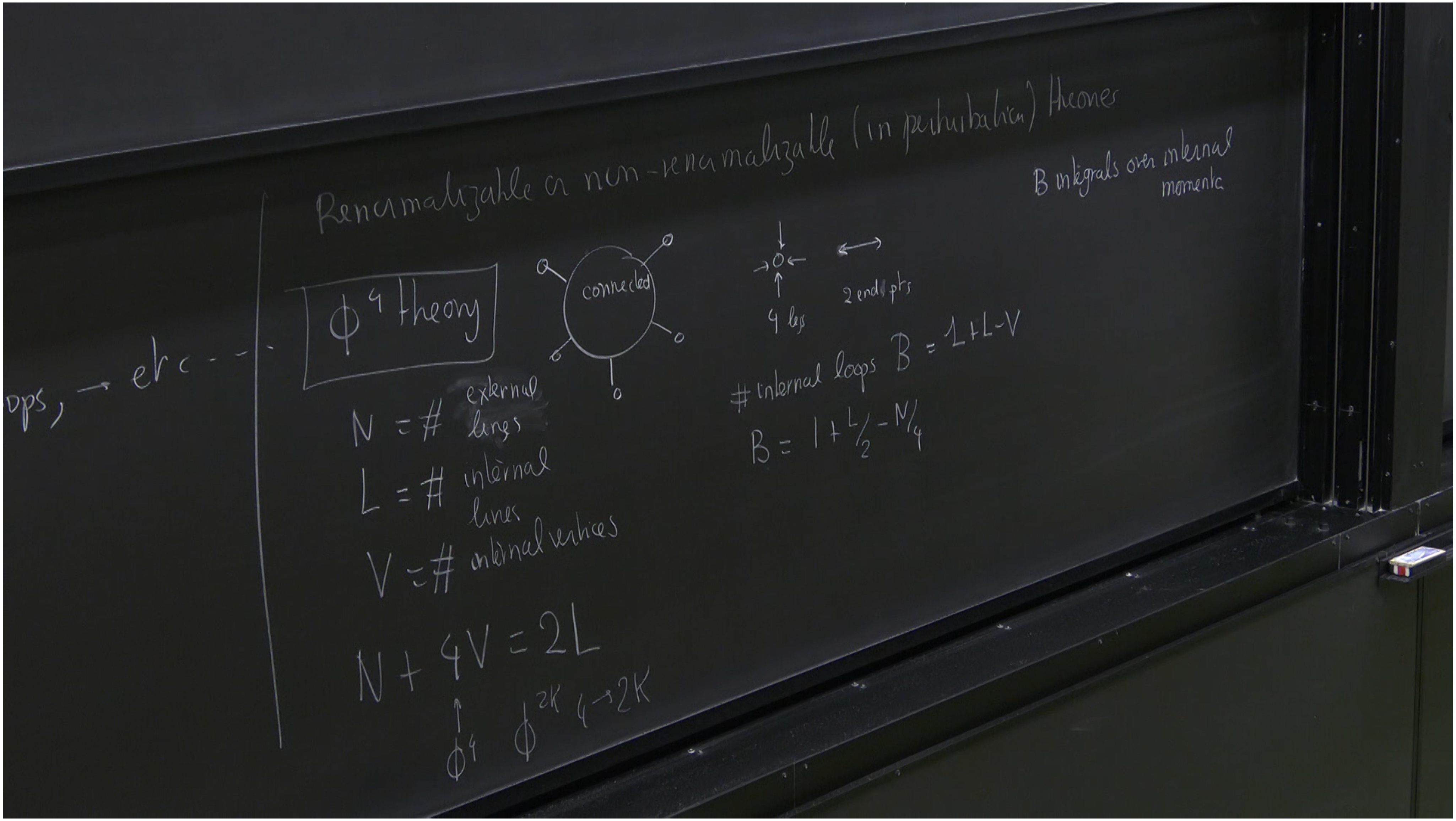
V = # internal vertices

internal loops $B = 1 + L - V$

$$B = 1 + L/2 - N/4$$

$$N + 4V = 2L$$

ϕ^4 $\quad \phi^{2K} \quad 4 \to 2K$



ops, → etc - - -

Renormalizable or non-renormalizable (in perturbation) theories

B integrals over internal momenta

$\boxed{\phi^4 \text{ theory}}$

N = # external lines

L = # internal lines

V = # internal vertices

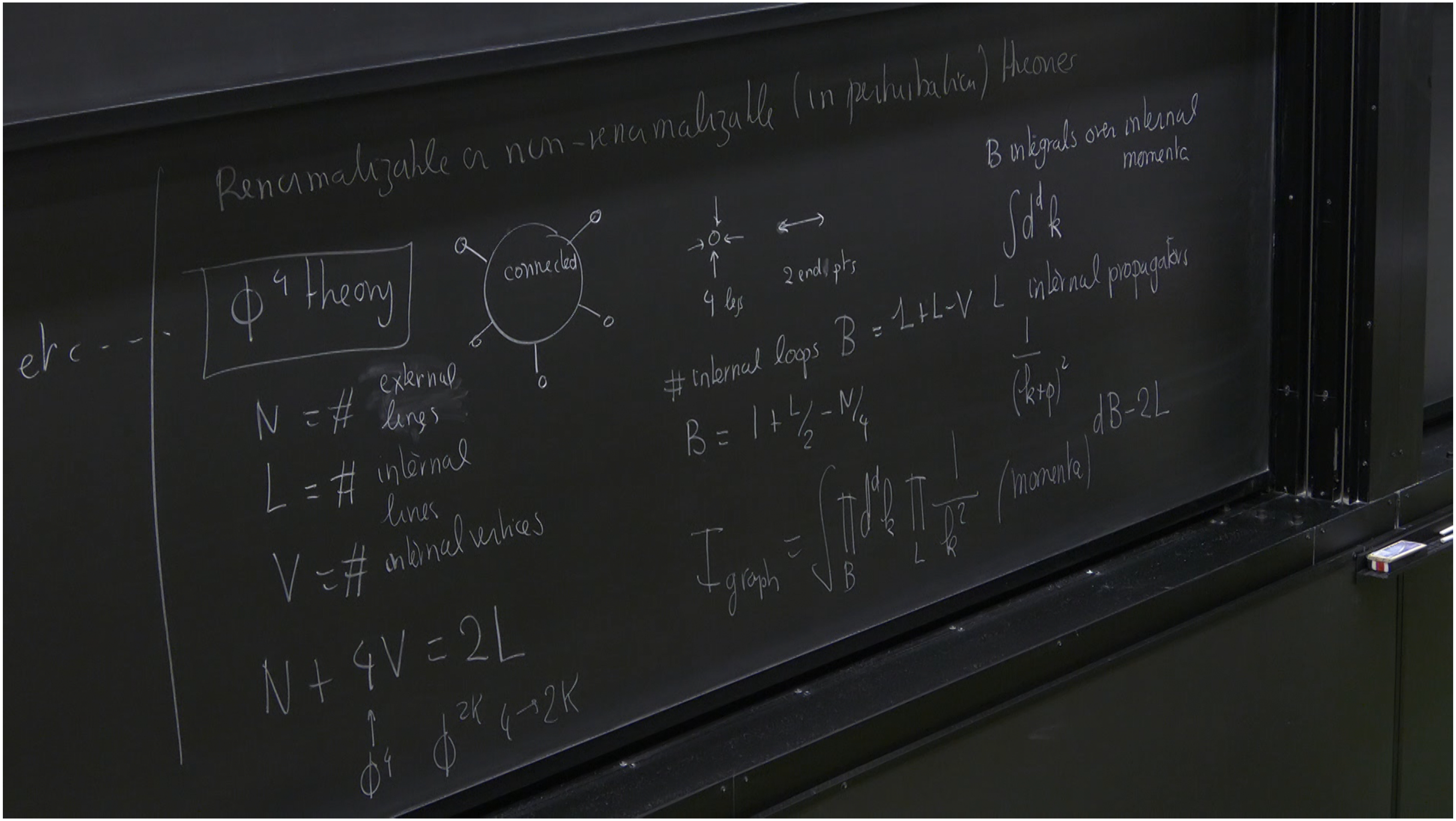
$N + 4V = 2L$

$\uparrow$

ϕ^4 ϕ^{2K} $4 \to 2K$

internal loops $B = 1 + L - V$

