Title: QFT2 Lecture - 120423

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Collection: Quantum Field Theory 2 2023/24

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Renormalization Theory

$x \to y \quad \phi(x)\,\phi(y)$ is singular

$$\int d^d x \; \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$$

Scalar Free Field

$$G_0(x-y) = \frac{1}{4\pi^2}|x-y|^{-2} \;,\; -\frac{1}{2\pi}\log\left(\frac{|x-y|}{2}m\right) \;,\; |x-y|^{2-d}$$

Euclidean $\quad d=4 \qquad d=2 \qquad d$

Renormalization Theory

$x \to y$ $\quad \phi(x)\,\phi(y)$ is singular

$$S[\phi] = \int d^d x \; \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 \quad \text{action}$$

$[\phi \cdot \phi] = d - 2$

$x \qquad y$

Scalar Free Field

$$G_0(x-y) = \frac{1}{4\pi^2}|x-y|^{-2} \;, \quad -\frac{1}{2\pi}\log\left(\frac{|x-y|}{2}m\right) \;, \quad |x-y|^{2-d}$$

Euclidean $\qquad d=4 \qquad\qquad d=2 \qquad\qquad d$ dim of spacetime ar

Dimensional analysis : dim of l^{-1} or E/k

$\hbar = c = 1$ $\quad [S] = 0$ dimension

$[x] = -1 \quad [m] = 1$

$[\phi] = \frac{d-2}{2}$ scalar field

can we make sense of

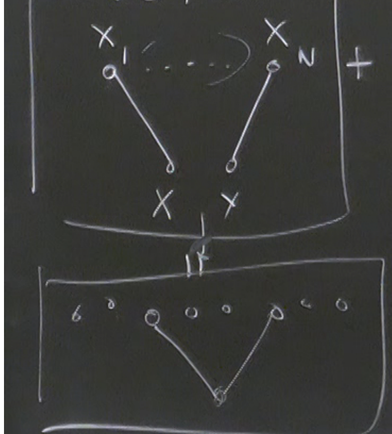
$\phi^2(x)$ as a local operator? Composite operator

1

pacetime

$$\langle \phi(x)\,\phi(y)\,\underbrace{\phi(x_1)\cdots\phi(x_N)}_{\text{fixed, any of them}}\rangle_0 \underset{x\to y}{=} |x-y|^{2-d}\,\langle \mathbb{1}\left(\tfrac{x+y}{2}\right)\cdots\rangle + \langle :\phi^2:\left(\tfrac{x+y}{2}\right)\phi(x_1)\cdots\phi(x_N)\rangle$$

Wick Theorem



$$\underset{x\quad y}{\circ\!\!-\!\!\circ} \longrightarrow |x-y|^{2-d}\times \mathbb{1}\left(\tfrac{x+y}{2}\right)$$

UV singular

$$\underbrace{:\phi^2:\left(\tfrac{x+y}{2}\right)}_{\text{well defined operator}} = \lim_{x\to y}\Big[\underbrace{\phi(x)\phi(y)}_{\text{operator}} - \underbrace{\langle\phi(x)\phi(y)\rangle}_{C(x-y)\ \text{coefficient}}\,\underbrace{\mathbb{1}\left(\tfrac{x+y}{2}\right)}_{\text{operator}}\Big]$$

Normal product or normal ordered product of two ϕ operators

$$\ldots \rangle + \langle :\phi^2:\left(\tfrac{x+y}{2}\right) \phi(x_1) \cdots \phi(x_N) \rangle$$

$$\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle \, \mathbb{1}\left(\tfrac{x+y}{2}\right)$$

operator — $C(x-y)$ coefficient — operator

...nal ordered product of two ϕ operators

$$\phi(x)\phi(y) \xrightarrow[x \to y]{} C(x-y)\, \mathbb{1}\left(\tfrac{x+y}{2}\right) + :\phi^2:\left(\tfrac{x+y}{2}\right)$$

Operator Product Expansion (at small distances)

