Title: Monster Lie Algebra: Friend or Foe?

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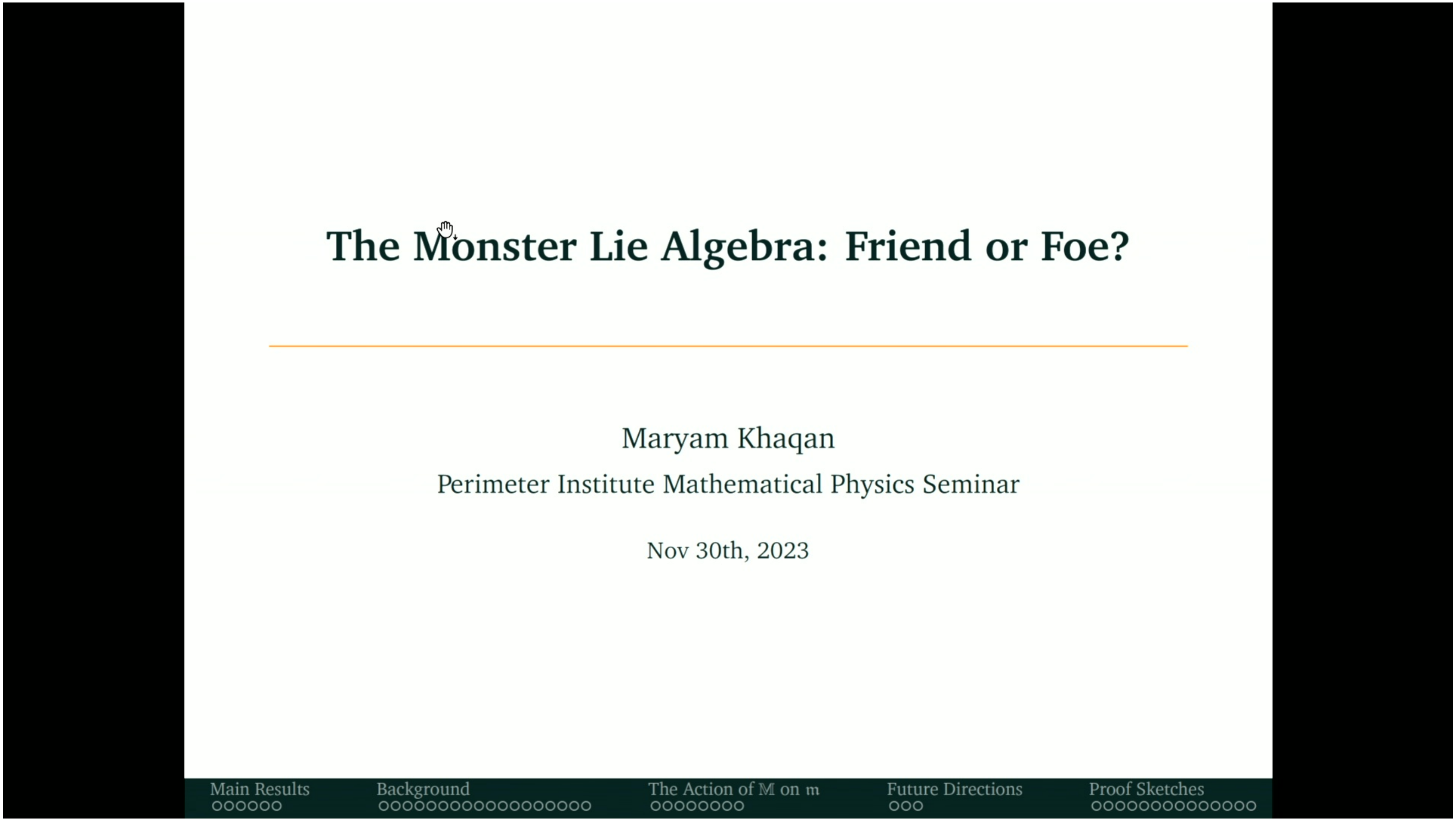
Abstract: The Monster Lie Algebra $\mathfrak m$ has two well-known avatars: It is a Borcherds' algebra that is also a quotient of the physical space of a specific tensor product of vertex algebras. In this talk, I will discuss a construction of vertex algebra elements that project to bases for subalgebras of $\mathfrak m$ isomorphic to $\mathfrak{gl}_2$, corresponding to each of the imaginary simple roots of the Monster Lie algebra.