$B = 1 + L/2 - N/4$



Renormalizable or non-renormalizable (in perturbation) theories

etc - - -

$\boxed{\phi^4 \text{ theory}}$

connected

$N = \#$ external lines

$L = \#$ internal lines

$V = \#$ internal vertices

$N + 4V = 2L$

ϕ^4 $\quad \phi^{2K}$ $4 \to 2K$

4 legs

$\longleftrightarrow$ 2 end pts

$\#$ internal loops $B = 1 + L - V$

$B = 1 + \frac{L}{2} - \frac{N}{4}$

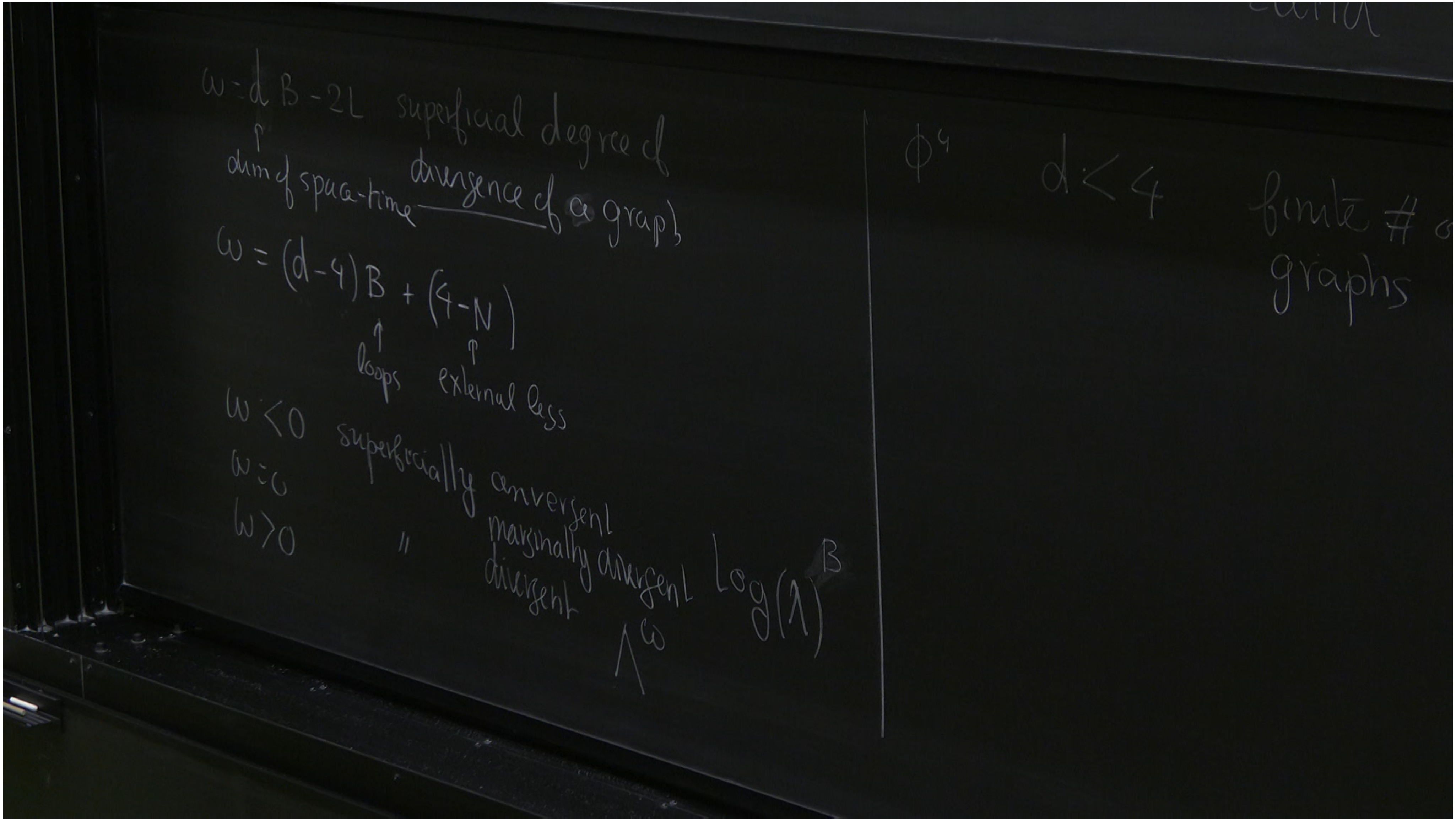
$$I_{graph} = \int \prod_B d^d k \prod_L \frac{1}{k^2} \sim (\text{momenta})^{dB-2L}$$