OPE

$$\ldots \rangle + \langle :\phi^2:\left(\tfrac{x+y}{2}\right) \phi(x_1)\cdots\phi(x_N) \rangle$$

$$\phi(x)\,\phi(y) - \langle \phi(x)\,\phi(y) \rangle\, \mathbb{1}\left(\tfrac{x+y}{2}\right)$$

operator

$C(x-y)$ coefficient

operator

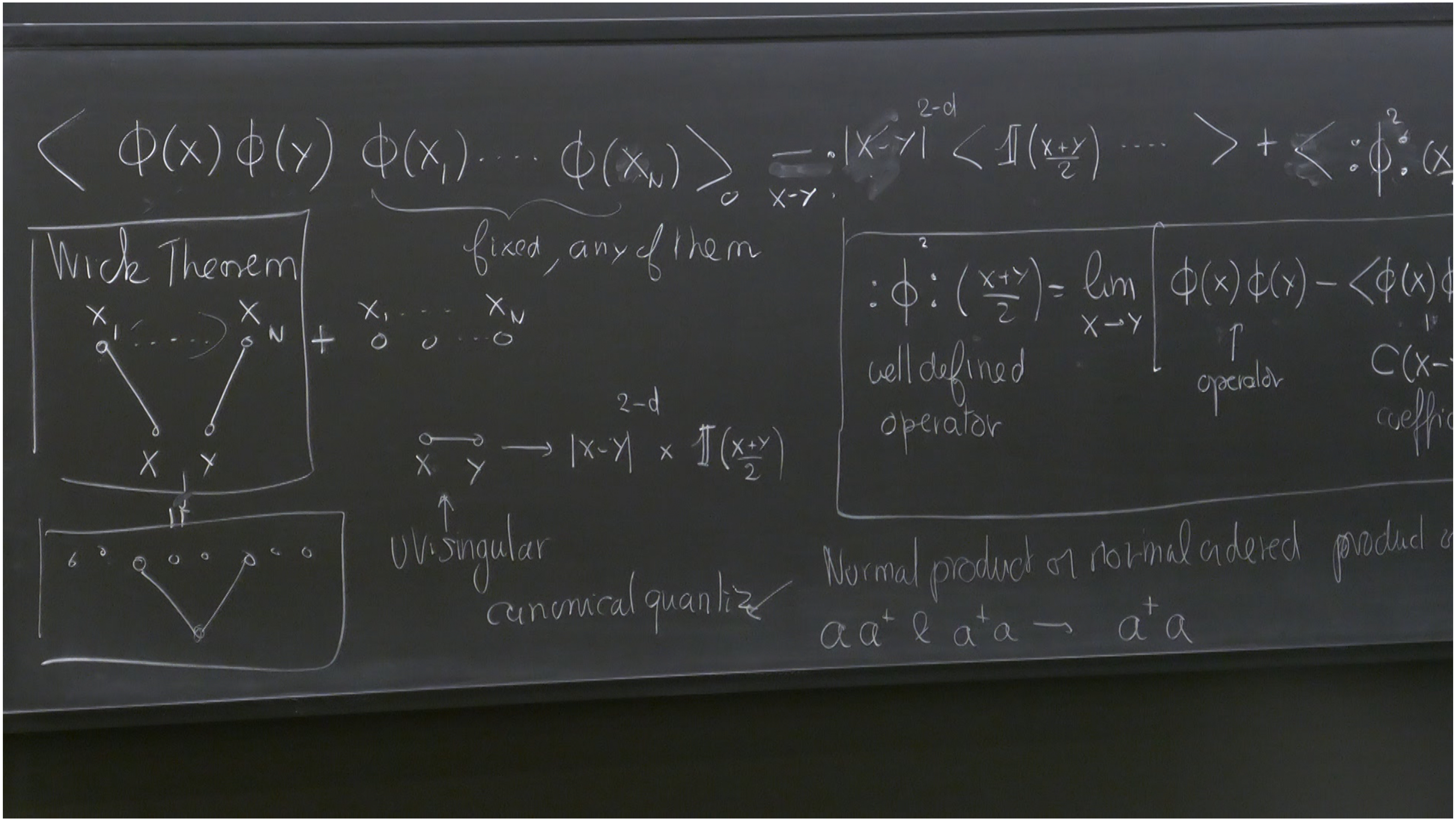
...al ordered product of two ϕ operators

$$\phi(x)\,\phi(y) \xrightarrow[x \to y]{} C(x-y)\, \mathbb{1}\left(\tfrac{x+y}{2}\right) + :\phi^2:\left(\tfrac{x+y}{2}\right)$$

