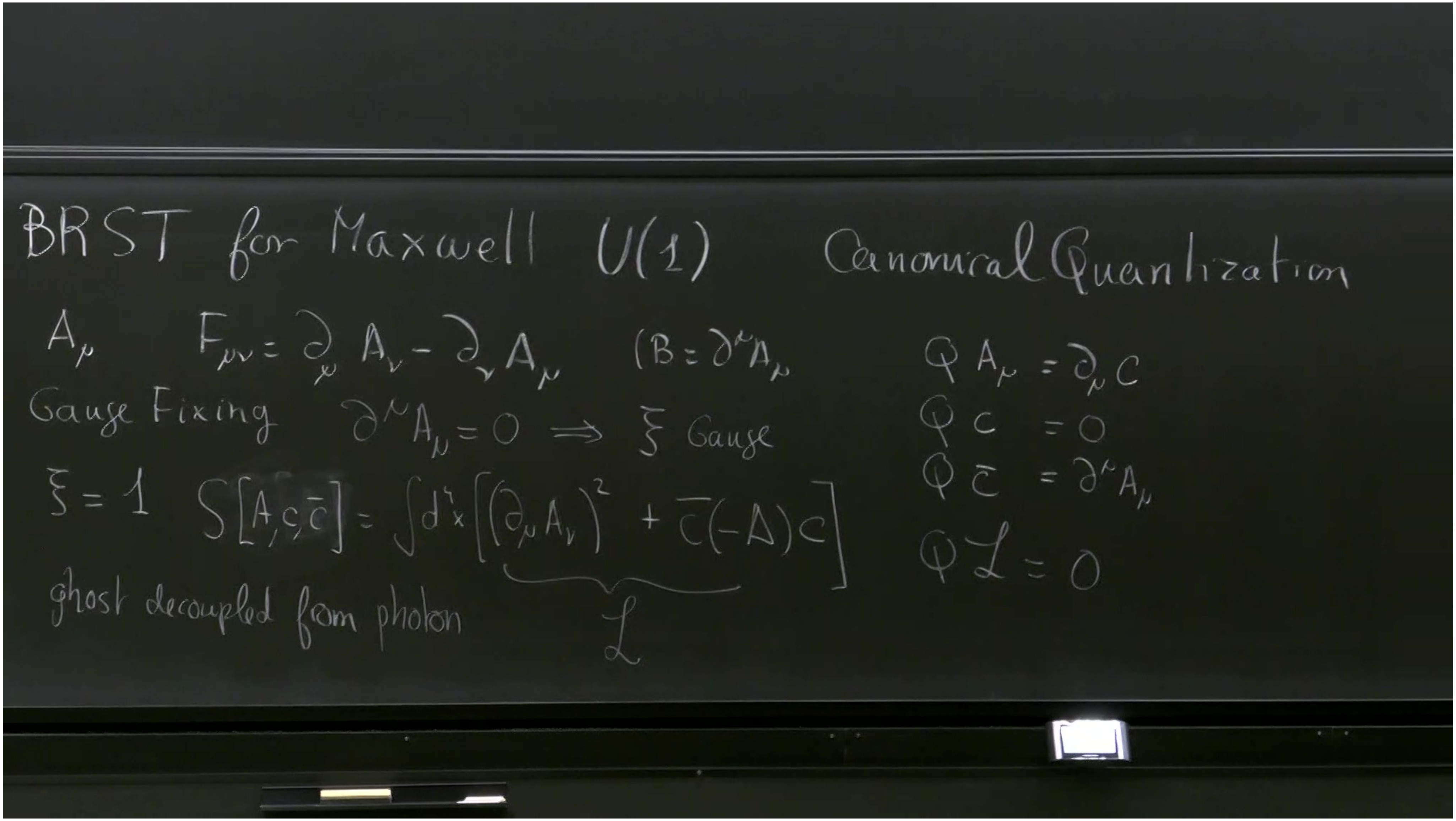
Title: QFT2 Lecture - 120123

Speakers: Francois David

Collection: Quantum Field Theory 2 2023/24

Date: December 01, 2023 - 9:00 AM

URL: https://pirsa.org/23120004



BRST for Maxwell $U(1)$

A_μ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\quad (B = \partial^\mu A_\mu$

Gauge Fixing $\quad \partial^\mu A_\mu = 0 \Longrightarrow \xi$ Gauge

$$\xi = 1 \quad S[A, c, \bar{c}] = \int d^4x \left[\underbrace{(\partial_\mu A_\nu)^2 + \bar{c}(-\Delta)c}_{\mathcal{L}} \right]$$

ghost decoupled from photon

Canonical Quantization

$Q A_\mu = \partial_\mu c$

$Q c = 0$

$Q \bar{c} = \partial^\mu A_\mu$

$Q \mathcal{L} = 0$

anlization

C

A_μ

$|\Omega\rangle$

massless $\quad K = (k^0, \vec{k}) = (|\vec{k}|, \vec{k})$

on shell $\quad K^2 = -(k^0)^2 + (\vec{k})^2 = 0$

$|k| = E(\vec{k})$

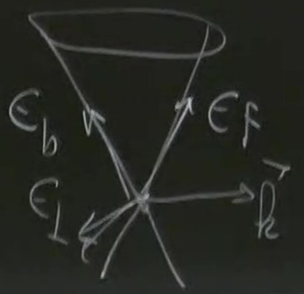
A_μ polarization $\quad$ space momentum $\vec{k} = |k|\,\vec{u} \quad \vec{u}^2 = 0$ 3-vector

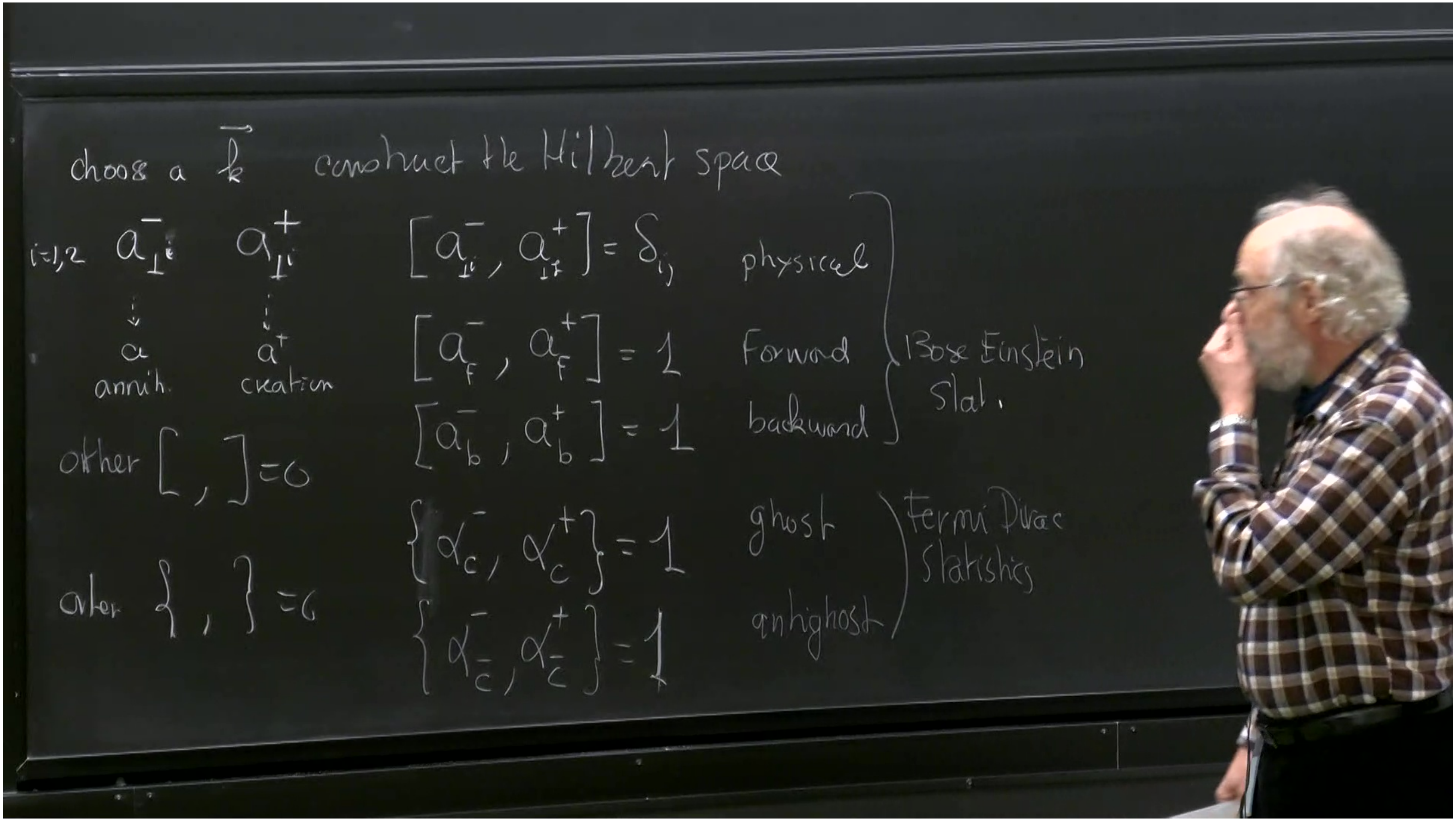
$A_\mu = \epsilon_\mu e^{i k \cdot x}$

transverse pa $\quad \epsilon_\perp = (0, \vec{e}) \quad \vec{\epsilon} \cdot \vec{u} = 0$ ☺

longitunal modes $\quad \epsilon_F = (1, \vec{u})$ forward ☹

$\epsilon_b\ (1, -\vec{u})$ backward





choose a $\vec{k}$ construct the Hilbert space

$i=1,2$ $\quad a^-_{\perp i} \quad a^+_{\perp i}$

$\quad\downarrow \qquad \downarrow$

$\quad a \qquad a^+$

annih. creation

$[a^-_{\perp i}, a^+_{\perp j}] = \delta_{ij}$ physical

$[a^-_f, a^+_f] = 1$ Forward

$[a^-_b, a^+_b] = 1$ backward

Bose Einstein Stat.

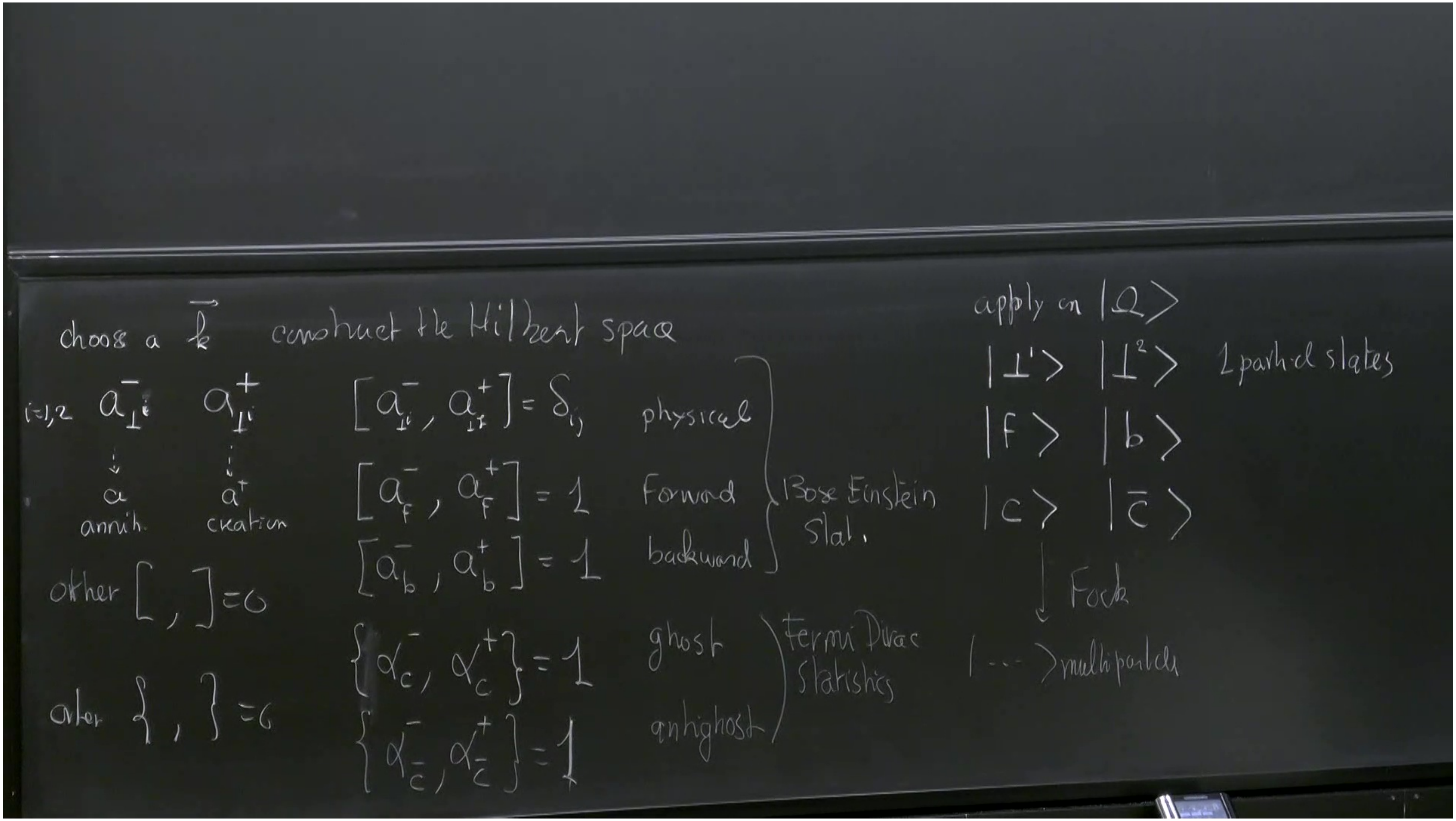
other $[\ ,\ ] = 0$

$\{\alpha^-_c, \alpha^+_c\} = 1$ ghost

$\{\alpha^-_{\bar{c}}, \alpha^+_{\bar{c}}\} = 1$ antighost

Fermi Dirac Statistics

other $\{\ ,\ \} = 0$



choose a $\vec{k}$ construct the Hilbert space

$i=1,2$ $a^-_{\perp i}$ $a^+_{\perp i}$

$\downarrow$ a annih. $\downarrow$ a^+ creation

$[a^-_{\perp i}, a^+_{\perp j}] = \delta_{ij}$ physical

$[a^-_f, a^+_f] = 1$ Forward

$[a^-_b, a^+_b] = 1$ backward

$\Big\}$ Bose Einstein Stat.