Furthermore, for a fixed imaginary simple root, I will illustrate how the action of the Monster simple group on the Moonshine module induces an action of the Monster group on the set of the $\mathfrak{gl}_2$ subalgebras constructed this way. I will discuss this action and related open questions.

This talk is based on joint work with Darlayne Addabbo, Lisa Carbone, Elizabeth Jurisich, and Scott H. Murray.

Zoom link TBA



The Monster Lie Algebra: Friend or Foe?

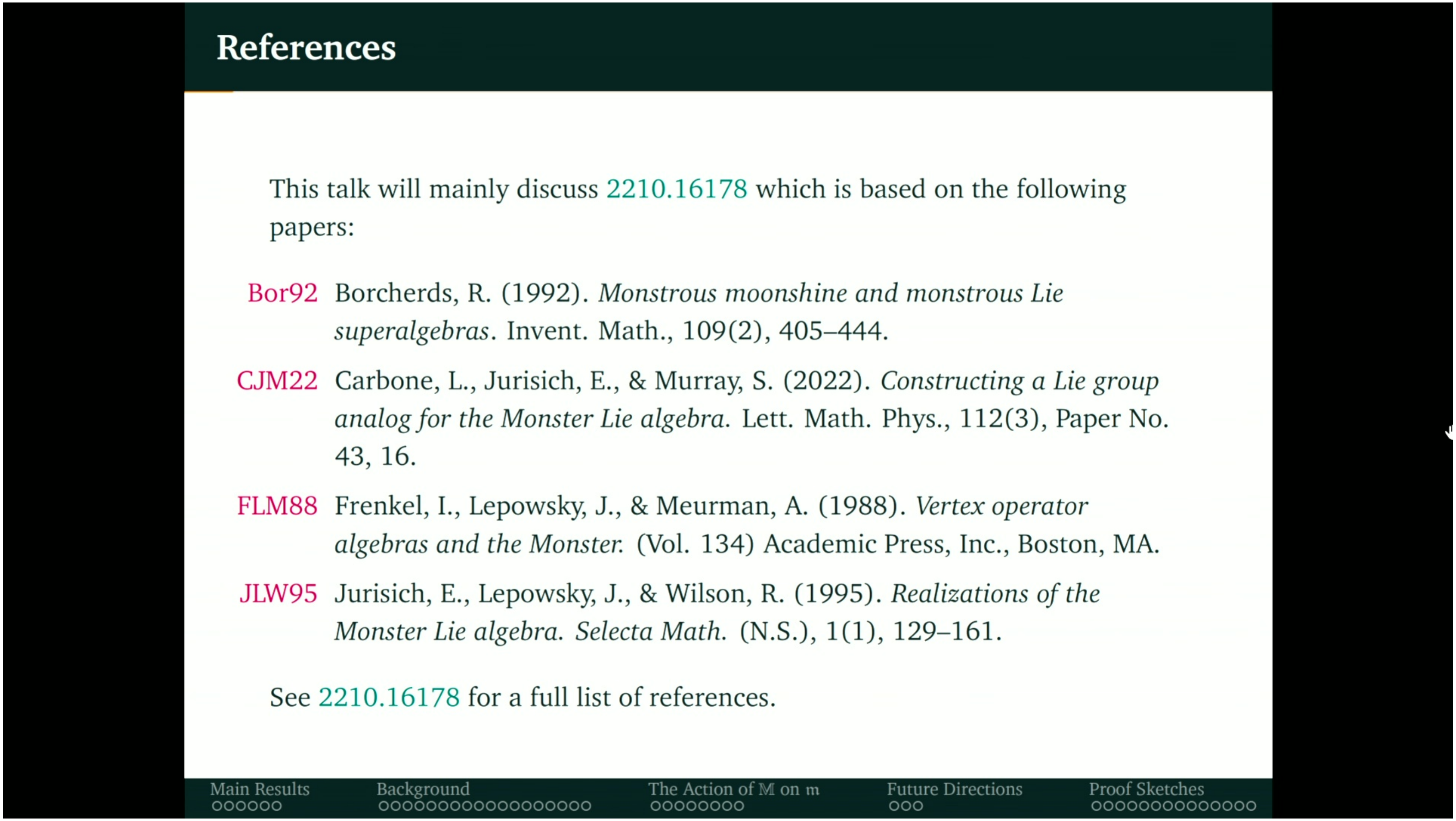
Maryam Khaqan
Perimeter Institute Mathematical Physics Seminar