Operator Product Expansion (at small distances)

OPE

K. Wilson '69-'70



$\langle \phi(x)\phi(y)\,\phi(x_1)\cdots\phi(x_N)\rangle_0 \underset{x\to y}{=} |x-y|^{2-d}\langle \mathbb{1}(\tfrac{x+y}{2})\cdots\rangle + \langle :\phi^2:(x$
fixed, any of them
Wick Theorem
x_1 x_N
x y
$x_1 \cdots x_N$
$|x-y|^{2-d} \times \mathbb{1}(\tfrac{x+y}{2})$
UV singular
canonical quantiz
$:\phi^2:(\tfrac{x+y}{2}) = \lim_{x\to y}[\phi(x)\phi(y) - \langle\phi(x)\phi$
well defined operator
operator
Normal product or normal ordered product
$aa^+ \,\&\, a^+a \to a^+a$

$\rangle + \langle :\phi^2: \left(\frac{x+y}{2}\right) \phi(x_1) \cdots \phi(x_N) \rangle$

$\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle \, \mathbb{1}\left(\frac{x+y}{2}\right)$

operator

$C(x-y)$

coefficient

operator

ordered product of two ϕ operators

$$\phi(x)\phi(y) \xrightarrow[x \to y]{} C(x-y) \cdot \mathbb{1}\left(\frac{x+y}{2}\right) + :\phi^2:\left(\frac{x+y}{2}\right)$$

Operator Product Expansion (at small distances)

OPE

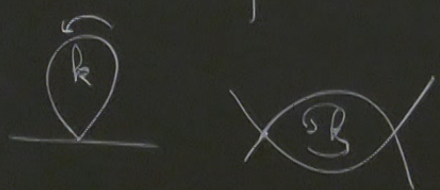
K. Wilson '69-'70

general feature of QFT

...ean $d=4$ $d=2$ d dim of spacetim...

(3) $S[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right)$ interaction theory

1 loop order, ...



$\int d^dk \, \frac{1}{k^2}$ or $\frac{1}{k^2(p+k)^2}$

UV div — $d \geq 2$ — $d \geq 4$

Improve short distance / high e...
behaviour of the theory ⟶ li...
"Trick" of regularization ⟶ co...
not black magic or "un...

$$\Big)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\Big)$$ interaction theory

or $\dfrac{1}{k^2(p+k)^2}$

$d \geq 4$

Improve short distance (high energy/momentum)
behaviour of the theory ⟶ limit
"Trick" of regularization ⟶ continuum limit
<u>not</u> black magic or "under the rug trick"

...crementum)

...um limit

..e rug tuch"

- $\dfrac{1}{k^2+m^2} \Leftarrow$ for $|k^2| < \underline{\Lambda}^2$ UV cut-off scale

  $0 \Leftarrow |k^2| > \Lambda^2$ sharp cut off

- Pauli-Villars regularization

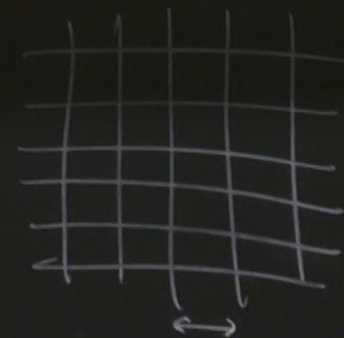
$$\frac{1}{k^2+m^2} \cdot \frac{\Lambda^2}{k^2+\Lambda^2} \quad \Lambda \gg m \quad |k| \ll \Lambda \quad \frac{1}{k^2}$$

$$|k| \gg \Lambda \quad \frac{\Lambda^2}{(k^2)^2}$$

Poincaré Invariant

$$\left(\frac{1}{k^2+m^2} - \frac{1}{k^2+\Lambda^2}\right)\left(\frac{\Lambda^2}{\Lambda^2-m^2}\right)$$ Unitarity?