other $[\;,\;]=0$

$\{\alpha^-_c, \alpha^+_c\} = 1$ ghost

$\{\alpha^-_{\bar{c}}, \alpha^+_{\bar{c}}\} = 1$ antighost

$\Big\}$ Fermi Dirac Statistics

other $\{\;,\;\}=0$

apply on $|\Omega\rangle$

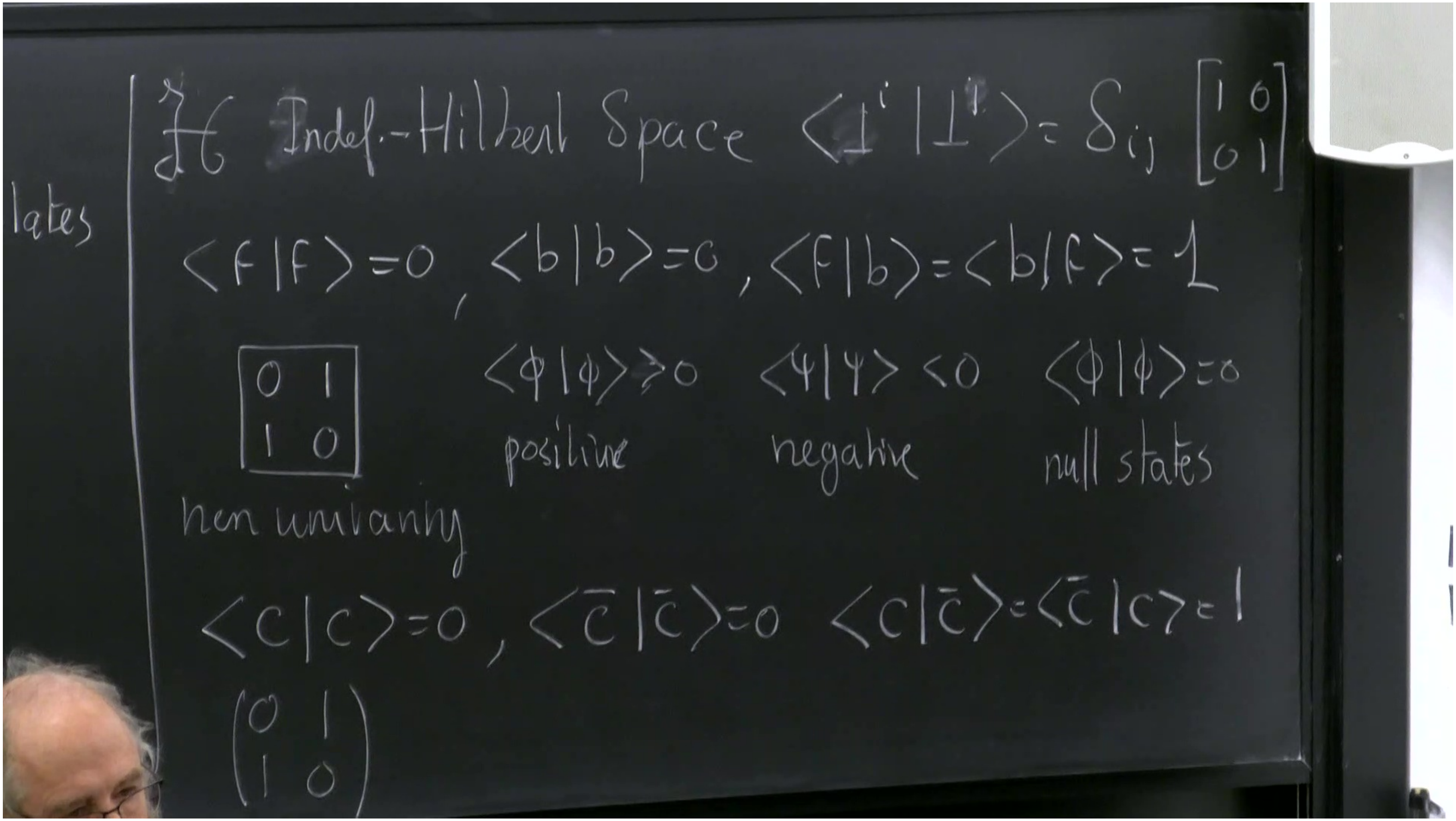
$|\perp^1\rangle$ $|\perp^2\rangle$ 1 particle states

$|f\rangle$ $|b\rangle$

$|c\rangle$ $|\bar{c}\rangle$

$\downarrow$ Fock

$|\ldots\rangle$ multiparticle



lates

$\mathcal{H}$ Indef.-Hilbert Space $\langle 1^i | 1^j \rangle = \delta_{ij}$ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\langle f|f\rangle = 0$, $\langle b|b\rangle = 0$, $\langle f|b\rangle = \langle b|f\rangle = 1$

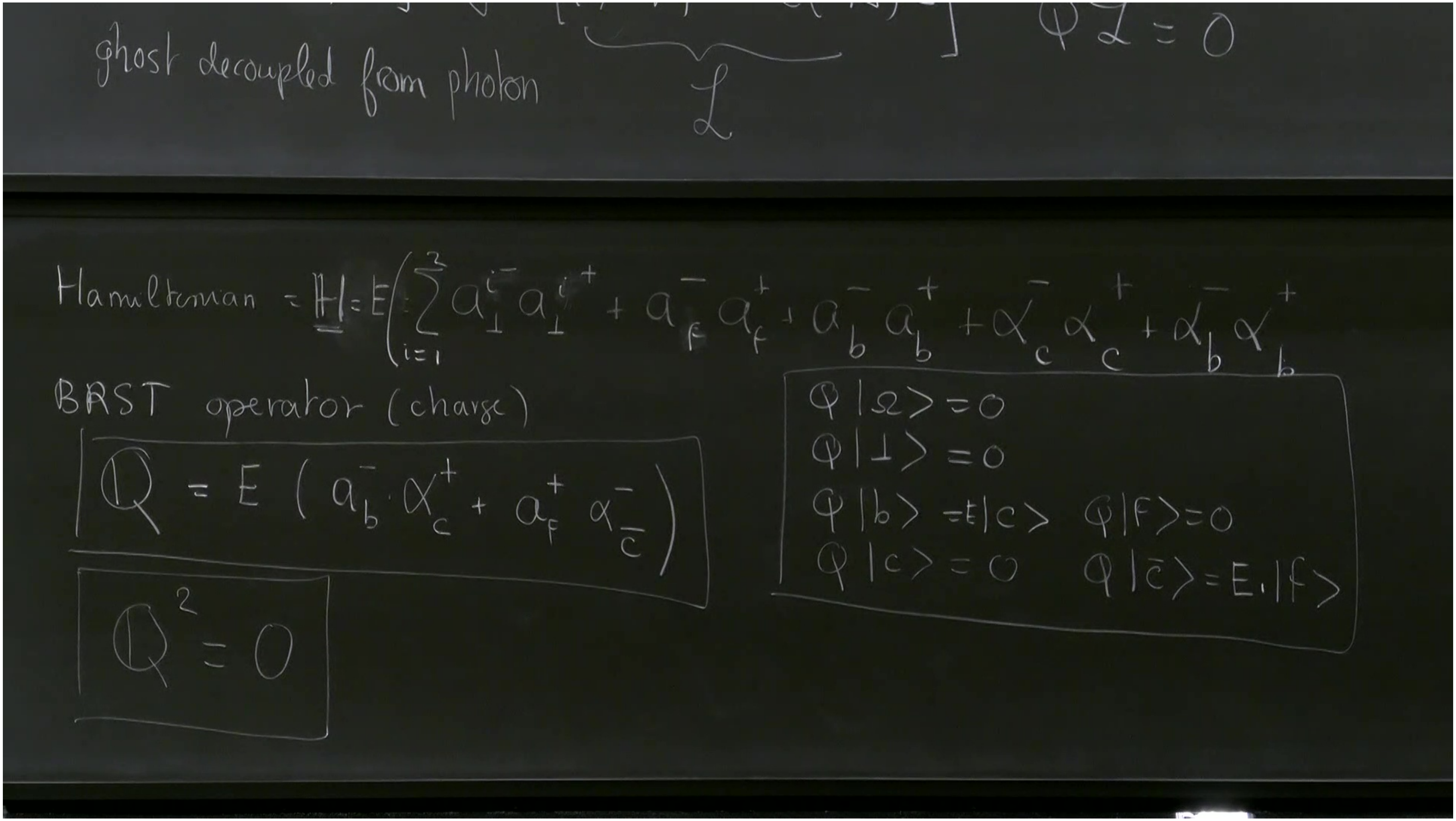
$\boxed{\begin{matrix}0 & 1\\ 1 & 0\end{matrix}}$

$\langle\phi|\phi\rangle > 0$ positive $\quad\langle\psi|\psi\rangle < 0$ negative $\quad\langle\phi|\phi\rangle = 0$ null states

non unitarity

$\langle c|c\rangle = 0$, $\langle\bar{c}|\bar{c}\rangle = 0$ $\quad\langle c|\bar{c}\rangle = \langle\bar{c}|c\rangle = 1$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$



ghost decoupled from photon $\underbrace{\qquad}_{\mathcal{L}}$ $\mathbb{Q}\mathcal{L} = 0$

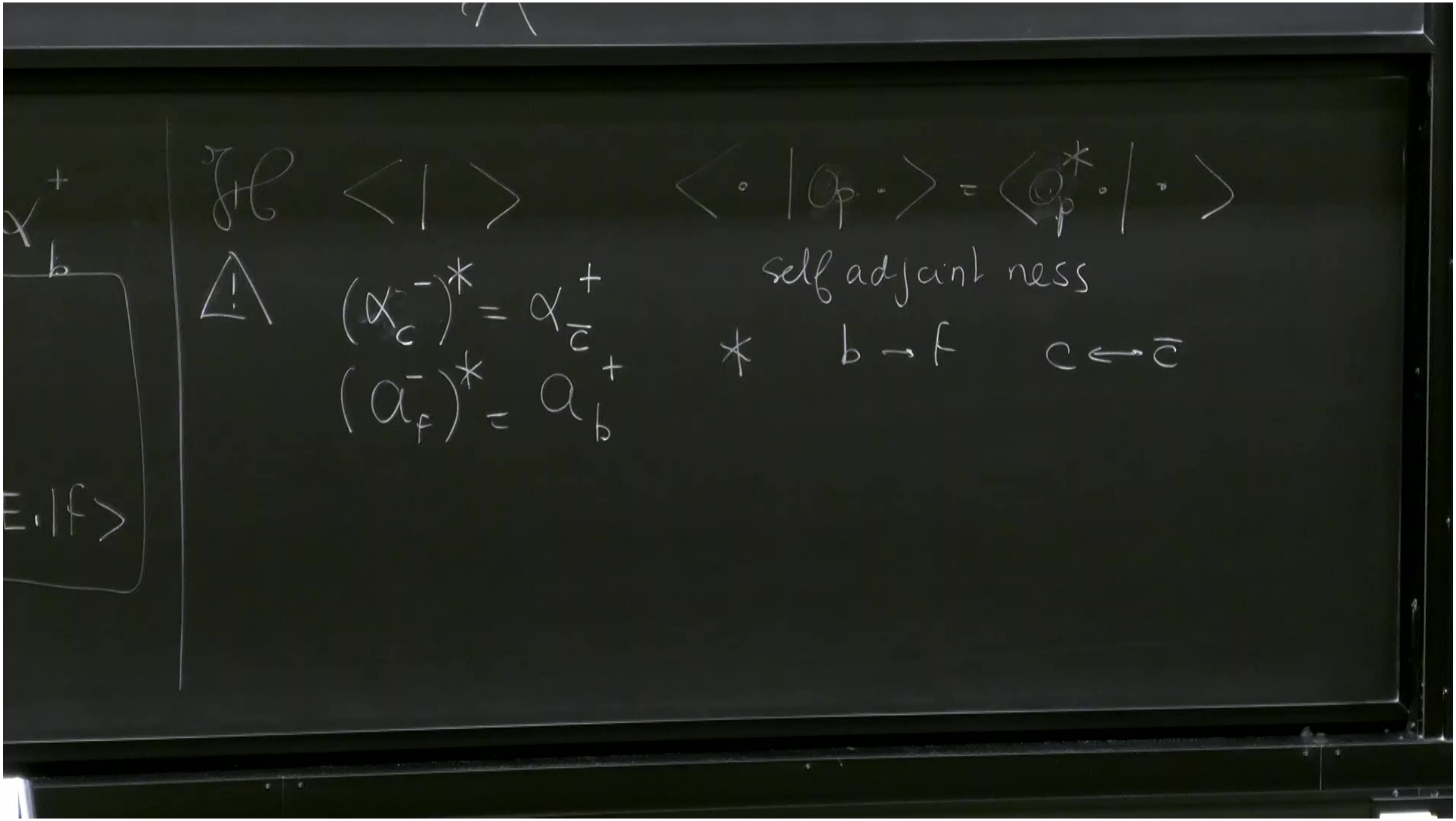
Hamiltonian $= \underline{\underline{\mathbb{H}}} = E\left(\sum_{i=1}^{2} a_\perp^{i-} a_\perp^{i+} + a_f^- a_f^+ + a_b^- a_b^+ + \alpha_c^- \alpha_c^+ + \alpha_b^- \alpha_b^+\right.$