B integrals over internal momenta

$\int d^d k$

L internal propagators

$\frac{1}{(k+p)^2}$



$\omega = d\,B - 2L$ superficial degree of divergence of a graph

$\uparrow$ dim of space-time

$\omega = (d-4)B + (4-N)$

$\uparrow$ loops $\uparrow$ external legs

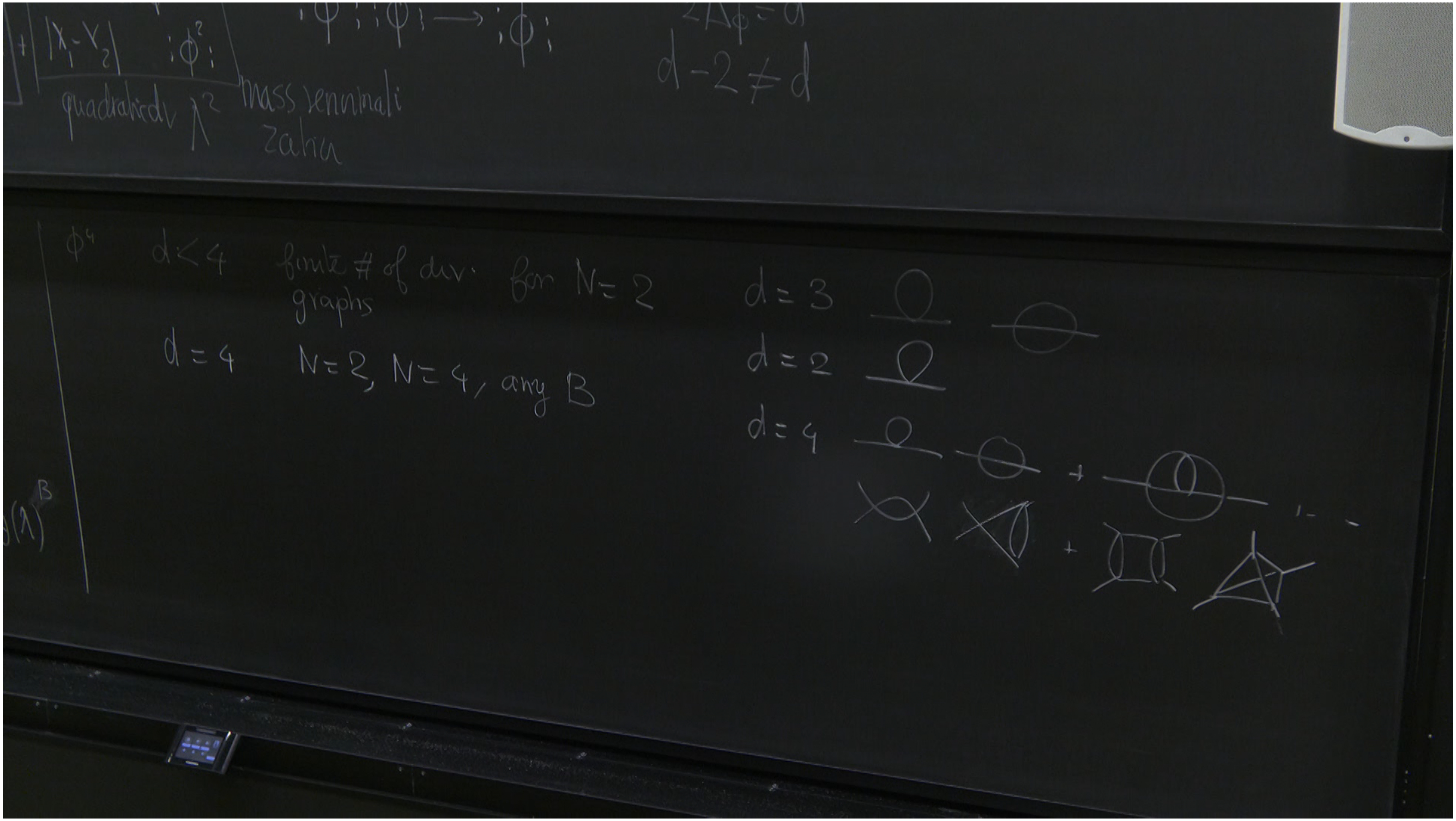
$\omega < 0$ superficially convergent

$\omega = 0$ " marginally convergent $\mathrm{Log}(\Lambda)^B$

$\omega > 0$ " divergent Λ^ω

ϕ^4 $d < 4$ finite # o

graphs



$|x_1 - x_2|$ $:\phi^2:$

quadratic div Λ^2 mass renormalization

$:\phi::\phi: \longrightarrow :\phi:$

$2\Delta_\phi = d$

$d-2 \neq d$

ϕ^4 $d<4$ finite # of div. graphs for $N=2$

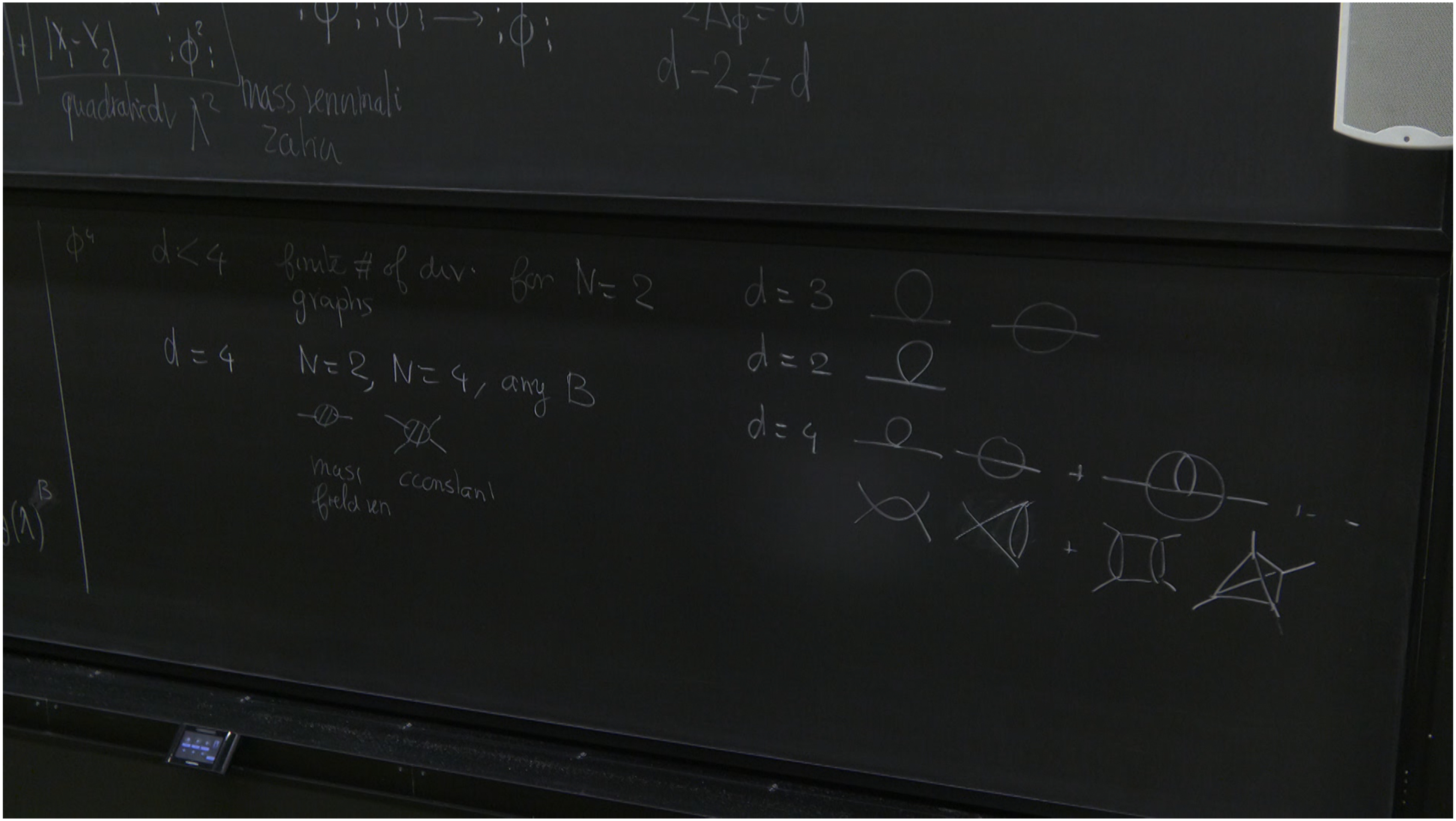
$d=4$ $N=2$, $N=4$, any B

$d=3$

$d=2$

$d=4$

$g(\lambda)^B$



$+ |x_1 - x_2|$ $:\phi^2:$ $:\phi::\phi: \longrightarrow :\phi:$ $2\Delta_\phi = d$