$M^{1,d}$ or $\mathbb{R}^d$ $\xrightarrow{\text{Lattice}}$



$\mathcal{L} = \mathbb{Z}^d$

mesh a

$a \sim \frac{1}{\Lambda}$ short distance regulator

Poincaré invariance $\longrightarrow$ Discre Lattice Symm

Wilson + Wegner Lattice Gauge Theory

Normal product or normal ordered product of two ϕ operators

K. Wilson '69 - '70

general feature of QFT

aa^{+} & $a^{+}a \to a^{+}a$

"Analytic" regularizations

dimensional regularization

't Hooft, Veltman $\to$ Gauge theories

$d=4 \to d=4+\epsilon$ not integer (complex)

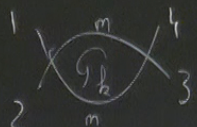
Feynman integrals & scattering amplitudes

UV $\log \Lambda$ divergence (or $\log \frac{1}{a}$)

$\downarrow$

pole at $d=4$ $\Rightarrow$ meromorphic function of d

(only poles)



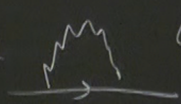
$p = p_1 + p_2$ $\quad B(p; m; d) = \dfrac{1}{4-d} + \text{reg part of } d$

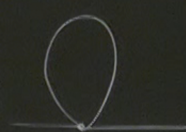
analytic function of d

ϕ^4 $\quad [g] = 4 - d = 0$ if dim 4 $\Leftrightarrow \log \Lambda$ singular terms

$[m^2] = 2 \Leftrightarrow \Lambda^2$ singular terms

Massless ϕ^4 theory : 1-particle quantum as $M_{phys} = 0$

QED $\longrightarrow + e^2$  $+ \cdots$ ← radiative correction

Irreducible 2 point function: $\frac{1}{p^2+m^2} + g\frac{1}{2}$  $+ \cdots$

Poincaré invariant

$$\left(\frac{1}{k^2+m^2} - \frac{1}{k^2+\Lambda^2}\right)\left(\frac{\Lambda^2}{\Lambda^2-m^2}\right) \quad \text{Unitarity?} \quad (k^2)^2$$

Idea: Masses, coupling constants, .. $\Leftarrow$ must specify how they are "measured"

QFT as in QT

$$\int \mathcal{D}[\phi] \exp(-S_R[\phi]) \qquad S_R[\phi] = \int d^4x \, \frac{A}{2}(\partial_\mu \phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$$

Euclidean

Renormalized action: you have to use it in order to construct physical observables

I.e. 2pt function $\Gamma^{(2)}(p) = 0$ at $p^2 = 0$ Massless physical condition

⑥ $\Gamma^{(2)}(p) = \mathcal{A}\, p^2 + \mathcal{B} + \mathcal{C}\,\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{\mathcal{A} k^2+\mathcal{B}}\right) = p^2 + \mathcal{B} + \underbrace{\frac{\mathcal{C}}{2}\Big((4}_{=0}$

$\boxed{\mathcal{A} = 1}$

$$T = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\mathcal{B}} = \frac{2\pi^2}{(4\pi)^4}\int_0^\Lambda d|k|\,|k|^3\,\frac{1}{|k|^2+\mathcal{B}} = \frac{1}{(4\pi)^2}\Lambda^2 + \underbrace{\mathcal{B}^2 \log\left(\frac{\Lambda}{\mathcal{B}}\right)^2 + \dots}_{\text{unimportant}}$$