BRST operator (charge)

$$\boxed{\mathbb{Q} = E\left(a_b^- \cdot \alpha_c^+ + a_f^+ \alpha_{\bar{c}}^-\right)}$$

$$\boxed{\mathbb{Q}^2 = 0}$$

$$\boxed{\begin{array}{ll} \mathbb{Q}|\Omega\rangle = 0 & \\ \mathbb{Q}|\perp\rangle = 0 & \\ \mathbb{Q}|b\rangle = E|c\rangle & \mathbb{Q}|f\rangle = 0 \\ \mathbb{Q}|c\rangle = 0 & \mathbb{Q}|\bar{c}\rangle = E\cdot|f\rangle \end{array}}$$



α_b^+

$E, |f\rangle$

$\mathcal{H}$ $\langle \ | \ \rangle$

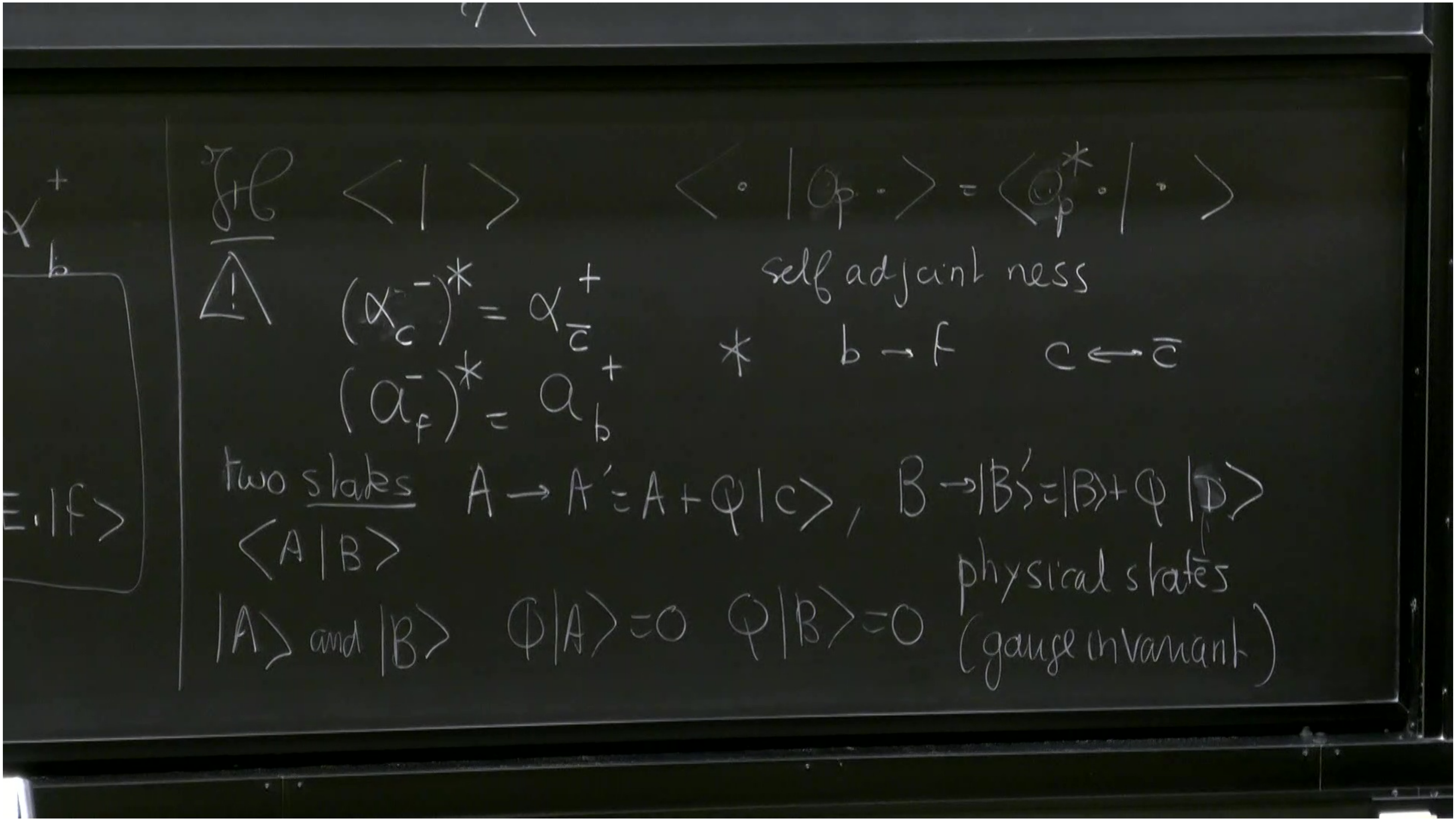
$\langle \cdot \,|\, \mathcal{O}_p \cdot \rangle = \langle \mathcal{O}_p^* \cdot \,|\, \cdot \rangle$

self adjointness

⚠ $(\alpha_c^-)^* = \alpha_{\bar{c}}^+$

$(a_f^-)^* = a_b^+$

$*$ $b \to f$ $c \leftrightarrow \bar{c}$



α_b^+

$E.|f\rangle$

$\underline{\mathcal{H}}$ $\langle\,|\,\rangle$ $\qquad \langle\cdot|\mathcal{O}_p\cdot\rangle = \langle\mathcal{O}_p^*\cdot|\cdot\rangle$

self adjointness

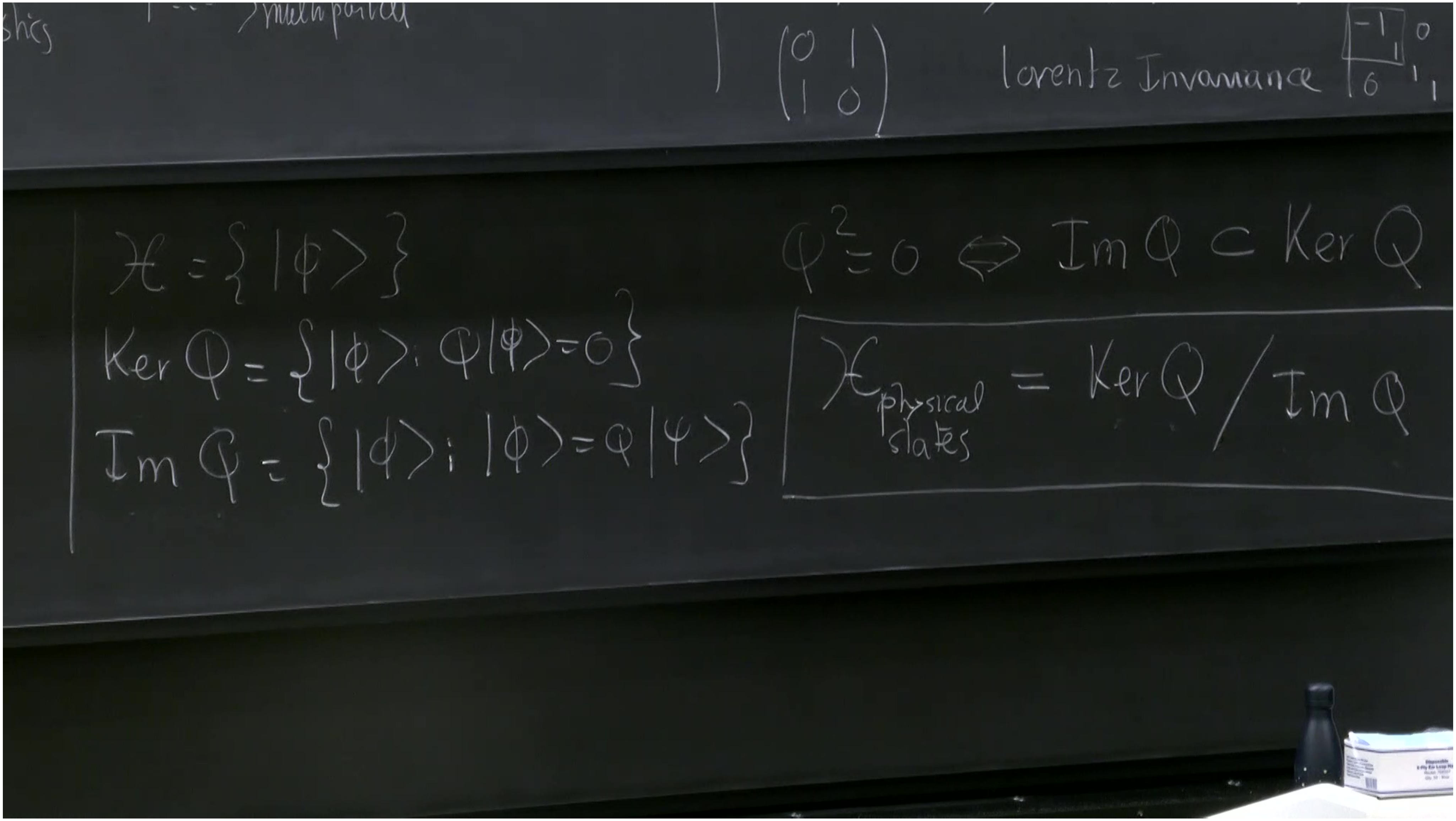
⚠ $(\alpha_c^-)^* = \alpha_{\bar{c}}^+ \qquad * \quad b\to f \quad c\leftrightarrow\bar{c}$

$(a_f^-)^* = a_b^+$

two $\underline{\text{states}}$ $A\to A' = A + Q|c\rangle,\ B\to|B'\rangle = |B\rangle + Q|D\rangle$

$\langle A|B\rangle$

$|A\rangle$ and $|B\rangle$ $\quad Q|A\rangle=0 \quad Q|B\rangle=0$ physical states (gauge invariant)



shics

> multiparticle

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

lorentz Invariance

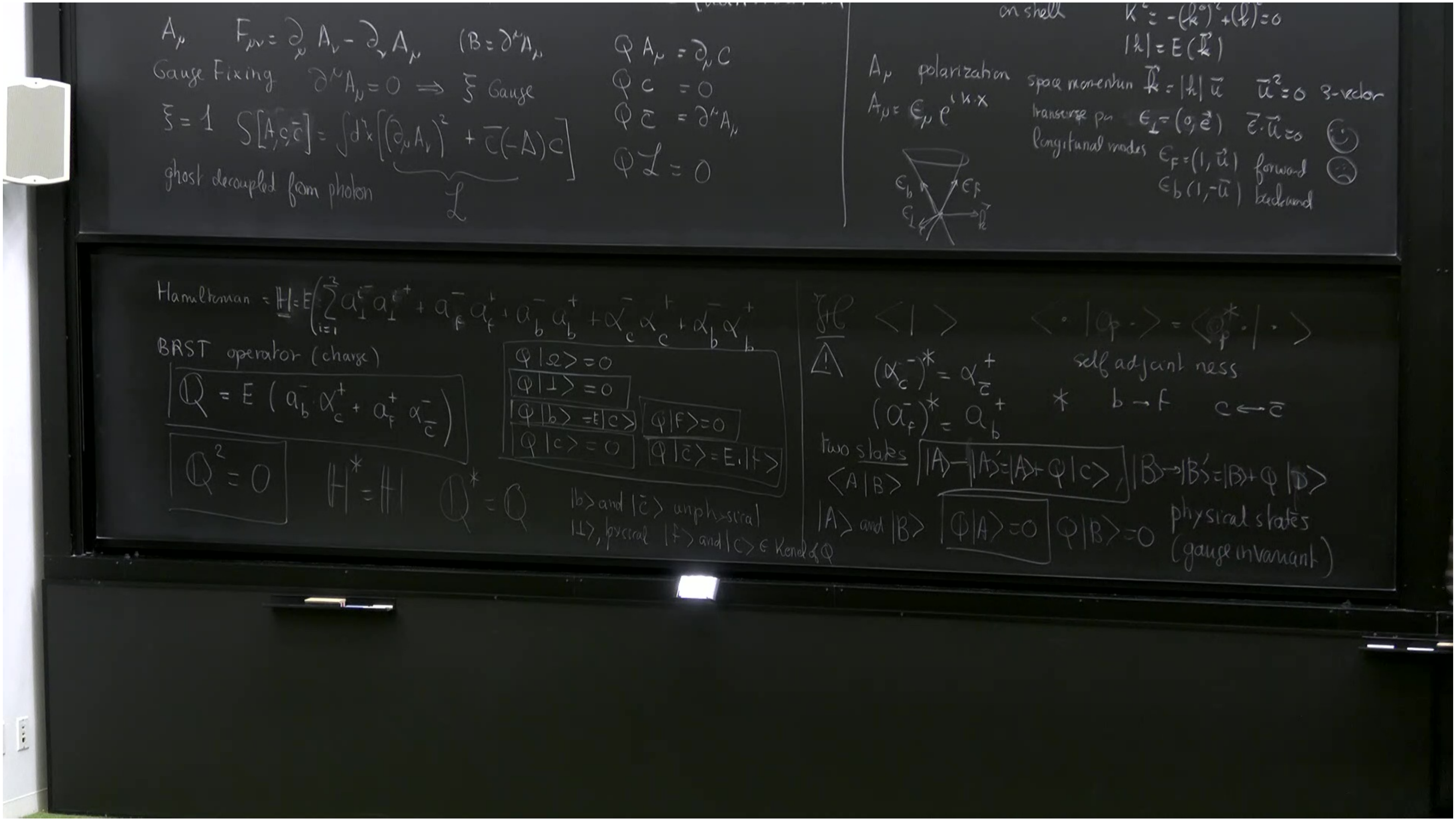
$$\mathcal{H} = \{ |\phi\rangle \}$$

$$\operatorname{Ker} Q = \{ |\phi\rangle : Q|\phi\rangle = 0 \}$$

$$\operatorname{Im} Q = \{ |\phi\rangle : |\phi\rangle = Q|\psi\rangle \}$$

$$Q^2 = 0 \Leftrightarrow \operatorname{Im} Q \subset \operatorname{Ker} Q$$

$$\boxed{\mathcal{H}_{\substack{\text{physical}\\ \text{states}}} = \operatorname{Ker} Q \,/\, \operatorname{Im} Q}$$