$d - 2 \neq d$

quadratic div Λ^2 mass renormalization

ϕ^4 $d < 4$ finite # of div. graphs for $N = 2$

$d = 4$ $N = 2$, $N = 4$, any B

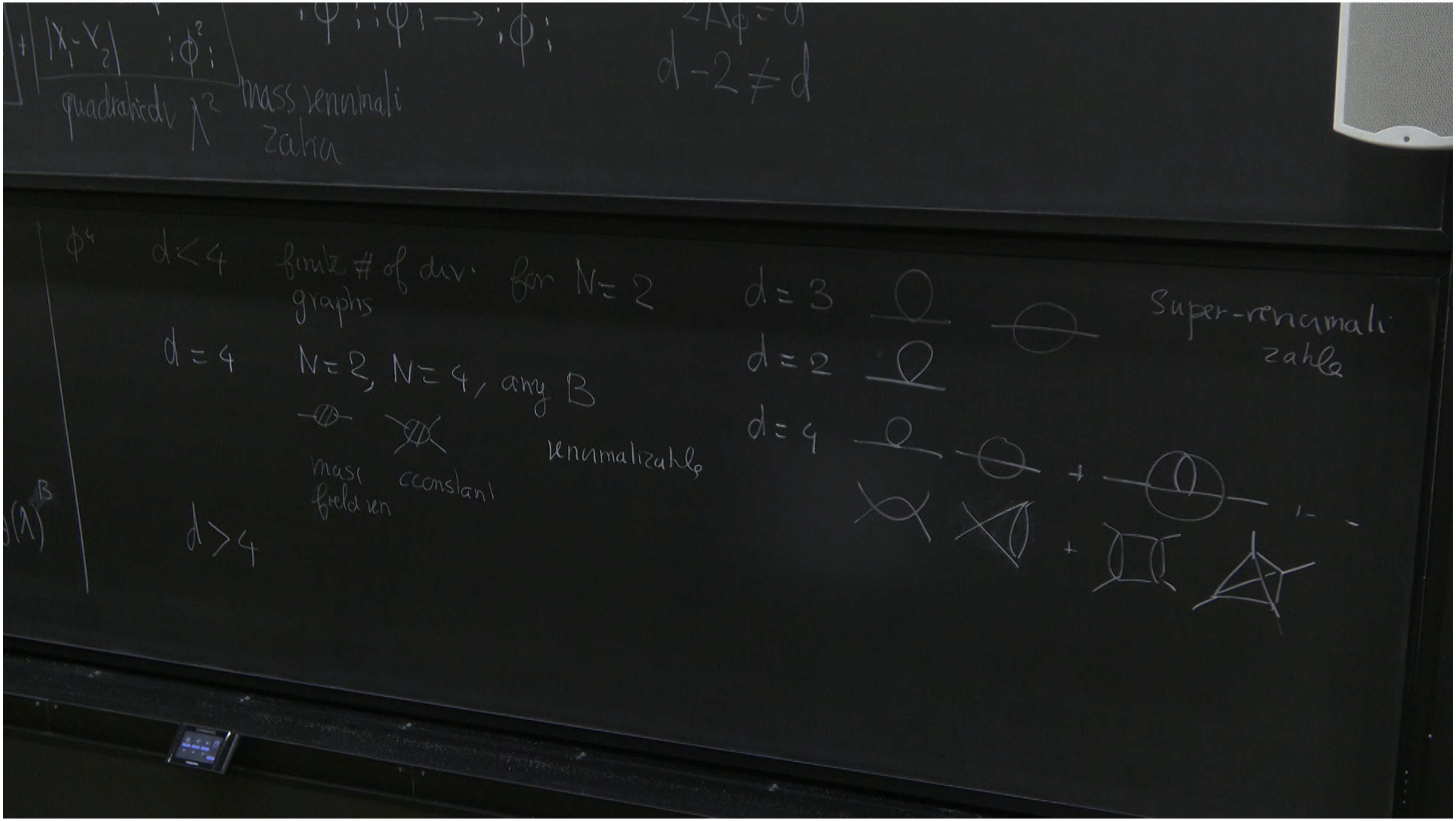
mass field ren c constant

$(\lambda)^B$

$d = 3$

$d = 2$

$d = 4$ + + , - -



$|x_1 - x_2|$ $:\phi^2:$

quadratically Λ^2 mass renormalization

$:\phi::\phi: \longrightarrow :\phi:$

$2\Delta_\phi = d$

$d-2 \neq d$

ϕ^4

$d<4$ finite # of div. graphs for $N=2$

$d=4$ $N=2$, $N=4$, any B

mass field ren constant

renormalizable

$d>4$

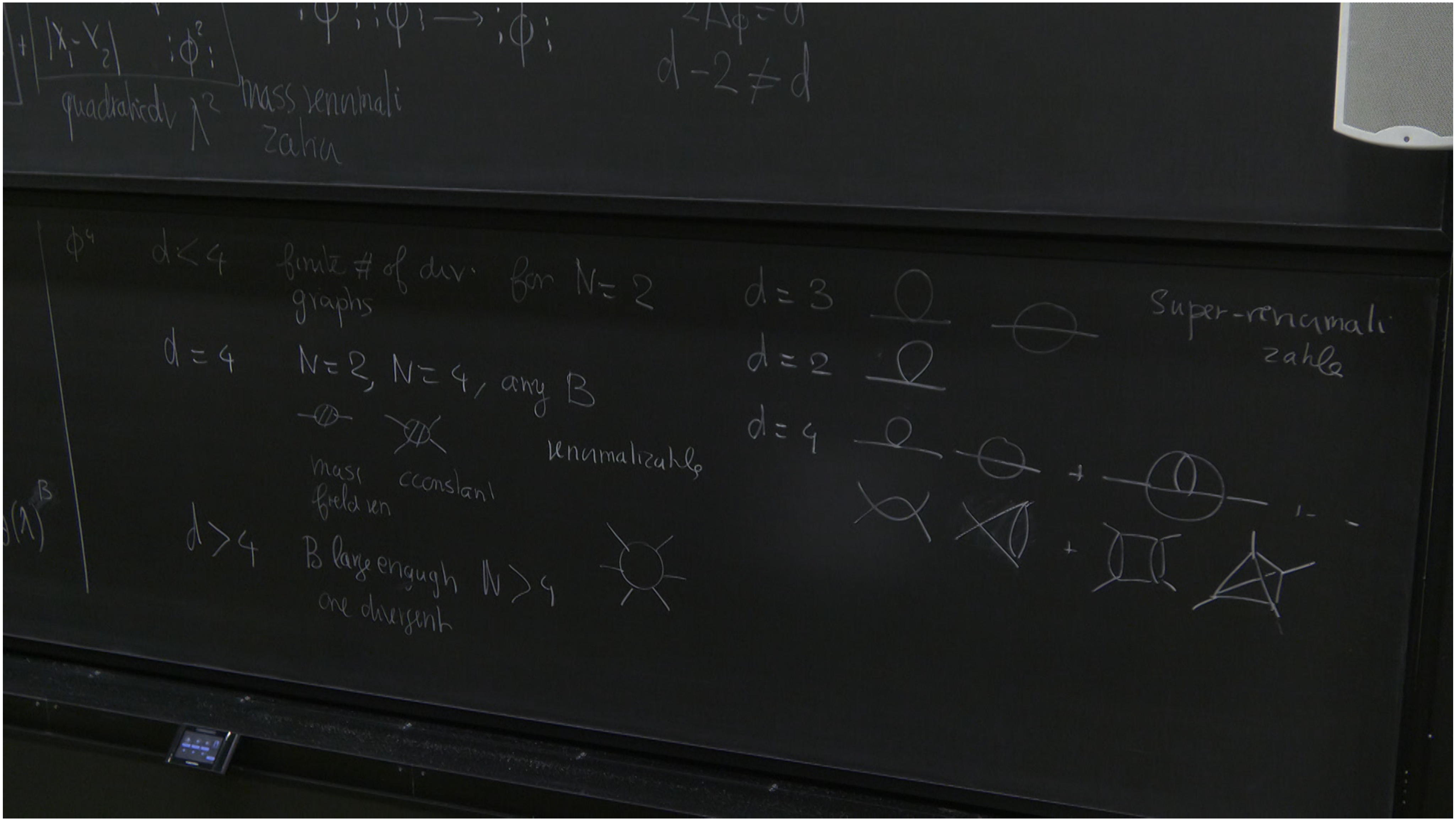
$(\lambda)^B$

$d=3$

$d=2$

$d=4$

Super-renormalizable



$|x_1 - x_2|$ $:\phi^2:$

quadratically Λ^2 mass renormalization

$:\phi::\phi: \longrightarrow :\phi:$

$2\Delta_\phi = d$

$d-2 \neq d$

ϕ^4

$g(\lambda)^B$

$d<4$ finite # of div. graphs for $N=2$

$d=4$ $N=2$, $N=4$, any B

mass constant field ren

renormalizable

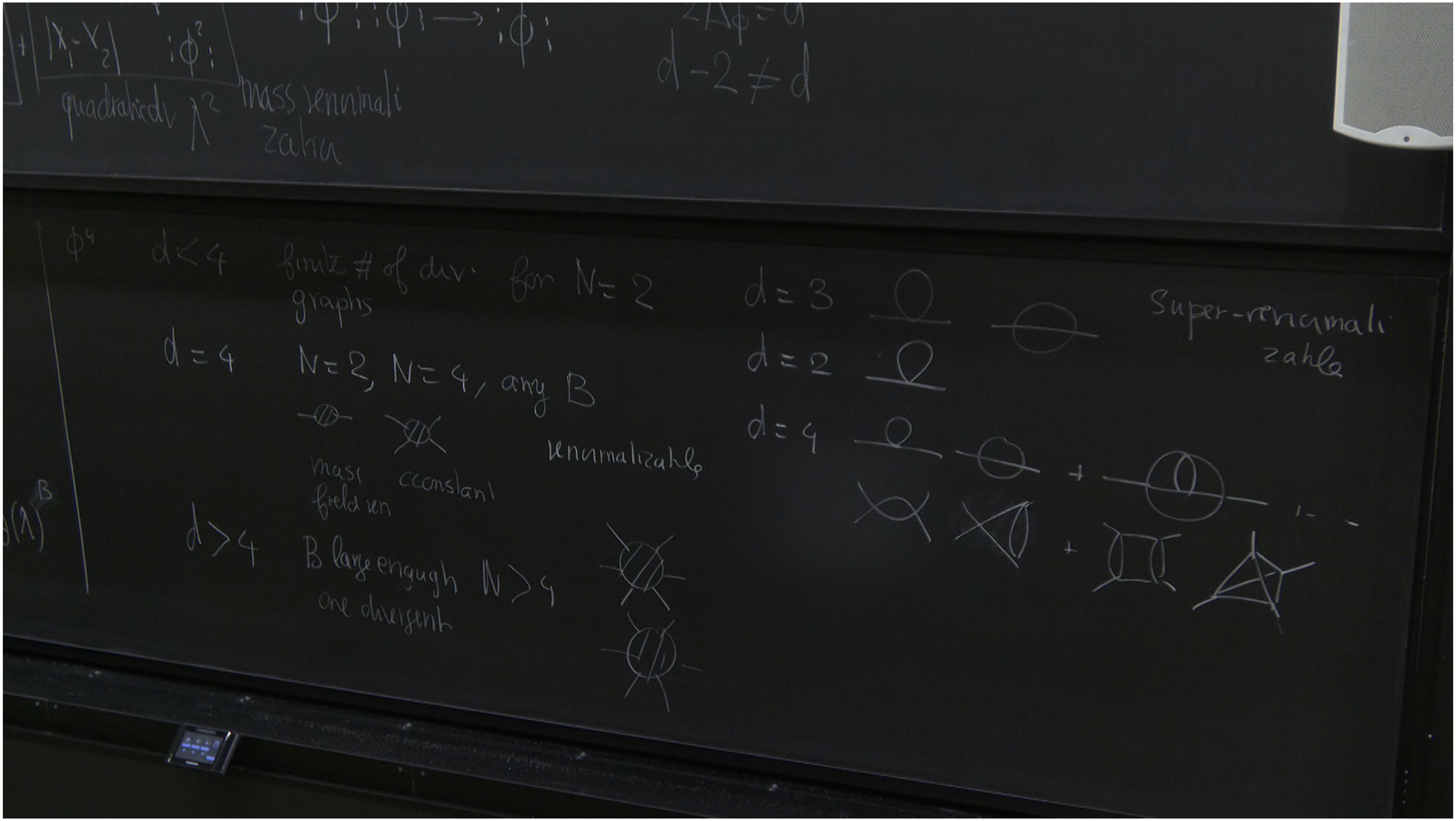
$d>4$ B large enough $N>4$ one divergent