Nov 30th, 2023

The Monster Lie Algebra: Friend or Foe?

“Vertex operators for imaginary $\mathfrak{gl}_2$ subalgebras of the Monster Lie Algebra”

w/ Darlayne Addabbo, Lisa Carbone, Elizabeth Jurisich, & Scott H. Murray



References

This talk will mainly discuss 2210.16178 which is based on the following papers:

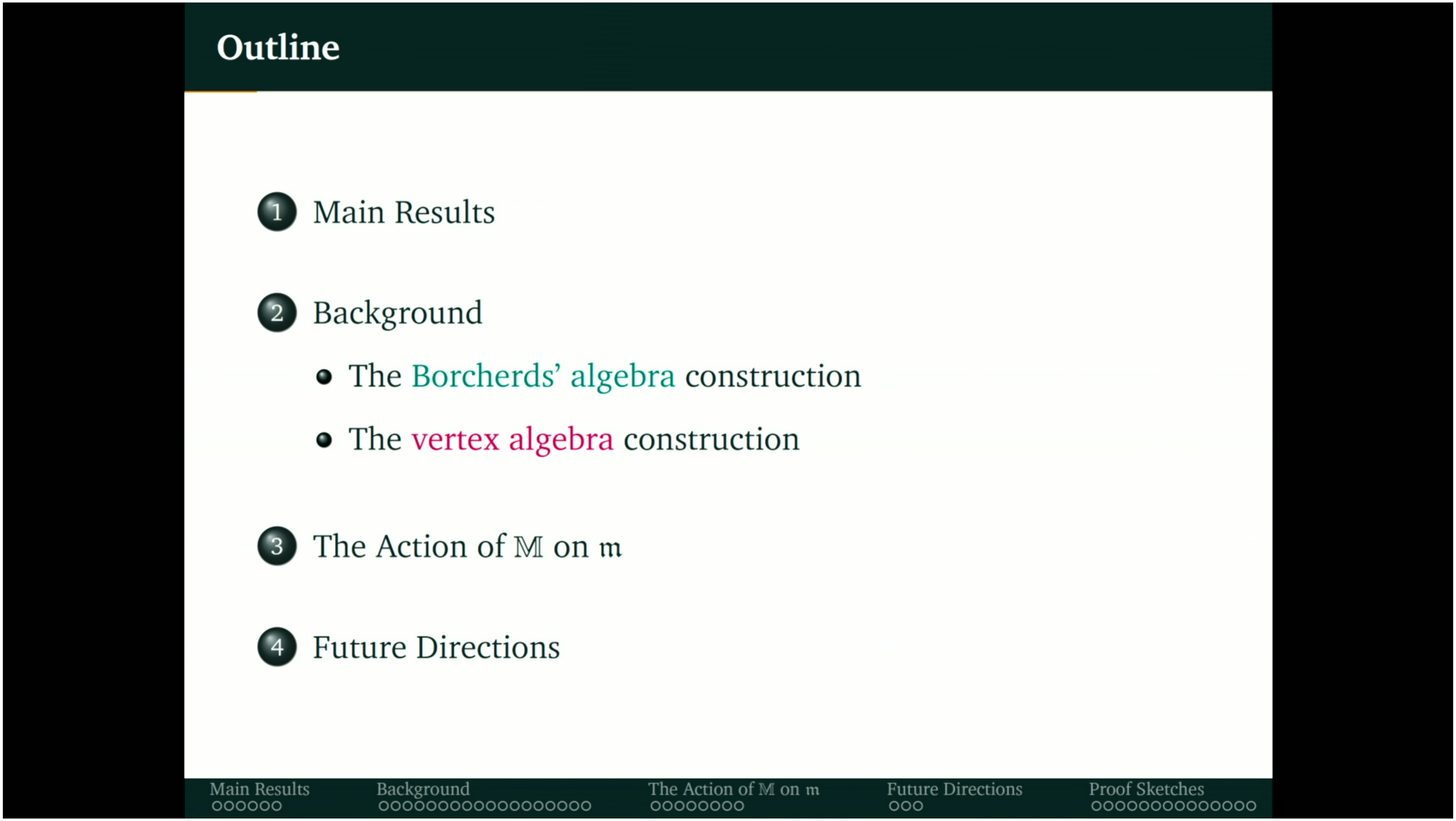
Bor92 Borcherds, R. (1992). *Monstrous moonshine and monstrous Lie superalgebras*. Invent. Math., 109(2), 405–444.