A_μ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $(B = \partial^\mu A_\mu$

Gauge Fixing $\partial^\mu A_\mu = 0 \Rightarrow \xi$ Gauge

$\xi = 1$ $S[A, c, \bar{c}] = \int d^4x \Big[\underbrace{(\partial_\mu A_\nu)^2 + \bar{c}(-\Delta) c}_{\mathcal{L}} \Big]$

ghost decoupled from photon

$Q A_\mu = \partial_\mu c$

$Q c = 0$

$Q \bar{c} = \partial^\mu A_\mu$

$Q \mathcal{L} = 0$

on shell $k^2 = -(k^0)^2 + (\vec{k})^2 = 0$

$|k| = E(\vec{k})$

A_μ polarization

$A_\mu = \epsilon_\mu e^{ik \cdot x}$

space momentum $\vec{k} = |k| \vec{u}$ $\vec{u}^2 = 0$ 3-vector

transverse pol $\epsilon_\perp = (0, \vec{e})$ $\vec{e} \cdot \vec{u} = 0$ ☺

longitudinal modes $\epsilon_F = (1, \vec{u})$ forward ☹

$\epsilon_b (1, -\vec{u})$ backward

Hamiltonian $= \mathbb{H} = E \left(\sum_{i=1}^{2} a_\perp^{i-} a_\perp^{i+} + a_F^- a_F^+ + a_b^- a_b^+ + \alpha_c^- \alpha_c^+ + \alpha_b^- \alpha_b^+ \right.$

BRST operator (charge)

$$\boxed{\mathbb{Q} = E \left(a_b^- \alpha_c^+ + a_F^+ \alpha_{\bar{c}}^- \right)}$$

$$\boxed{\mathbb{Q}^2 = 0} \qquad \mathbb{H}^* = \mathbb{H} \qquad \mathbb{Q}^* = \mathbb{Q}$$

$Q |\Omega\rangle = 0$

$\boxed{Q |\perp\rangle = 0}$

$\boxed{Q |b\rangle = E |c\rangle}$ $\boxed{Q |F\rangle = 0}$

$\boxed{Q |c\rangle = 0}$ $\boxed{Q |\bar{c}\rangle = E \cdot |F\rangle}$

$|b\rangle$ and $|\bar{c}\rangle$ unphysical

$|\perp\rangle$, physical $|F\rangle$ and $|c\rangle \in$ Kernel of Q

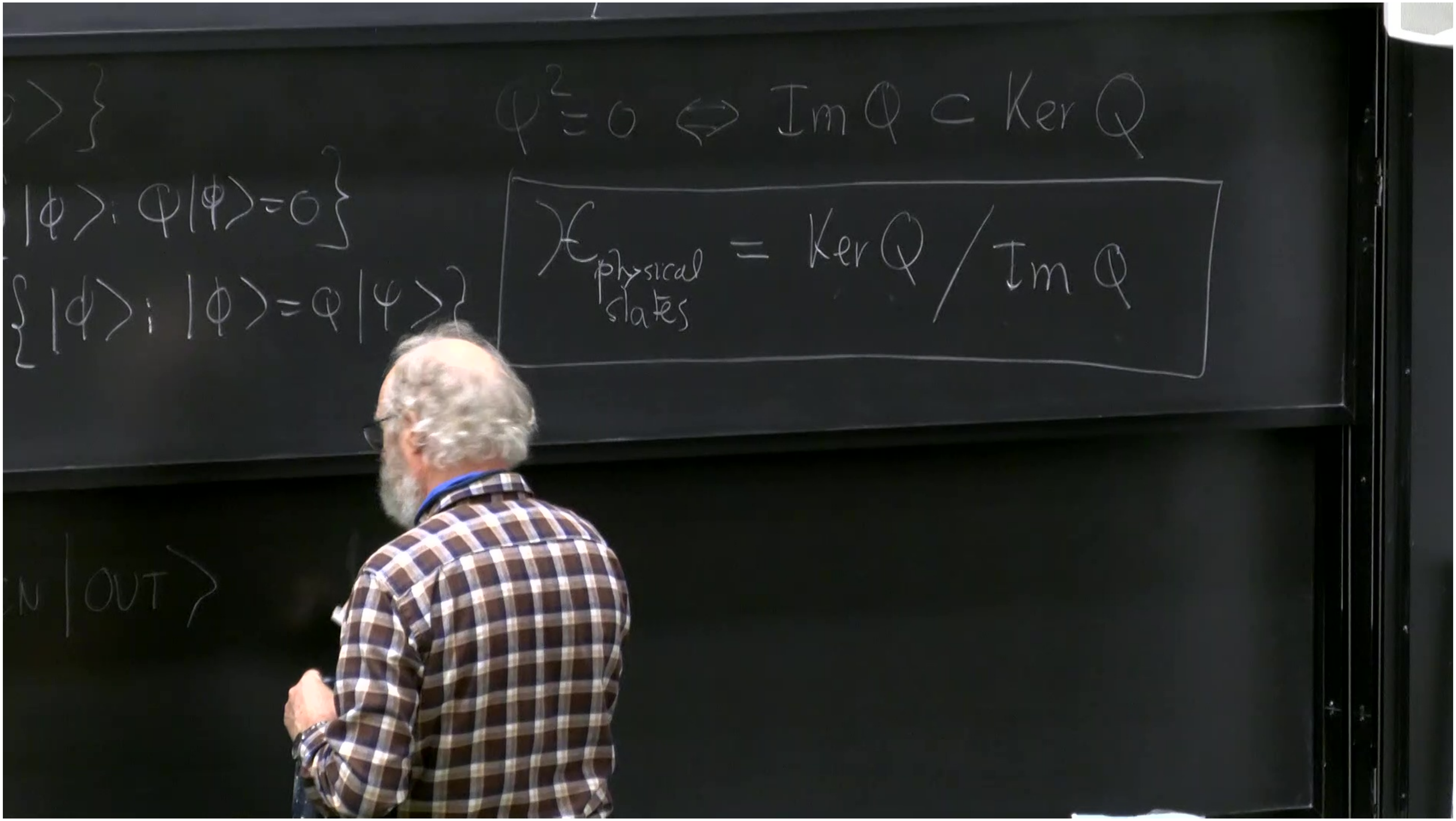
$\mathcal{H}$ $\langle \, | \, \rangle$ $\langle \cdot | Q \cdot \rangle = \langle Q^* \cdot | \cdot \rangle$

⚠ $(\alpha_c^-)^* = \alpha_{\bar{c}}^+$ self adjointness

$(a_F^-)^* = a_b^+$ $*$ $b \to F$ $c \leftrightarrow \bar{c}$

two states $\langle A | B \rangle$ $\boxed{|A\rangle \to |A'\rangle = |A\rangle + Q |c\rangle}$, $|B\rangle \to |B'\rangle = |B\rangle + Q |D\rangle$

$|A\rangle$ and $|B\rangle$ $\boxed{Q |A\rangle = 0}$ $Q |B\rangle = 0$ physical states (gauge invariant)



$\rangle \}$

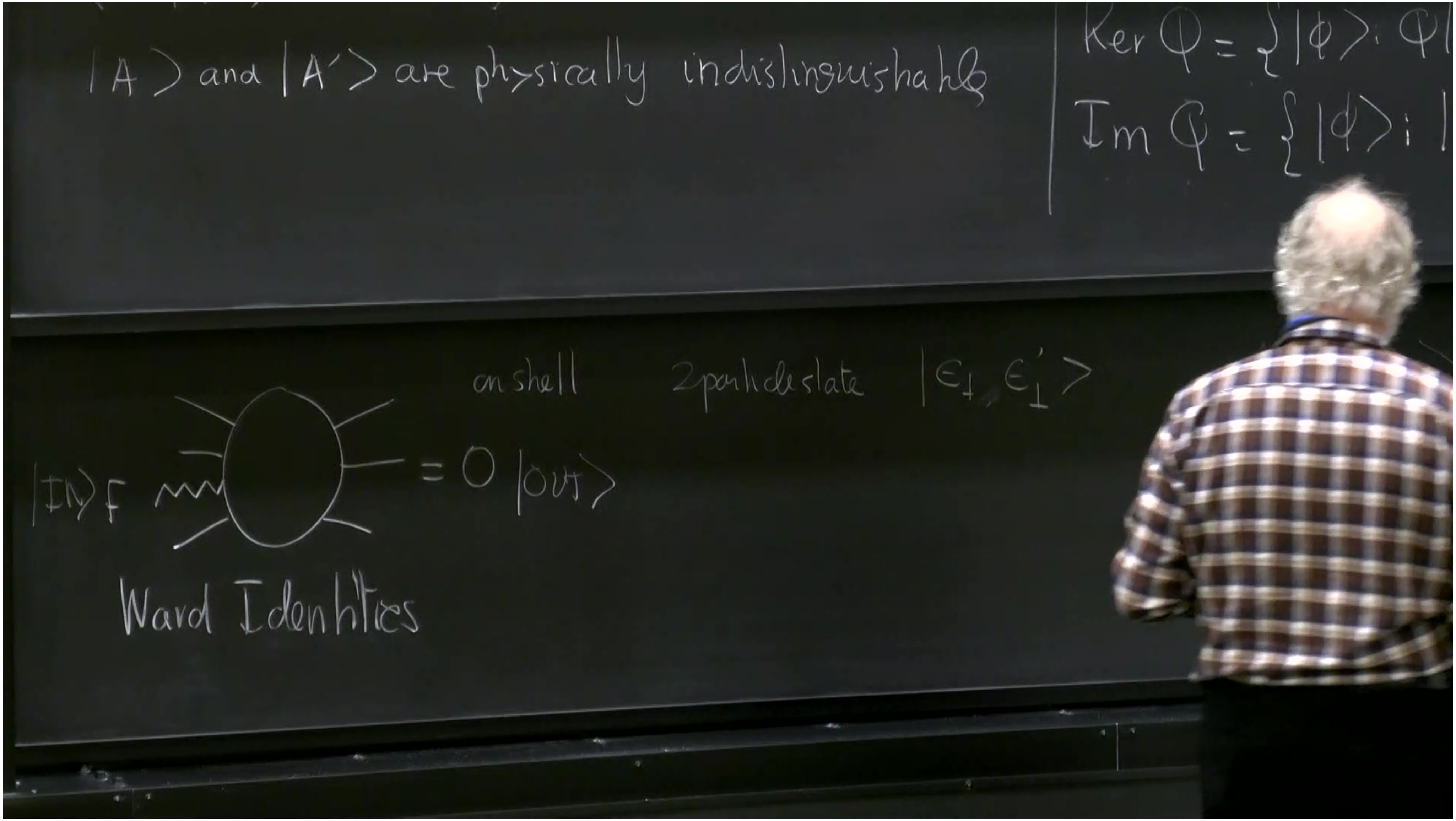
$|\varphi\rangle : Q|\varphi\rangle = 0 \}$

$\{ |\phi\rangle : |\phi\rangle = Q|\psi\rangle \}$

$$Q^2 = 0 \Leftrightarrow \mathrm{Im}\,Q \subset \mathrm{Ker}\,Q$$

$$\boxed{\mathcal{H}_{\substack{\text{physical}\\ \text{states}}} = \mathrm{Ker}\,Q / \mathrm{Im}\,Q}$$