$d=3$

$d=2$

Super-renormalizable

$d=4$



$|x_1 - x_2|$ $:\phi^2:$

quadratic div λ^2

mass renormalization

$:\phi::\phi: \longrightarrow :\phi:$

$2\Delta_\phi = d$

$d - 2 \neq d$

ϕ^4

$d < 4$ finite # of div. graphs for $N = 2$

$d = 4$ $N = 2$, $N = 4$, any B

mass field ren, coupling constant

renormalizable

$d > 4$ B large enough $N > 4$ one divergent

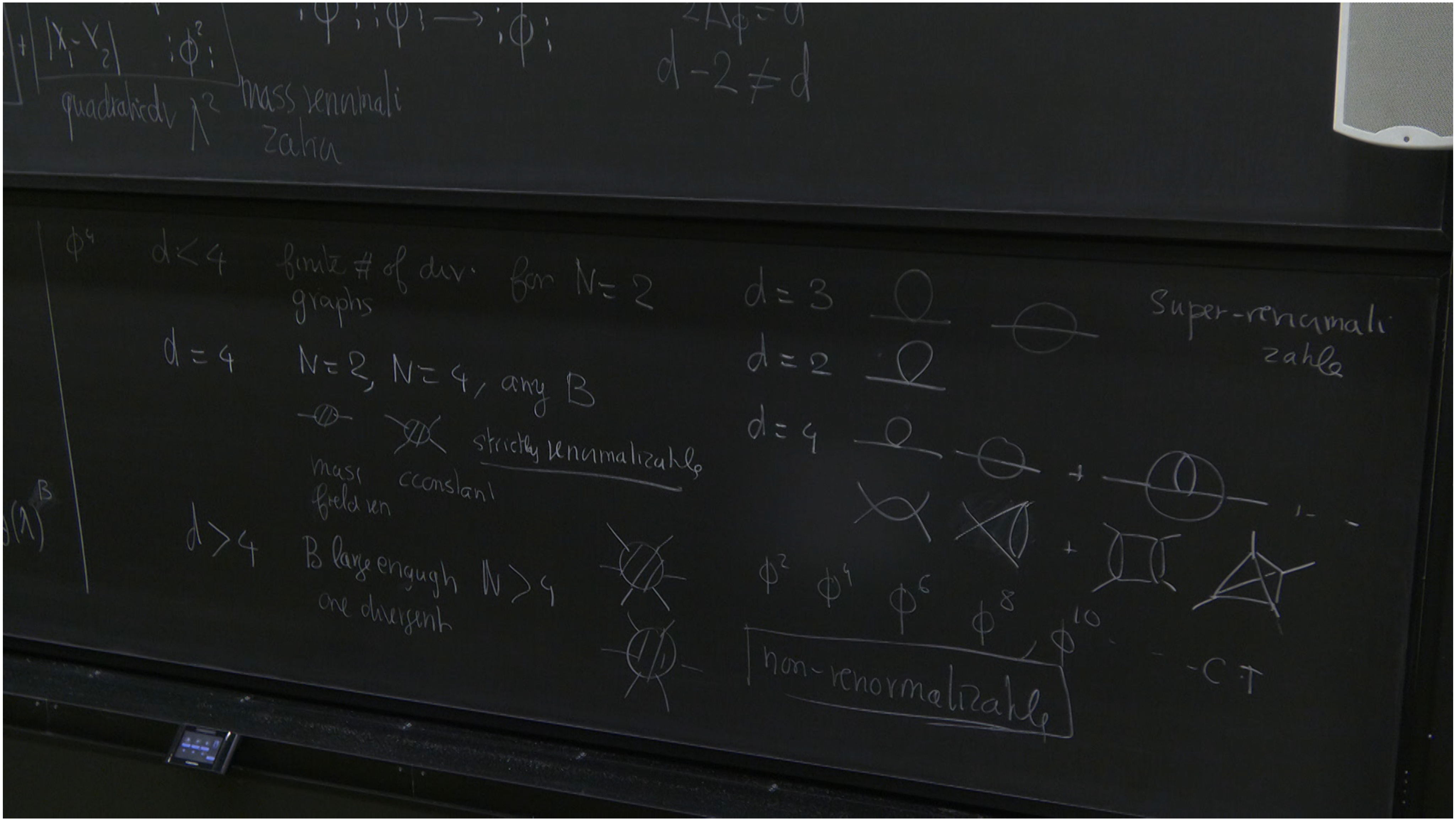
$g(\lambda)^B$

$d = 3$

$d = 2$

$d = 4$

Super-renormalizable



$|x_1-x_2|$ $:\phi^2:$

quadratically λ^2

mass renormalization

$:\phi::\phi: \longrightarrow :\phi:$

$2\Delta_\phi = d$

$d-2 \neq d$

ϕ^4

$d<4$ finite # of div. graphs for $N=2$