CJM22 Carbone, L., Jurisich, E., & Murray, S. (2022). *Constructing a Lie group analog for the Monster Lie algebra.* Lett. Math. Phys., 112(3), Paper No. 43, 16.

FLM88 Frenkel, I., Lepowsky, J., & Meurman, A. (1988). *Vertex operator algebras and the Monster.* (Vol. 134) Academic Press, Inc., Boston, MA.

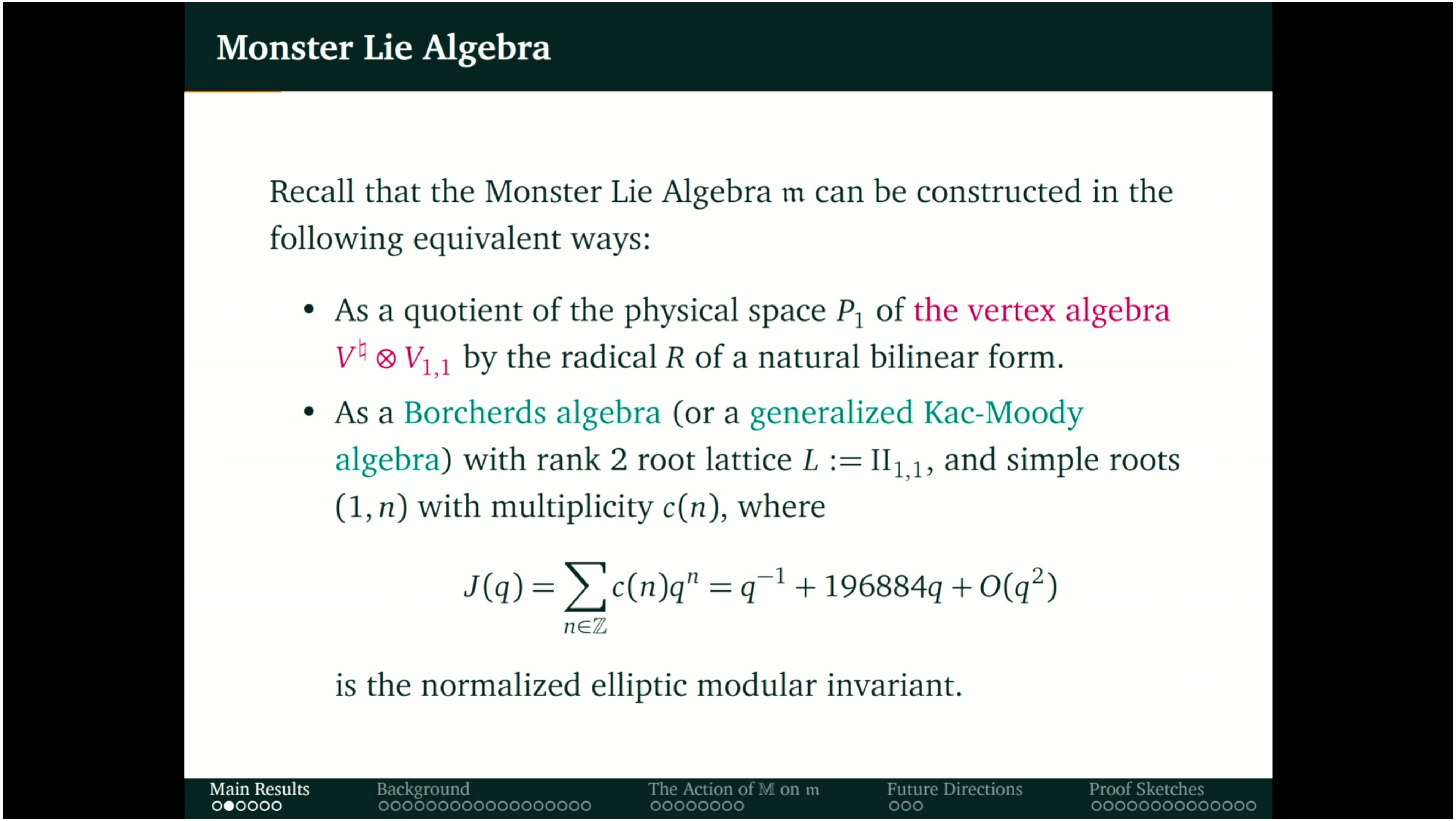
JLW95 Jurisich, E., Lepowsky, J., & Wilson, R. (1995). *Realizations of the Monster Lie algebra. Selecta Math.* (N.S.), 1(1), 129–161.

See 2210.16178 for a full list of references.



Outline

1. Main Results
2. Background
    - The Borcherds' algebra construction
    - The vertex algebra construction
3. The Action of $\mathbb{M}$ on $\mathfrak{m}$
4. Future Directions