$\vec{k} = \vec{v}\times|k|$

Vol. unit sphere in $\mathbb{R}^4$

divergence

$\Rightarrow \mathcal{B} =$

$\boxed{\mathcal{C} = g_R \quad \text{ren} \ldots \ \text{co}\ldots}$

Regulator $|k| < \Lambda$

$a \to a^{\dagger} a$

K. Wilson 69-70

general feature of QFT

$$= p^2 + \underbrace{\mathcal{B} + \frac{\mathcal{C}}{2}\left(\frac{1}{(4\pi)^2}\Lambda^2 + \dots\right)}_{=0}$$

$\Gamma^{(2)}(p) = 0$ at $p^2 = 0$

physical condition

$\boxed{\text{mass counterterm}}$

$$\Rightarrow \mathcal{B} = -\frac{1}{2}\frac{1}{(4\pi)^2}\mathcal{C}\cdot\Lambda^2 ; \quad \boxed{\mathcal{B} = -\frac{1}{2}\frac{\Lambda^2}{(4\pi)^2} g_R}$$

$\log\left(\frac{\Lambda}{\mathcal{B}}\right)^2 + \dots$

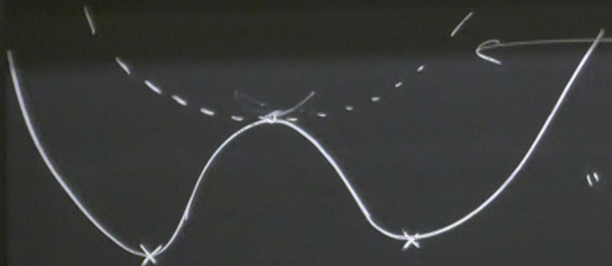
unimportant

$\boxed{\mathcal{C} = g_R}$ renormalized coupling constant

neglecting

$$\boxed{\Gamma^{(2)}(p) = p^2 +} \; O(g_R^2)$$

Regulator $|k| < \Lambda$



effective potential $\sim \varphi^4$ no φ^2 term. $M_{phys} = 0$

$\Gamma[\varphi]$

"Bare Potential

$$g_R\left(-\Lambda^2\phi^2 + \phi^4\right)$$

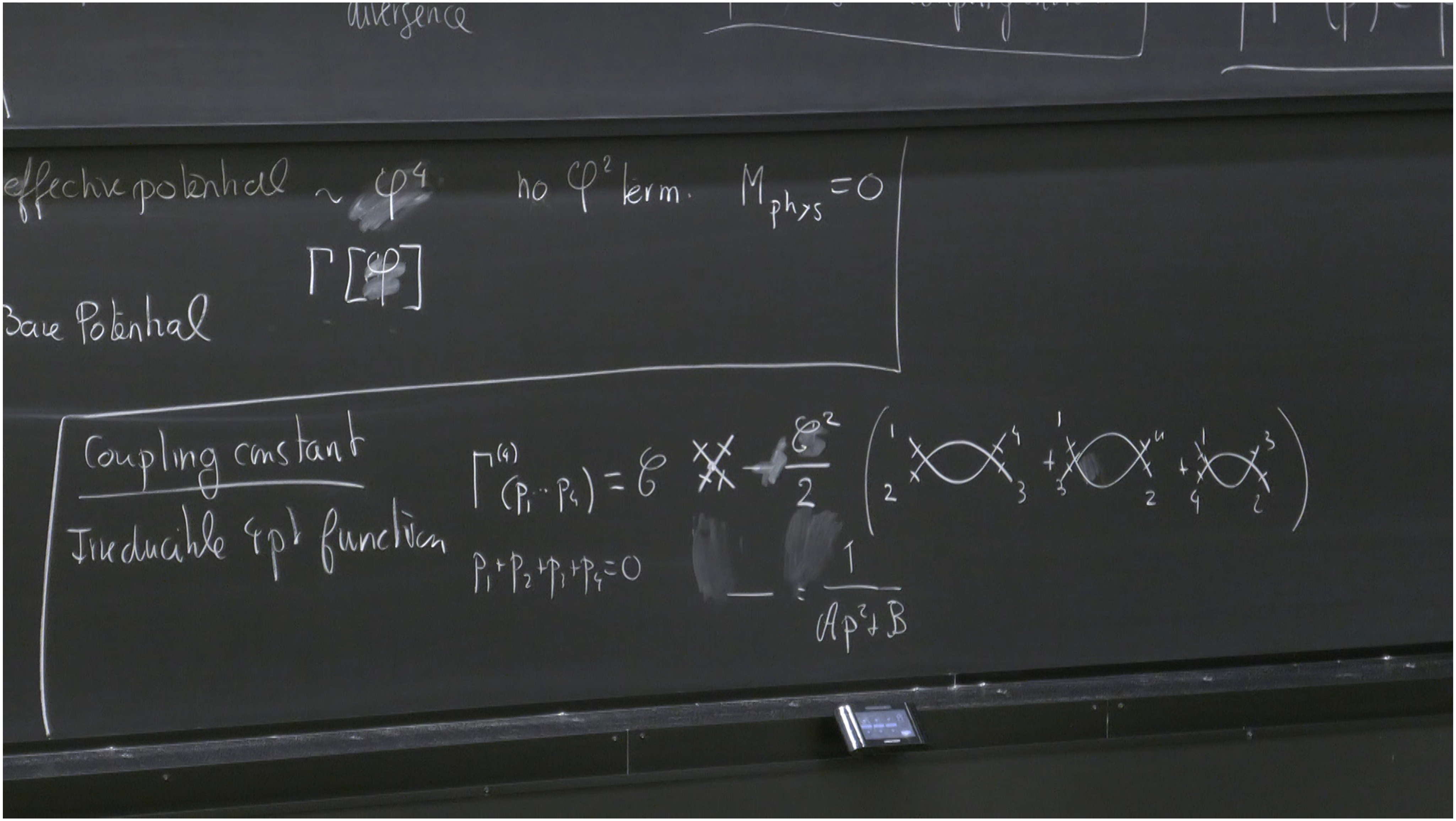