$d=4$ $N=2$, $N=4$, any B

strictly renormalizable

mass field ren

cconstant

$d>4$ B large enough $N>4$ one divergent

$g(\lambda)^B$

$d=3$

$d=2$

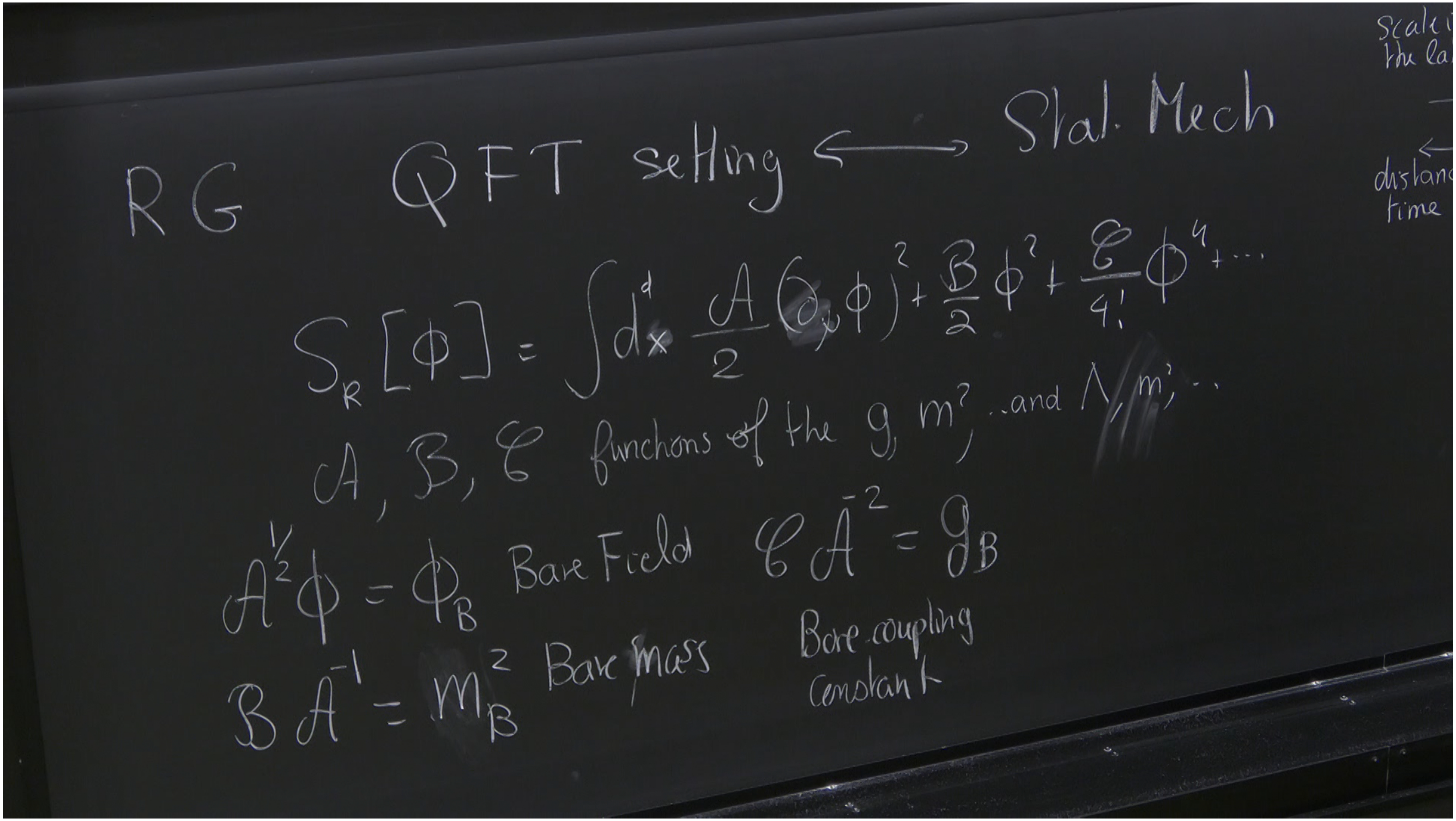
$d=4$

Super-renormalizable

ϕ^2 ϕ^4 ϕ^6 ϕ^8 ϕ^{10} ...

non-renormalizable

C.T



RG   QFT setting ⟷ Stat. Mech

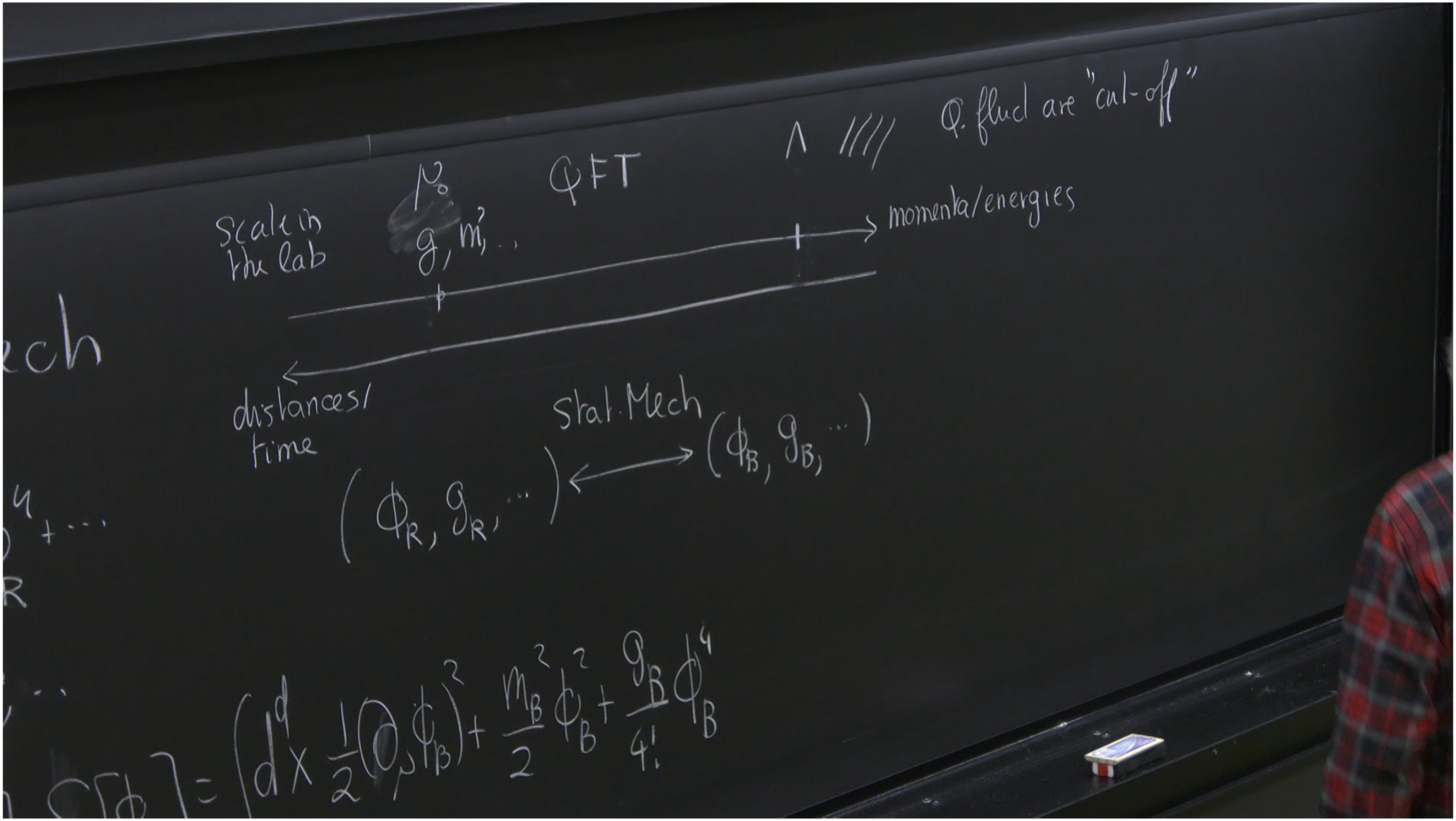
$$S_R[\phi] = \int d^dx\, \frac{\mathcal{A}}{2}(\partial_\mu \phi)^2 + \frac{\mathcal{B}}{2}\phi^2 + \frac{\mathcal{C}}{4!}\phi^4 + \dots$$

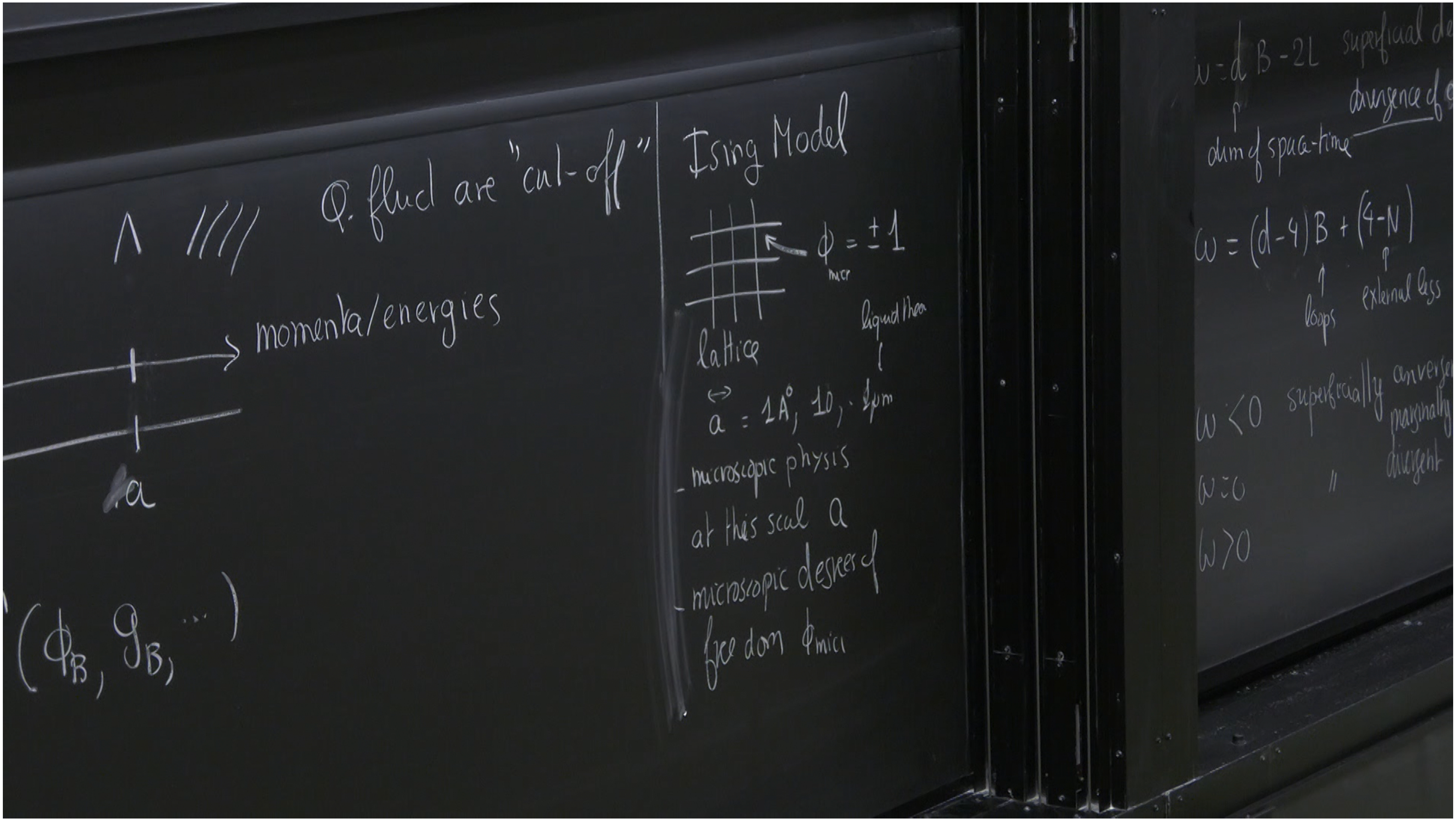
$\mathcal{A}, \mathcal{B}, \mathcal{C}$ functions of the $g, m^2, \dots$ and $\Lambda, m^2, \dots$