$N|\mathrm{OUT}\rangle$



$|A\rangle$ and $|A'\rangle$ are physically indistinguishable

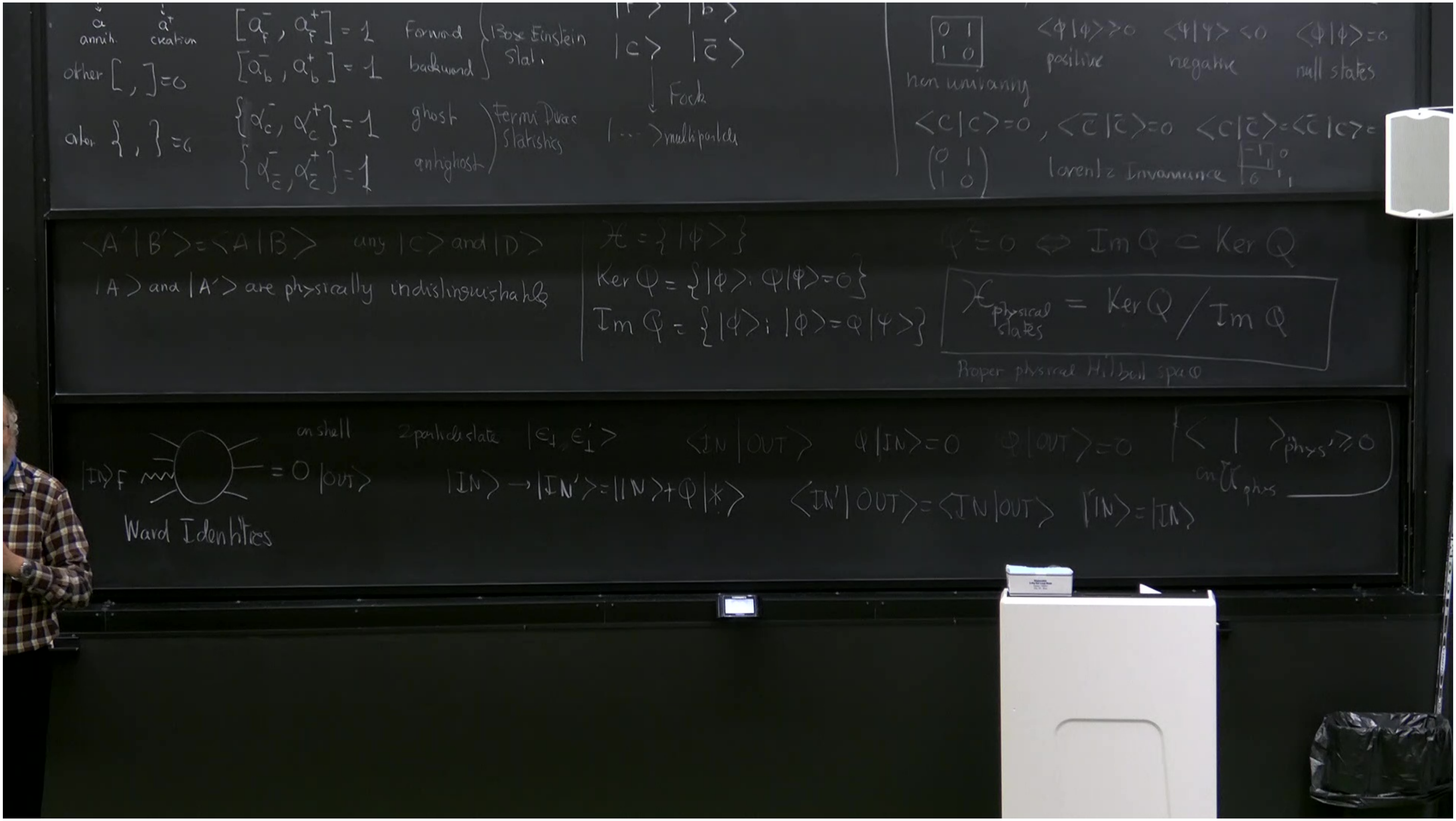
$\mathrm{Ker}\,\Phi = \{|\phi\rangle : \Phi|$

$\mathrm{Im}\,\Phi = \{|\phi\rangle : |$

on shell   2 particle state $|\epsilon_\perp, \epsilon'_\perp\rangle$

$|IN\rangle_F$ $= 0$ $|OUT\rangle$

Ward Identities



a annih. a^+ creation

other $[\ ,\ ]=0$

anti $\{\ ,\ \}=0$

$[a_f^-, a_f^+] = 1$ forward

$[a_b^-, a_b^+] = 1$ backward

Bose Einstein Stat.

$\{\alpha_c^-, \alpha_c^+\} = 1$ ghost

$\{\alpha_{\bar c}^-, \alpha_{\bar c}^+\} = 1$ antighost

Fermi Dirac Statistics

$|c\rangle \quad |\bar c\rangle$

$\downarrow$ Fock

$|\ldots\rangle$ multiparticle

$\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}$ non unitarity

$\langle\phi|\phi\rangle > 0$ positive

$\langle\psi|\psi\rangle < 0$ negative

$\langle\phi|\phi\rangle = 0$ null states

$\langle c|c\rangle = 0, \ \langle\bar c|\bar c\rangle = 0 \quad \langle c|\bar c\rangle = \langle\bar c|c\rangle =$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Lorentz Invariance

$\langle A'|B'\rangle = \langle A|B\rangle$ any $|C\rangle$ and $|D\rangle$

$|A\rangle$ and $|A'\rangle$ are physically indistinguishable

$\mathcal{H} = \{|\phi\rangle\}$

$\mathrm{Ker}\, Q = \{|\phi\rangle : Q|\phi\rangle = 0\}$

$\mathrm{Im}\, Q = \{|\phi\rangle : |\phi\rangle = Q|\psi\rangle\}$

$Q^2 = 0 \Leftrightarrow \mathrm{Im}\, Q \subset \mathrm{Ker}\, Q$

$$\boxed{\mathcal{H}_{\substack{\text{physical}\\ \text{states}}} = \mathrm{Ker}\, Q / \mathrm{Im}\, Q}$$

Proper physical Hilbert space

on shell

$|IN\rangle_F$ $= 0\ |OUT\rangle$

Ward Identities

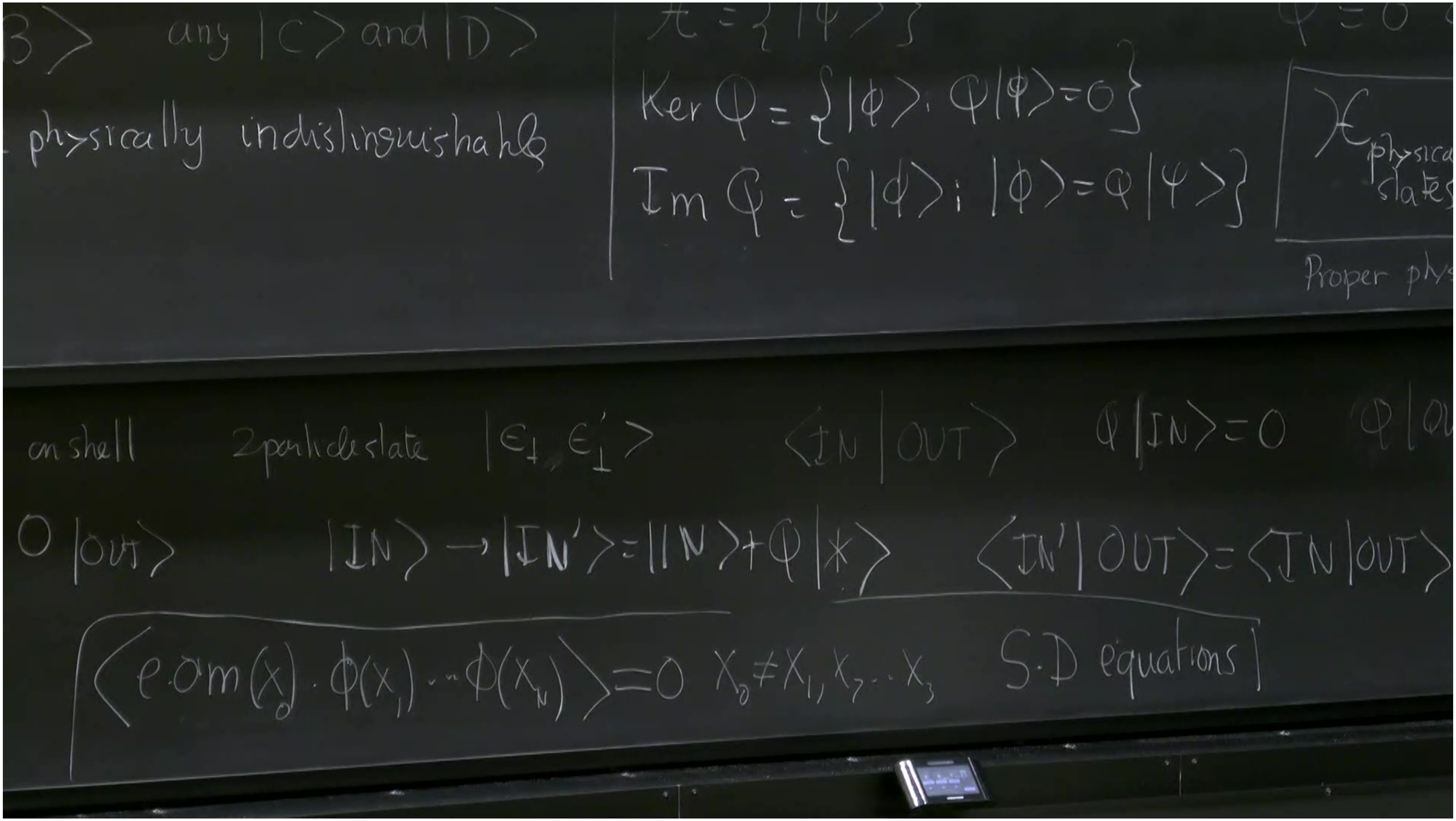
2 particle state $|\epsilon_1, \epsilon_1'\rangle$

$|IN\rangle \to |IN'\rangle = |IN\rangle + Q|*\rangle$

$\langle IN|OUT\rangle \quad Q|IN\rangle = 0 \quad Q|OUT\rangle = 0$

$\langle IN'|OUT\rangle = \langle IN|OUT\rangle \quad |\widetilde{IN}\rangle = |IN\rangle$

$\langle\ |\ \rangle_{\text{phys}'} \geq 0$ on $\mathcal{H}_{\text{phys}}$



$B\rangle$ any $|C\rangle$ and $|D\rangle$

physically indistinguishable

$\mathcal{H} = \{|\psi\rangle\}$

$\mathrm{Ker}\, Q = \{|\psi\rangle : Q|\psi\rangle = 0\}$

$\mathrm{Im}\, Q = \{|\phi\rangle : |\phi\rangle = Q|\psi\rangle\}$

$Q = 0$

$\boxed{\mathcal{H}_{\text{physical states}}}$

Proper phys

on shell 2 particle state $|\epsilon_+, \epsilon'_\perp\rangle$

$\langle IN | OUT \rangle$

$Q|IN\rangle = 0$

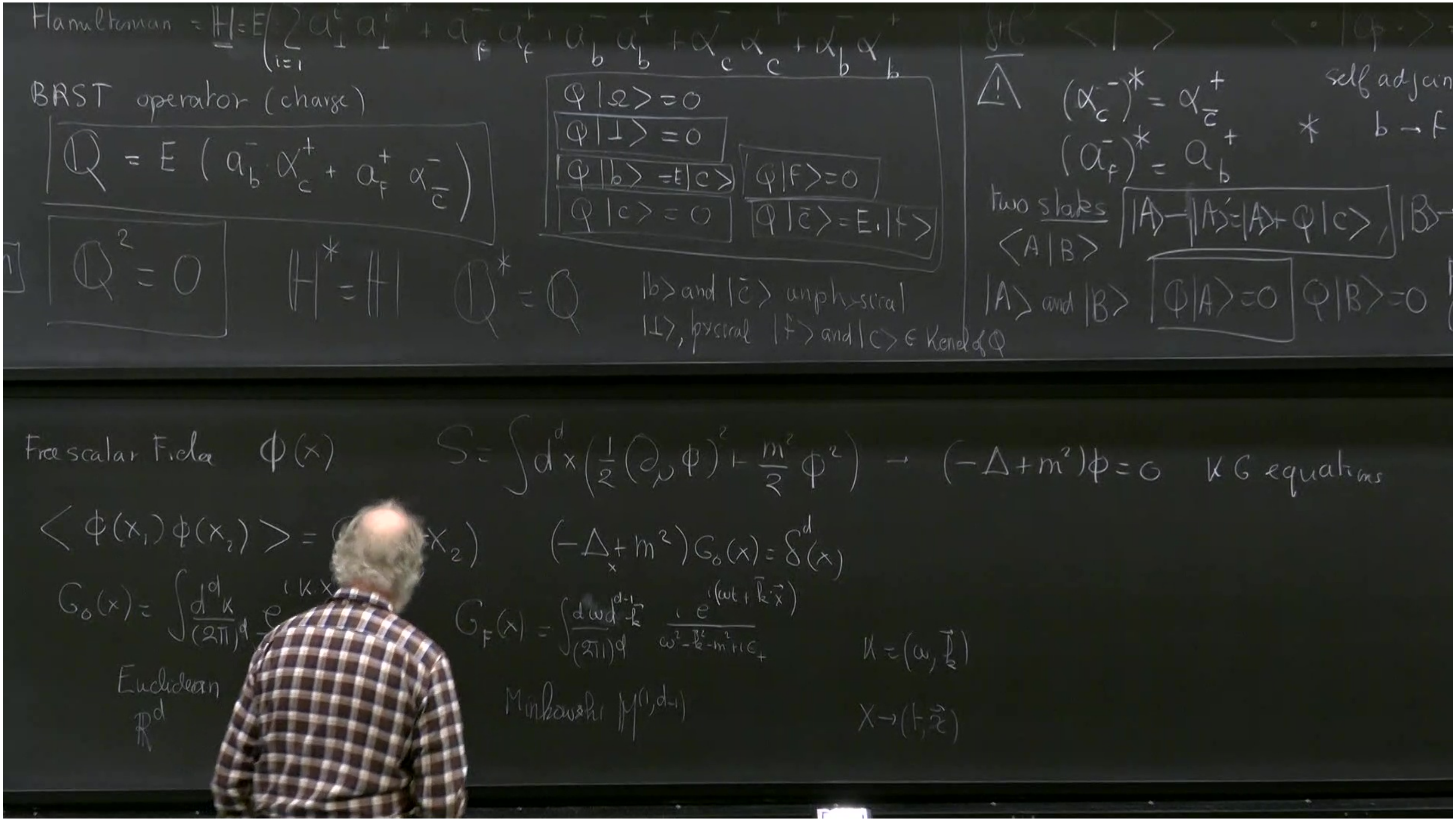
$Q|OU$

$0\ |OUT\rangle$

$|IN\rangle \to |IN'\rangle = |IN\rangle + Q|*\rangle$

$\langle IN' | OUT \rangle = \langle IN | OUT \rangle$

$\langle e.o.m(x_0) \cdot \phi(x_1) \cdots \phi(x_N) \rangle = 0$ $x_0 \neq x_1, x_2 \dots x_3$ S.D equations