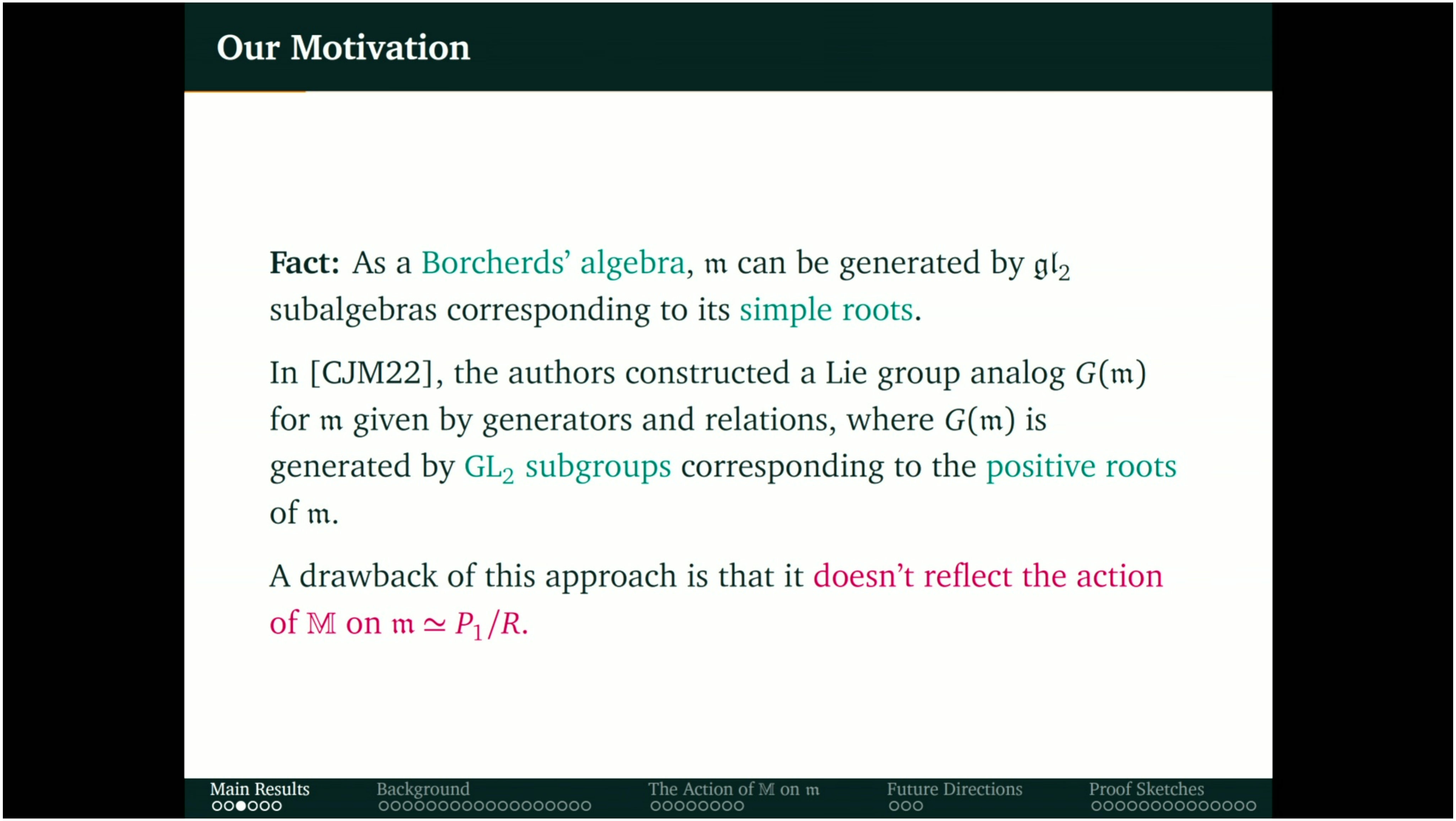
Monster Lie Algebra

Recall that the Monster Lie Algebra $\mathfrak{m}$ can be constructed in the following equivalent ways:

- As a quotient of the physical space P_1 of the vertex algebra $V^\natural \otimes V_{1,1}$ by the radical R of a natural bilinear form.
- As a Borcherds algebra (or a generalized Kac-Moody algebra) with rank 2 root lattice $L := \mathrm{II}_{1,1}$, and simple roots $(1, n)$ with multiplicity $c(n)$, where

$$J(q) = \sum_{n\in\mathbb{Z}} c(n)q^n = q^{-1} + 196884q + O(q^2)$$

is the normalized elliptic modular invariant.