$\mathcal{A}^{\frac{1}{2}}\phi = \phi_B$ Bare Field $\mathcal{C}\mathcal{A}^{-2} = g_B$

Bare coupling constant

$\mathcal{B}\mathcal{A}^{-1} = m_B^2$ Bare mass


Λ //// Q. fluct are "cut-off"
μ_0
QFT
Scale in the lab
$g, m^2, \ldots$
momenta/energies
distances/time
ech
Stat. Mech
$(\phi_R, g_R, \ldots) \longleftrightarrow (\phi_B, g_B, \ldots)$
$S[\phi] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi_B)^2 + \frac{m_B^2}{2}\phi_B^2 + \frac{g_B}{4!}\phi_B^4$



Λ //// Q. fluct are "cut-off"

momenta/energies

a

$(\phi_B, g_B, \dots)$

Ising Model

$\phi_{micr} = \pm 1$

liquid theo

lattice

$\overset{\leftrightarrow}{a} = 1\mathring{A}, 10, \cdot 1\mu m$

- microscopic physis at this scal a
- microscopic desrees of freedom ϕ_{mici}

$\omega = d\, B - 2L$ superficial de

divergence of

dim of space-time

$\omega = (d-4)B + (4-N)$

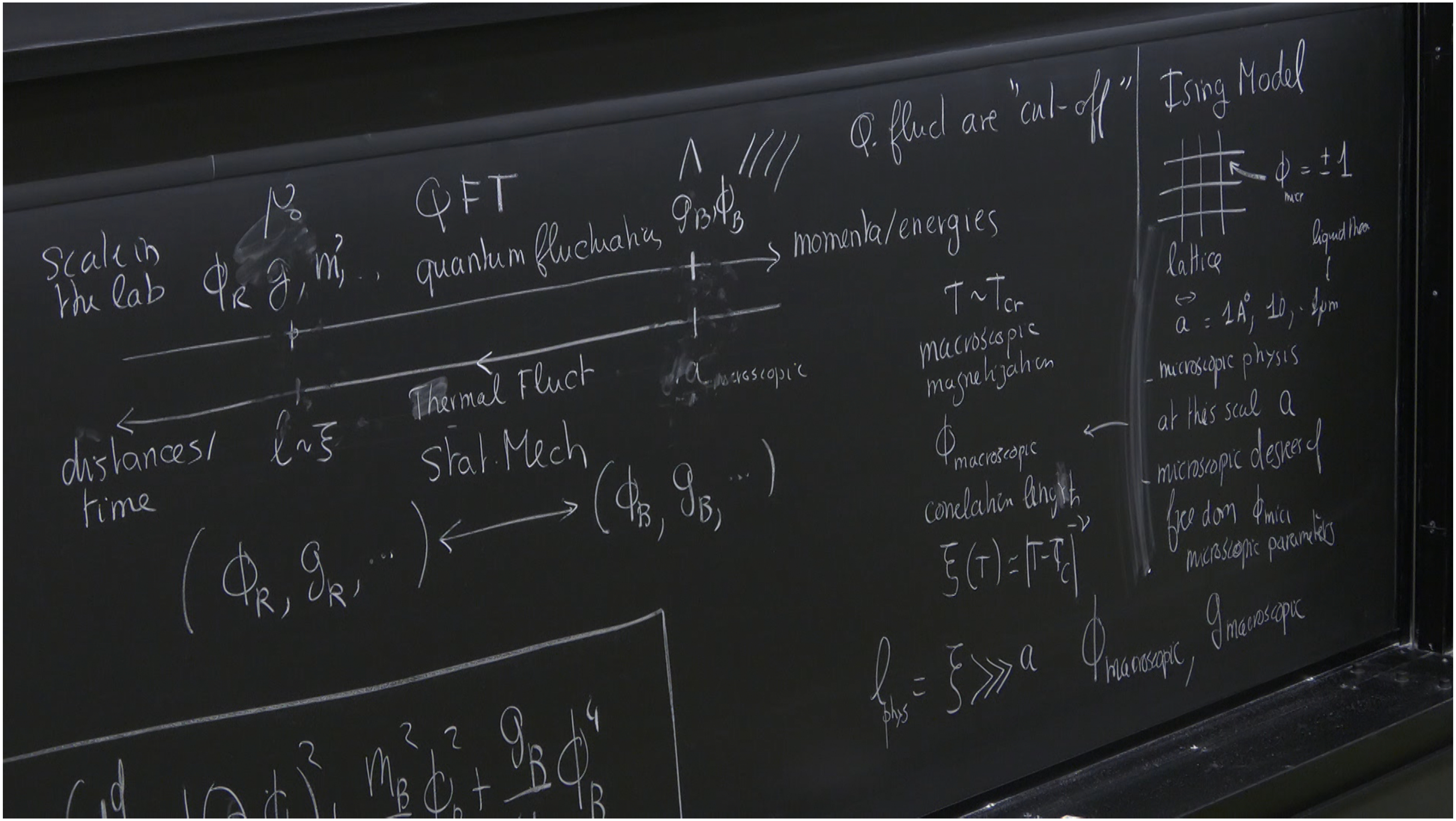
loops

external legs

$\omega < 0$ superficially converge

$\omega = 0$ 〃 marginally divergent

$\omega > 0$



QFT

Scale in the lab μ_0 ϕ_R $g, m^2, \ldots$ quantum fluctuations g_B, ϕ_B Λ → momenta/energies