$$[\phi] \sim [\Lambda]$$

<u>Coupling constant</u>

Irreducible 4pt function

$$\Gamma^{(4)}(p_1 \cdots p_4) =$$

$$p_1 + p_2 + p_3 + p_4 = 0$$


divergence
effective potential ~ φ⁴ no φ² term. M_phys = 0
Γ[φ]
Bare Potential
Coupling constant
Irreducible 4pt function
Γ⁽⁴⁾(p₁ ·· p₄) = g − g²/2 (+ +)
p₁+p₂+p₃+p₄=0
= 1/(Ap²+B)

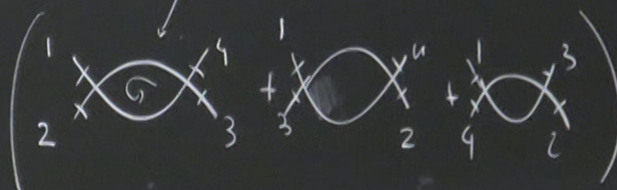
no φ^2 term. $M_{phys} = 0$

$$B(p_1+p_2) + B(p_1+p_3) + B(p_1+p_4)$$

$$B(p) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+B)((p+k)^2+B)}$$

$$= \log \Lambda$$

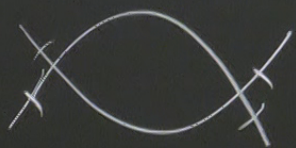
$$\Gamma^{(4)}(p_1 \cdots p_4) = g - \frac{g^2}{2}\left(\quad + \quad + \quad \right)$$



$p_1+p_2+p_3+p_4=0$

$$= \frac{1}{Ap^2+B}$$

(5) p^2+m^2

$\xrightarrow{P}$  $\simeq \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2} = \frac{\pi^2}{(4\pi)^2} \log \Lambda^2 = \frac{1}{(2\pi}$

$$\mathcal{C} = g_R + g_R^2 \cdot \text{Counterterm}$$

how we define g_R ?

$$= \frac{\pi^2}{(4\pi)^2} \log \Lambda^2 = \frac{1}{(2\pi)^2} \log \Lambda^2 \text{ divergence}$$

nter term