Hamiltonian = $\mathbb{H} = E\left(\sum_{i=1} a_\perp^{i-} a_\perp^{i+} + a_f^- a_f^+ + a_b^- a_b^+ + \alpha_c^- \alpha_c^+ + \alpha_b^- \alpha_b^+\right)$

BRST operator (charge)

$$\boxed{\mathbb{Q} = E\left(a_b^- \alpha_c^+ + a_f^+ \alpha_{\bar c}^-\right)}$$

$$\boxed{\mathbb{Q}^2 = 0} \qquad \mathbb{H}^* = \mathbb{H} \qquad \mathbb{Q}^* = \mathbb{Q}$$

$\mathbb{Q}|\Omega\rangle = 0$

$\boxed{\mathbb{Q}|\perp\rangle = 0}$

$\boxed{\mathbb{Q}|b\rangle = E|c\rangle}$ $\boxed{\mathbb{Q}|f\rangle = 0}$

$\boxed{\mathbb{Q}|c\rangle = 0}$ $\boxed{\mathbb{Q}|\bar c\rangle = E\cdot|f\rangle}$

$|b\rangle$ and $|\bar c\rangle$ unphysical

$|\perp\rangle$, pysical $|f\rangle$ and $|c\rangle \in$ Kenel of $\mathbb{Q}$

$\mathcal{H}$ $\langle \quad | \quad \rangle$ $\langle \cdot | \mathbb{Q} \cdot \rangle$

⚠ $(\alpha_c^-)^* = \alpha_{\bar c}^+$ self adjoin

$(a_f^-)^* = a_b^+$ $*$ $b \to f$

two states $\langle A|B\rangle$ $\boxed{|A\rangle \to |A'\rangle = |A\rangle + \mathbb{Q}|c\rangle,}$ $|B\rangle \to$

$|A\rangle$ and $|B\rangle$ $\boxed{\mathbb{Q}|A\rangle = 0}$ $\mathbb{Q}|B\rangle = 0$

Free scalar Field $\phi(x)$ $S = \int d^dx\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2\right) \to (-\Delta + m^2)\phi = 0$ KG equations

$\langle \phi(x_1)\phi(x_2)\rangle =$ $x_2)$ $(-\Delta_x + m^2)G_0(x) = \delta^d(x)$

$G_0(x) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x}$

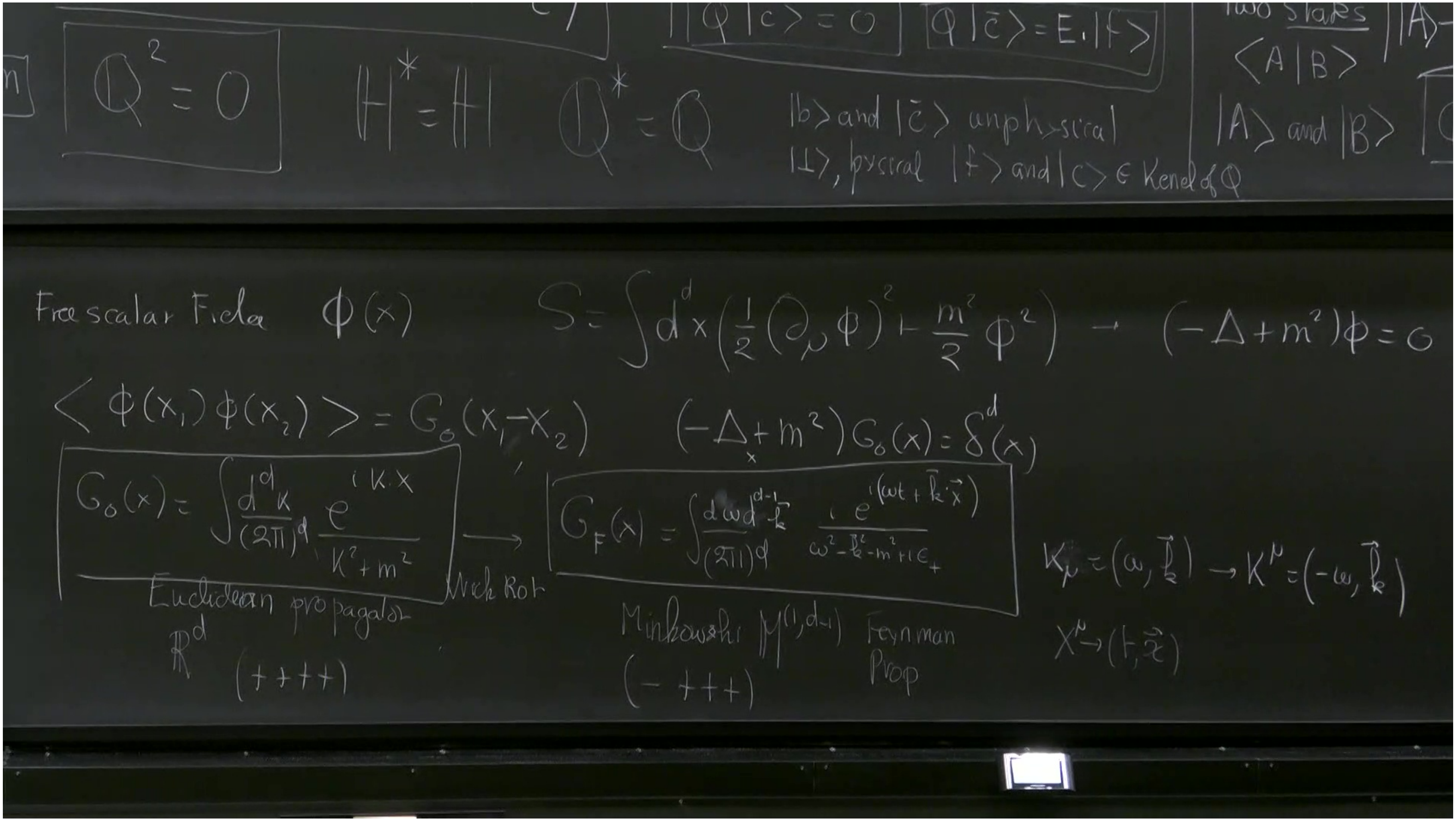
$G_F(x) = \int \frac{d\omega\, d^{d-1}\vec k}{(2\pi)^d} \frac{i\, e^{i(\omega t + \vec k\cdot\vec x)}}{\omega^2 - \vec k^2 - m^2 + i\epsilon_+}$

$k = (\omega, \vec k)$

Euclidean $\mathbb{R}^d$

Minkowski $\mathbb{M}^{(1,d-1)}$

$x \to (t, \vec x)$



$$\boxed{Q^2 = 0} \qquad \mathbb{H}^* = \mathbb{H} \qquad \mathbb{Q}^* = \mathbb{Q}$$

$$\boxed{Q|c\rangle = 0} \qquad \boxed{Q|\bar{c}\rangle = E.|f\rangle}$$

$|b\rangle$ and $|\bar{c}\rangle$ unphysical

$|\perp\rangle$, pysical $|f\rangle$ and $|c\rangle \in$ Kenel of Q

Two states

$\langle A|B\rangle$

$|A\rangle$ and $|B\rangle$

Free scalar Field $\phi(x)$

$$S = \int d^dx \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2\right) \to (-\Delta + m^2)\phi = 0$$

$$\langle \phi(x_1)\phi(x_2)\rangle = G_0(x_1 - x_2) \qquad (-\Delta_x + m^2)G_0(x) = \delta^d(x)$$

$$\boxed{G_0(x) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{k^2 + m^2}} \xrightarrow{} \boxed{G_F(x) = \int \frac{d\omega\, d^{d-1}\vec{k}}{(2\pi)^d} \frac{i\, e^{i(\omega t + \vec{k}\cdot\vec{x})}}{\omega^2 - \vec{k}^2 - m^2 + i\epsilon_+}}$$