Q. fluct are "cut-off"

← distances/ time $\ell \sim \xi$ Thermal Fluct $a_{\text{microscopic}}$

Stat. Mech

$$(\phi_R, g_R, \ldots) \longleftrightarrow (\phi_B, g_B, \ldots)$$

$$d^d \;\; (\partial\phi)^2 \;\; \frac{m_B^2}{}\phi_B^2 + \frac{g_B}{}\phi_B^4$$

$T \sim T_{cr}$

macroscopic magnetization

$\phi_{\text{macroscopic}}$

correlation length

$\xi(T) = |T-T_c|^{-\nu}$

Ising Model

$\phi_{\text{micr}} = \pm 1$

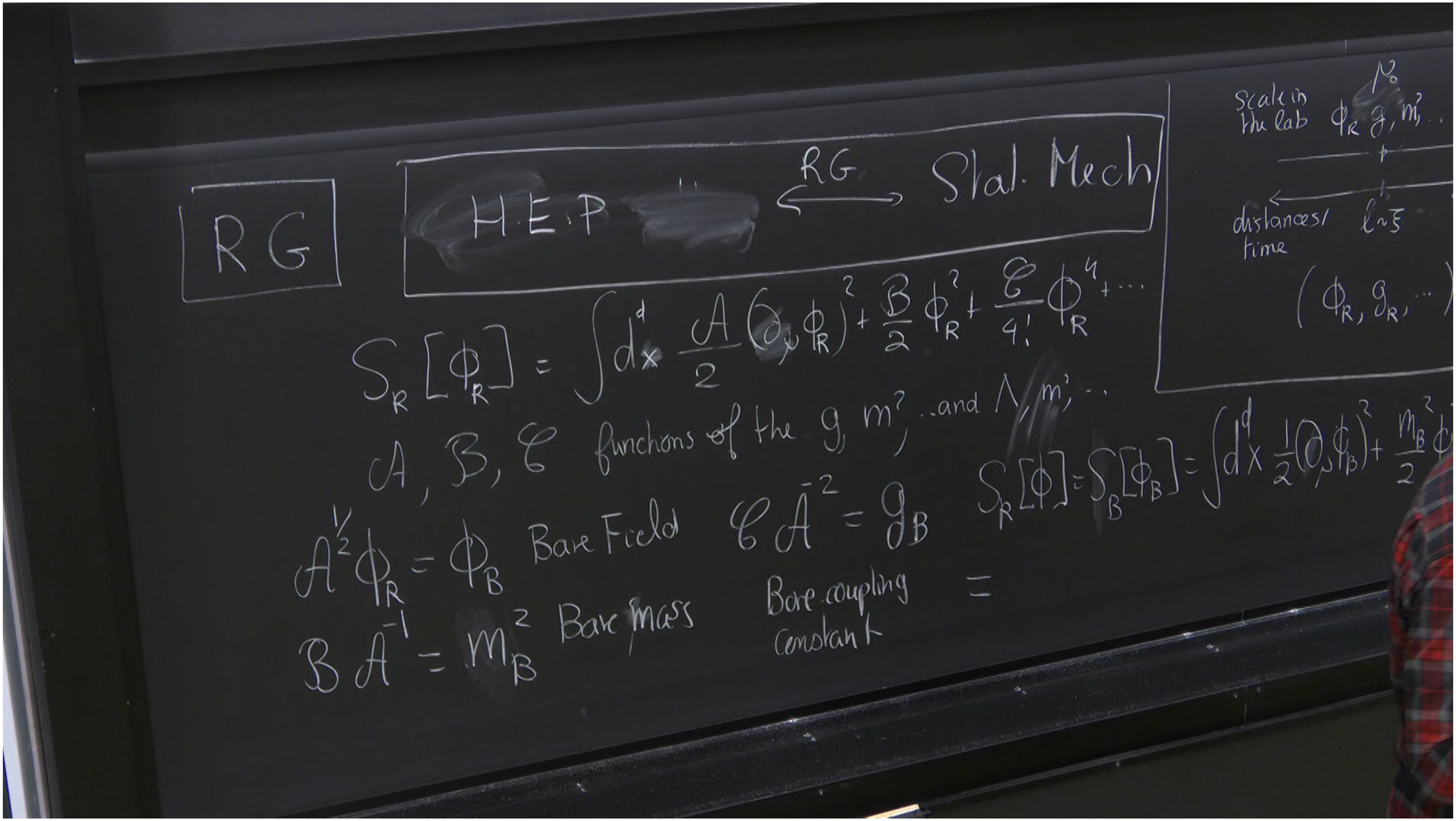
lattice

liquid theory

$\overset{\leftrightarrow}{a} = 1Å, 10, \ldots 1\mu m$

- microscopic physics at this scale a
- microscopic degrees of freedom ϕ_{micr}
- microscopic parameters

$\ell_{\text{phys}} = \xi \ggg a$ $\phi_{\text{macroscopic}}, g_{\text{macroscopic}}$



$\boxed{RG}$

$\boxed{H.E.P \xleftrightarrow{RG} \text{Stat. Mech}}$

$$S_R[\phi_R] = \int d^d x \, \frac{\mathcal{A}}{2}(\partial_\mu \phi_R)^2 + \frac{\mathcal{B}}{2}\phi_R^2 + \frac{\mathcal{C}}{4!}\phi_R^4 + \dots$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ functions of the $g, m^2, \dots$ and $\Lambda, m^2, \dots$

$\mathcal{A}^{1/2}\phi_R = \phi_B$ Bare Field

$\mathcal{C}\mathcal{A}^{-2} = g_B$ Bare coupling constant

$\mathcal{B}\mathcal{A}^{-1} = m_B^2$ Bare mass

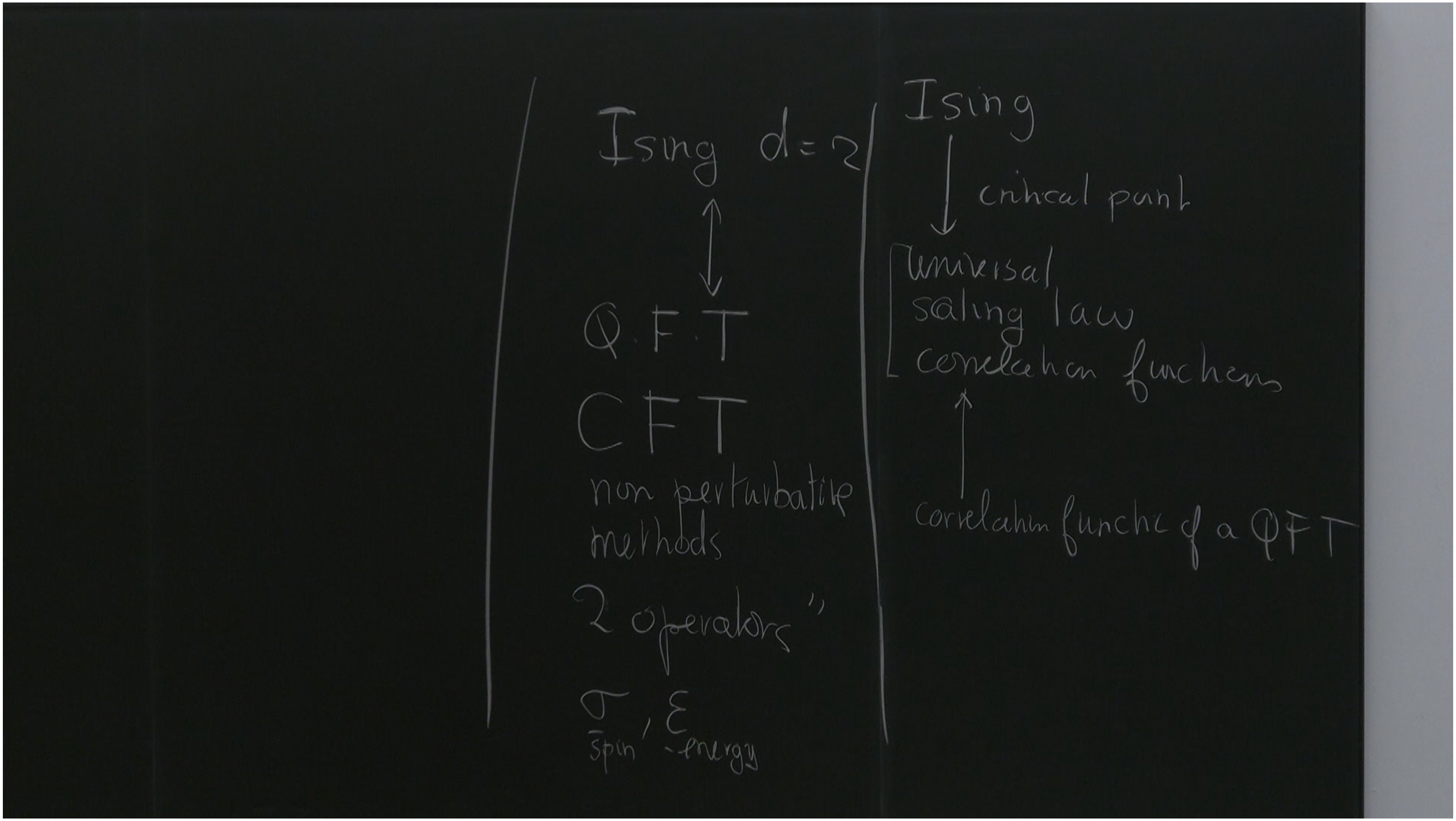
$$S_R[\phi] = S_B[\phi_B] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi_B)^2 + \frac{m_B^2}{2}\phi$$

$=$

Scale in the lab $\quad \mu_0 \quad \phi_R \; g, m^2, \dots$

distances/ time $\quad \ell \sim \xi$

$(\phi_R, g_R, \dots)$



Ising $d=2$

$\updownarrow$

Q.F.T

CFT

non perturbative methods

2 operators"

σ, ε
spin, energy

Ising

$\downarrow$ critical point

universal
scaling law
correlation functions

$\uparrow$

correlation functions of a QFT