Wick Rot

Euclidean propagator

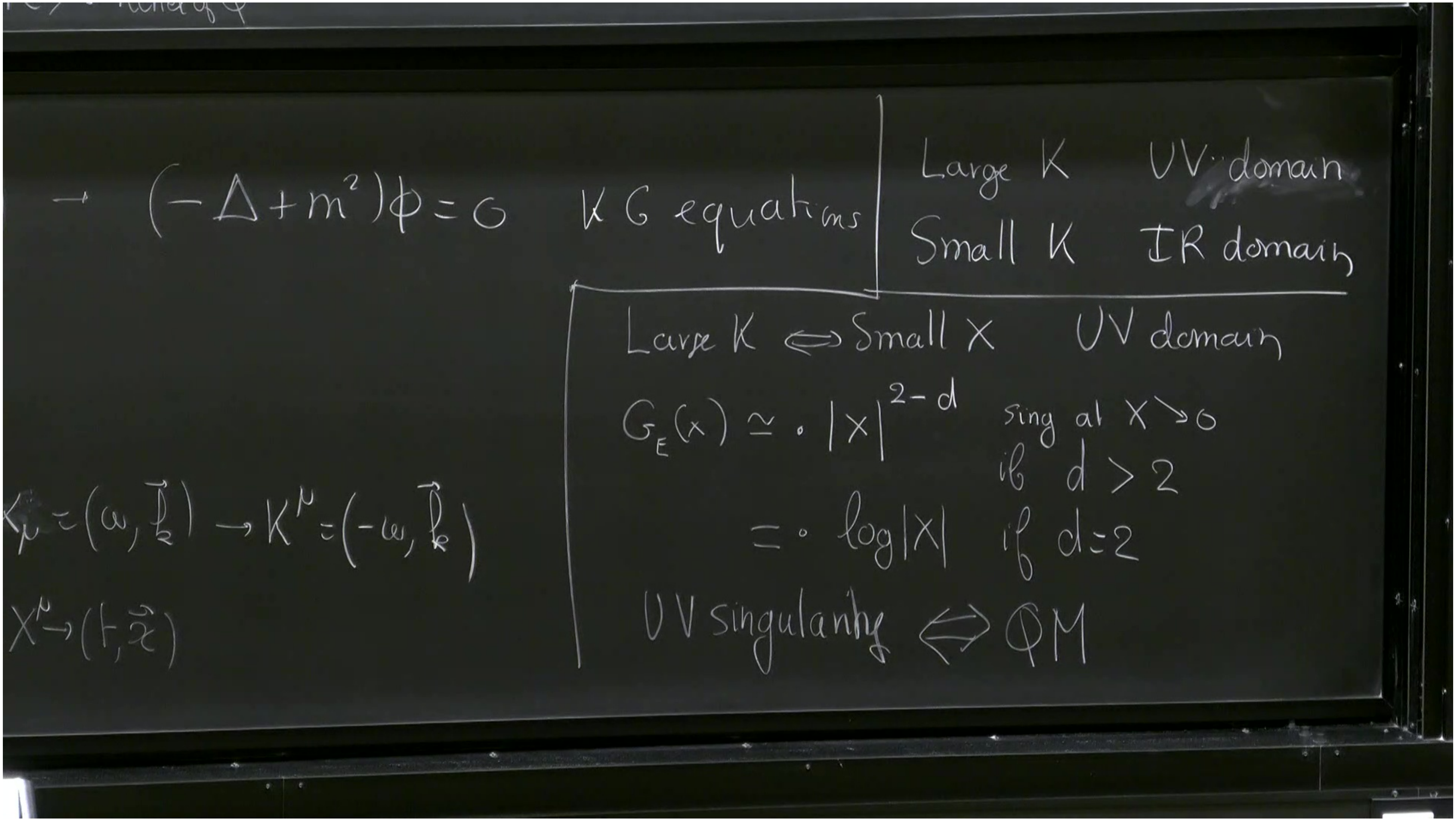
$\mathbb{R}^d$ $(++++)$

Minkowski $\mathbb{M}^{(1,d-1)}$ Feynman Prop

$(-+++)$

$$K_\mu = (\omega, \vec{k}) \to K^\mu = (-\omega, \vec{k})$$

$$X^\mu \to (t, \vec{x})$$



$\to \quad (-\Delta + m^2)\phi = 0$ KG equations

Large K UV domain

Small K IR domain

Large K $\Longleftrightarrow$ Small X UV domain

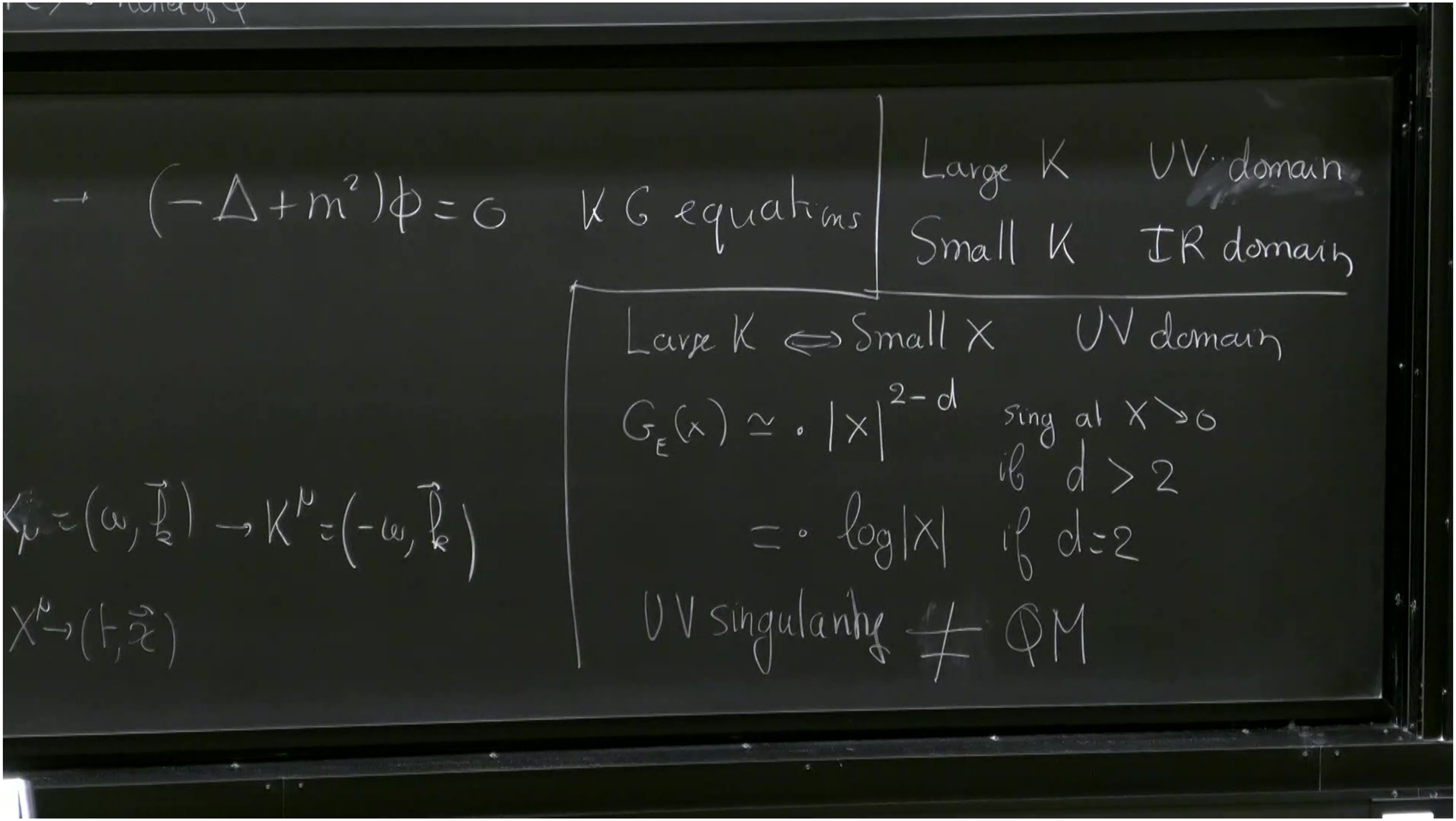
$G_E(x) \simeq \cdot\, |x|^{2-d}$ sing at $X \searrow 0$ if $d > 2$

$= \cdot\, \log|X|$ if $d = 2$

UV singularity $\Longleftrightarrow$ QM

$K_\mu = (\omega, \vec{k}) \to K^\mu = (-\omega, \vec{k})$

$X^\mu \to (t, \vec{x})$



$\rightarrow \quad (-\Delta + m^2)\phi = 0$ KG equations

Large K UV domain

Small K IR domain

Large K $\Longleftrightarrow$ Small X UV domain

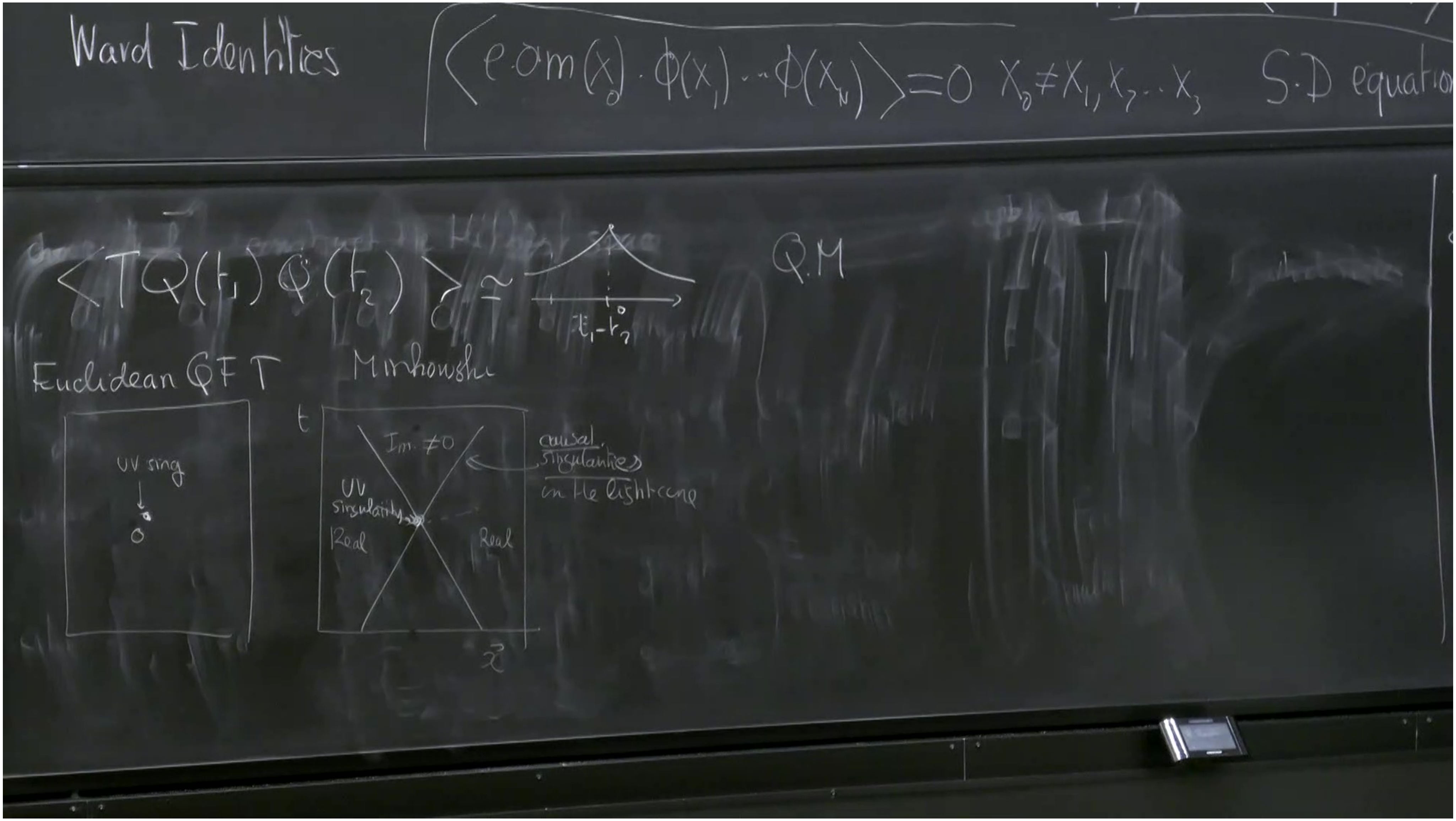
$G_E(x) \simeq \cdot |x|^{2-d}$ sing at $X \searrow 0$ if $d > 2$

$= \cdot \log|X|$ if $d = 2$

UV singularity $\neq$ QM

$K_\mu = (\omega, \vec{k}) \rightarrow K^\mu = (-\omega, \vec{k})$

$X^\mu \rightarrow (t, \vec{x})$



Ward Identities

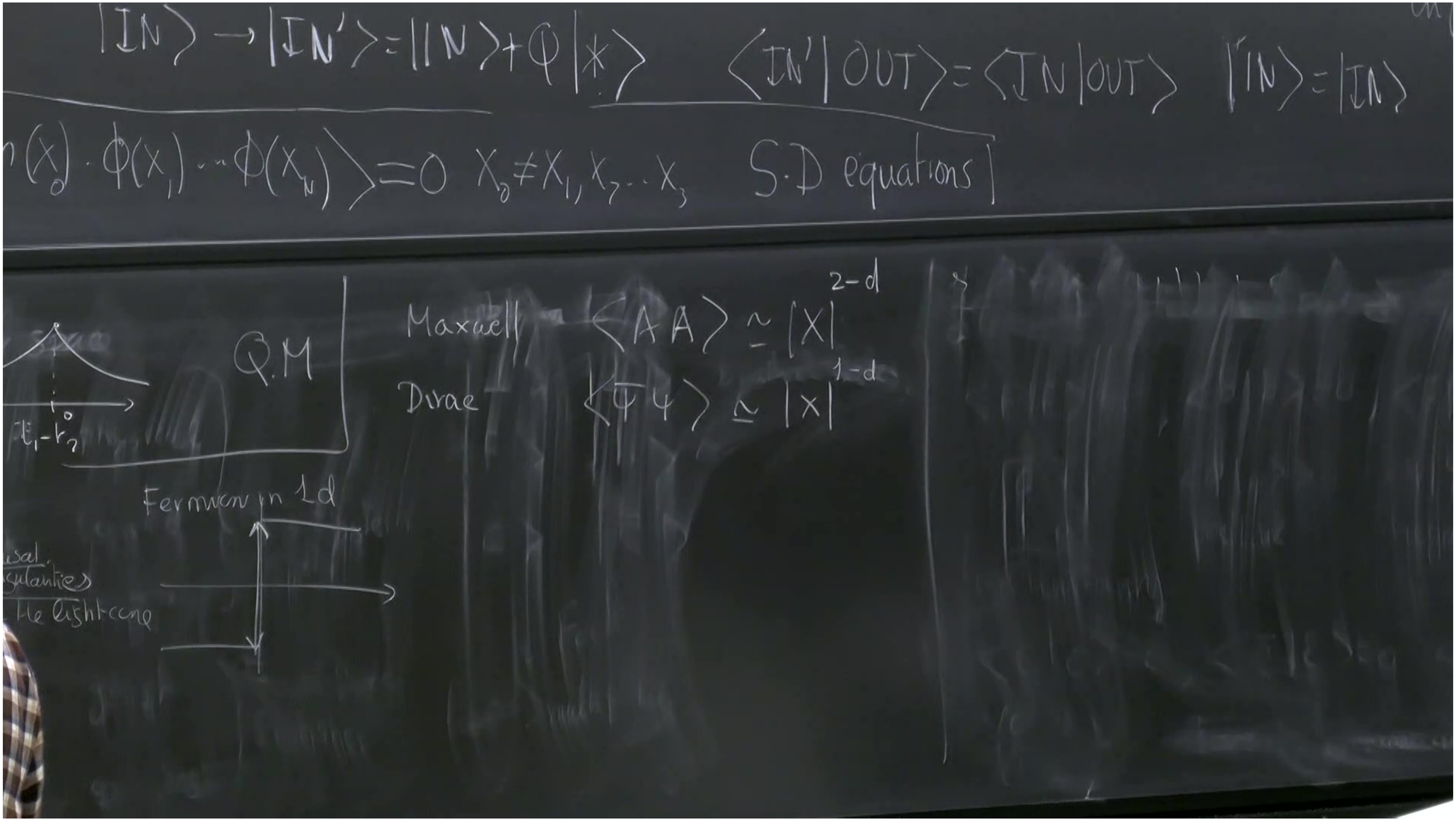
$\langle e \cdot \partial m(x_0) \cdot \phi(x_1) \cdots \phi(x_N) \rangle = 0 \quad x_0 \neq x_1, x_2 \ldots x_3$ S.D equatio

$\langle T Q(t_1) \dot{Q}(t_2) \rangle \simeq$

Q.M

Euclidean QFT

Minkowski



$|IN\rangle \to |IN'\rangle = |IN\rangle + \phi|*\rangle$ $\langle IN'|OUT\rangle = \langle IN|OUT\rangle$ $|IN\rangle = |IN\rangle$

$(X_0)\cdot\phi(X_1)\cdots\phi(X_N)\rangle = 0$ $X_0 \neq X_1, X_2 \dots X_3$ S.D equations

Q.M

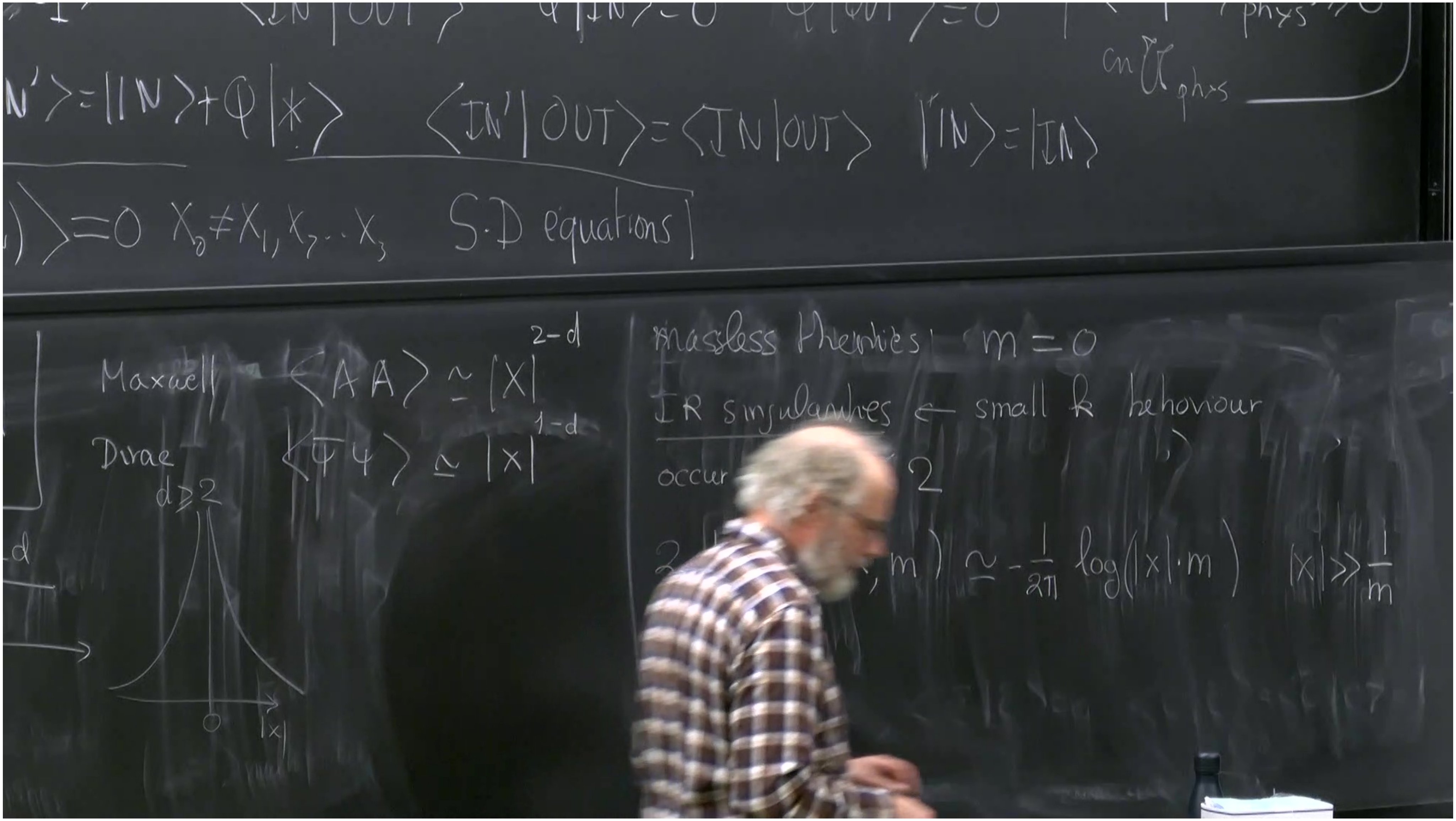
$t_1 - t_2$

Maxwell $\langle A A\rangle \simeq |X|^{2-d}$

Dirac $\langle \bar\Psi \Psi\rangle \simeq |X|^{1-d}$

Fermion in 1d

sal,
gularities
the lightcone



$$N'\rangle = |IN\rangle + Q|*\rangle \qquad \langle IN'|OUT\rangle = \langle IN|OUT\rangle \qquad |'IN\rangle = |IN\rangle$$

$$)\rangle = 0 \quad X_0 \neq X_1, X_2 \ldots X_3$$ S.D equations

on $\mathcal{H}_{phys}$

Maxwell $\langle A A\rangle \simeq |X|^{2-d}$

Dirac $\langle \bar\Psi \Psi\rangle \simeq |X|^{1-d}$

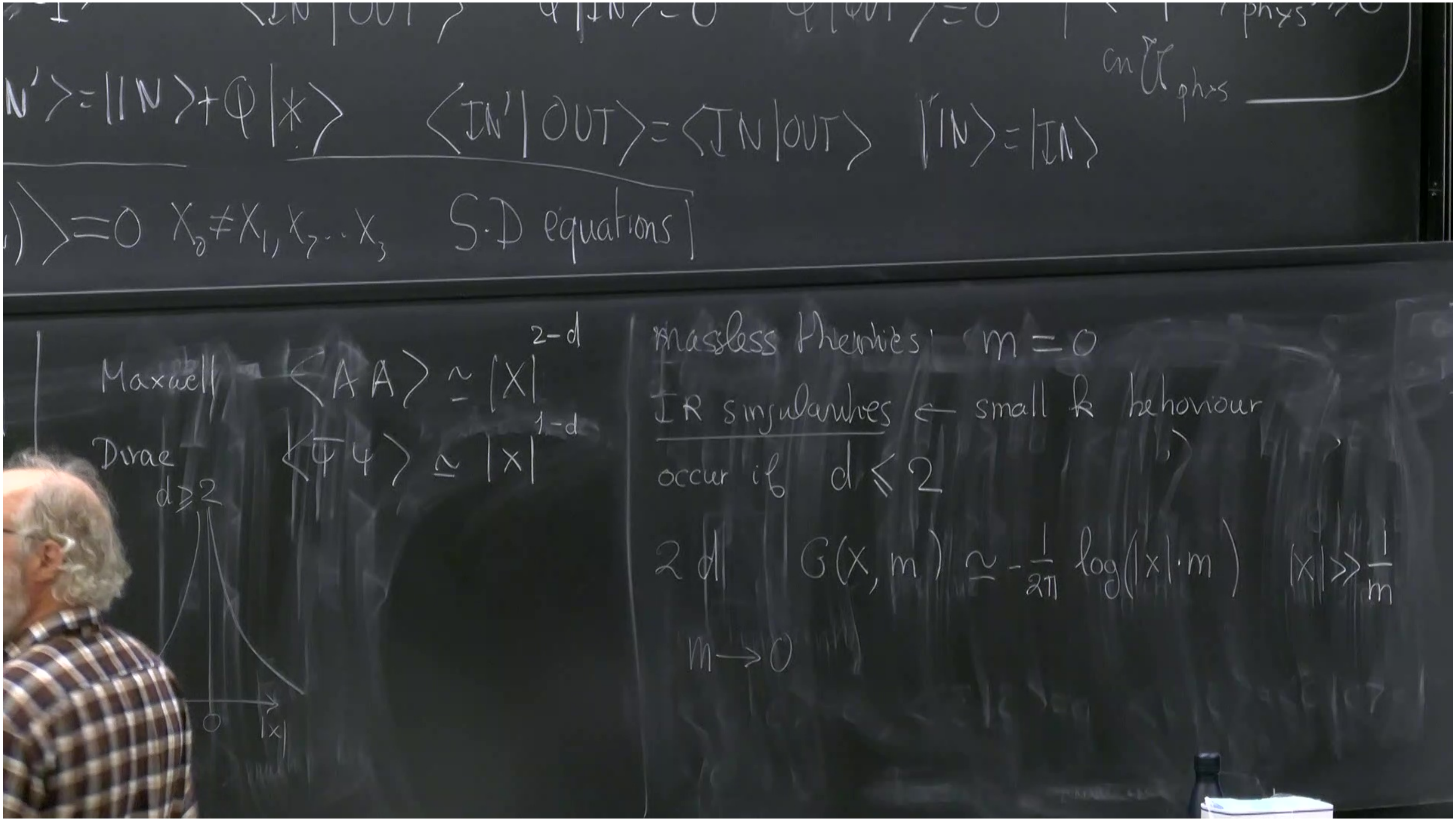
$d \geq 2$

massless theories : $m = 0$

IR singularities ⇐ small k behaviour

occur [illegible] 2

$$2 \ldots, m) \simeq -\frac{1}{2\pi}\log(|x|\cdot m) \qquad |x| \gg \frac{1}{m}$$



$\langle IN | OUT \rangle$ $Q|IN\rangle = 0$ $Q|OUT\rangle = 0$ phys

on $\mathcal{H}_{phys}$

$|IN'\rangle = |IN\rangle + Q|*\rangle$ $\langle IN'|OUT\rangle = \langle IN|OUT\rangle$ $|'IN\rangle = |IN\rangle$

$\rangle = 0$ $x_0 \neq x_1, x_2 \ldots x_3$ S.D equations

Maxwell $\langle A A \rangle \simeq |X|^{2-d}$

Dirac $\langle \bar{\Psi} \Psi \rangle \simeq |x|^{1-d}$

$d \geq 2$

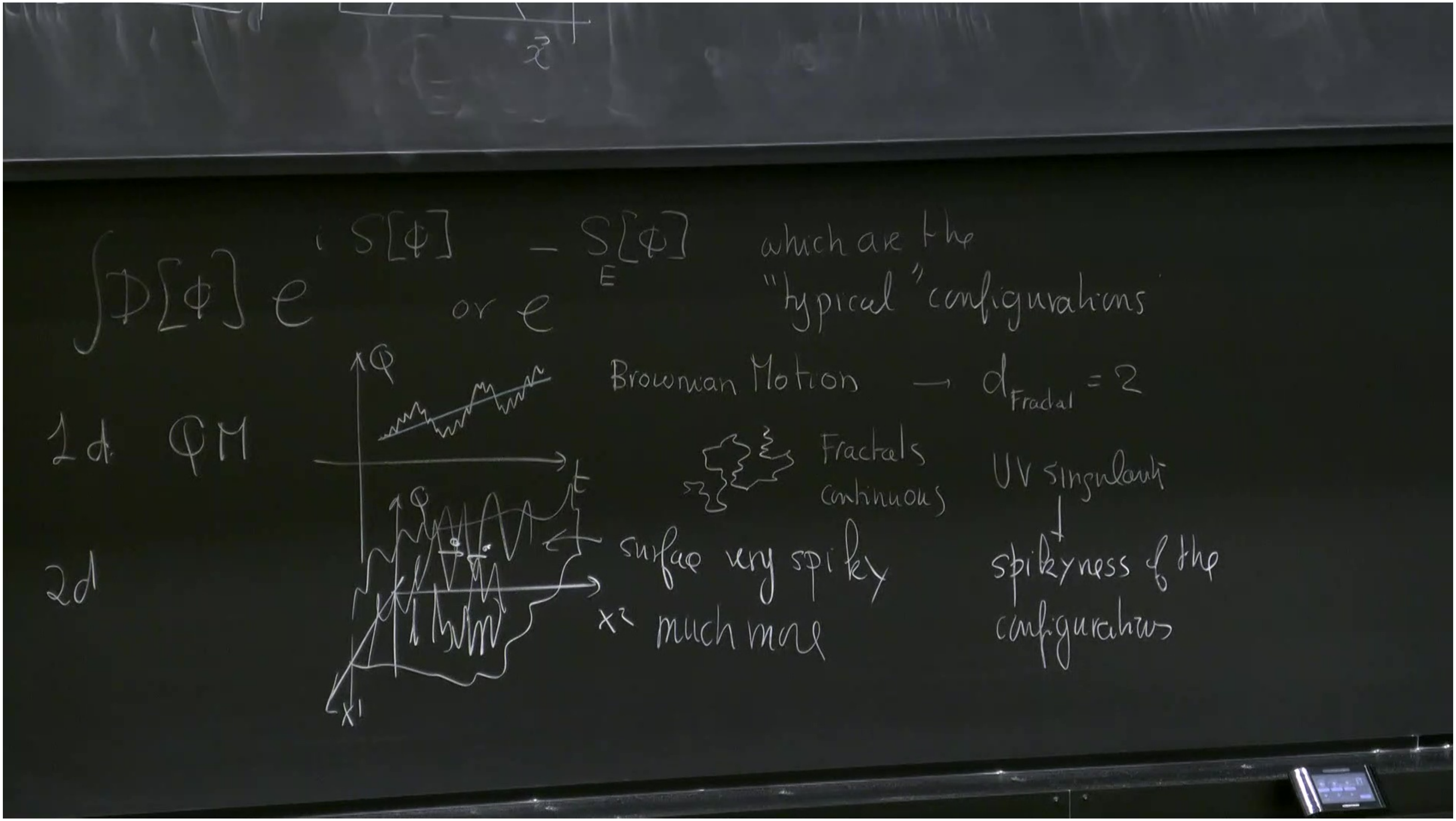
massless theories: $m = 0$

IR singularities ← small k behaviour

occur if $d \leq 2$

2 d $G(x, m) \simeq -\frac{1}{2\pi} \log(|x| \cdot m)$ $|x| \gg \frac{1}{m}$

$m \to 0$



$$\int \mathcal{D}[\phi]\, e^{iS[\phi]} \quad \text{or} \quad e^{-S_E[\phi]}$$ which are the "typical" configurations

1d. QM

Brownian Motion $\to$ $d_{\text{Fractal}} = 2$

Fractals continuous

UV singularit

$\downarrow$

spikyness of the configurations

2d

surface very spiky much more