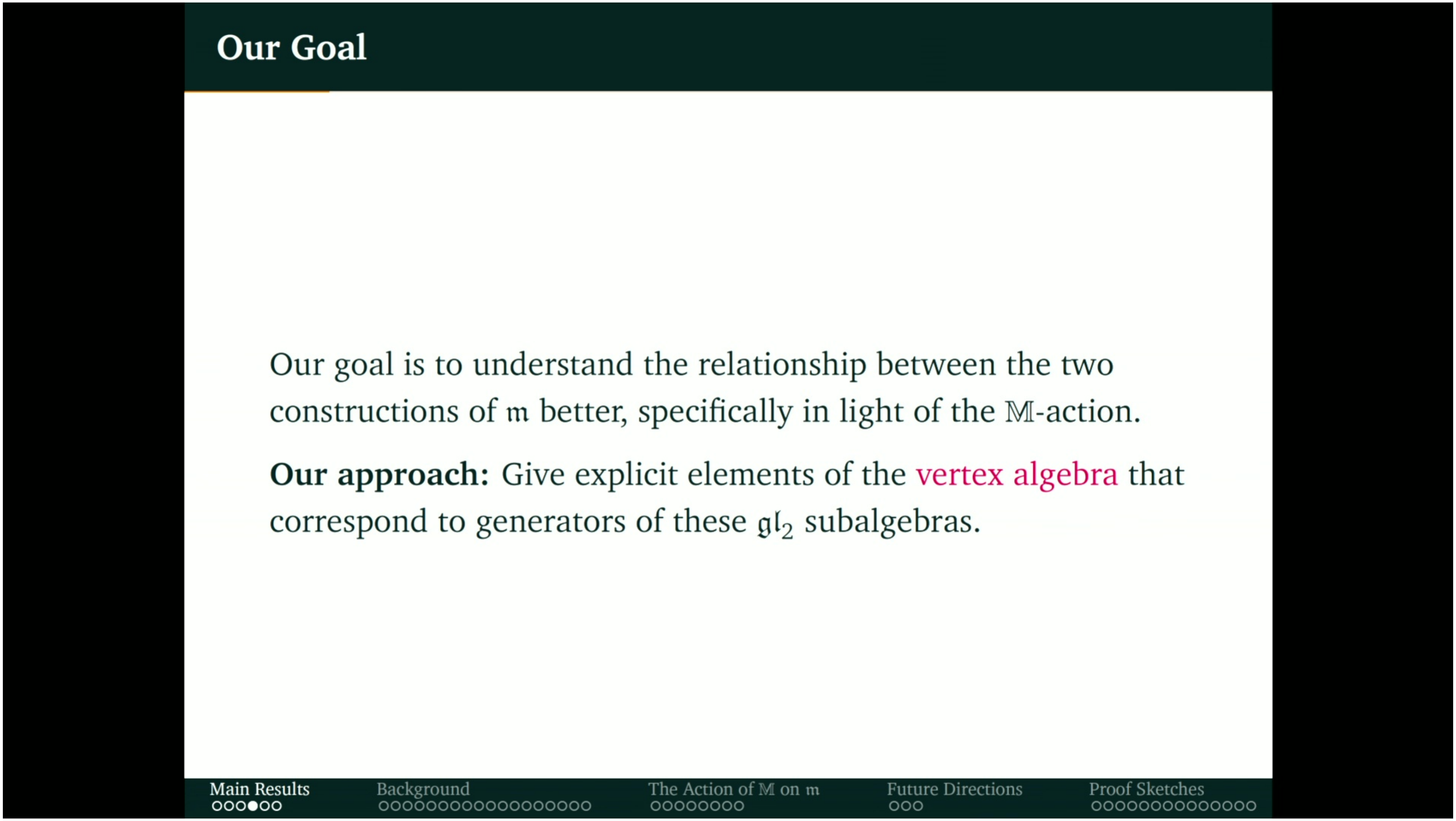


Our Motivation

Fact: As a Borcherds' algebra, $\mathfrak{m}$ can be generated by $\mathfrak{gl}_2$ subalgebras corresponding to its simple roots.

In [CJM22], the authors constructed a Lie group analog $G(\mathfrak{m})$ for $\mathfrak{m}$ given by generators and relations, where $G(\mathfrak{m})$ is generated by GL_2 subgroups corresponding to the positive roots of $\mathfrak{m}$.

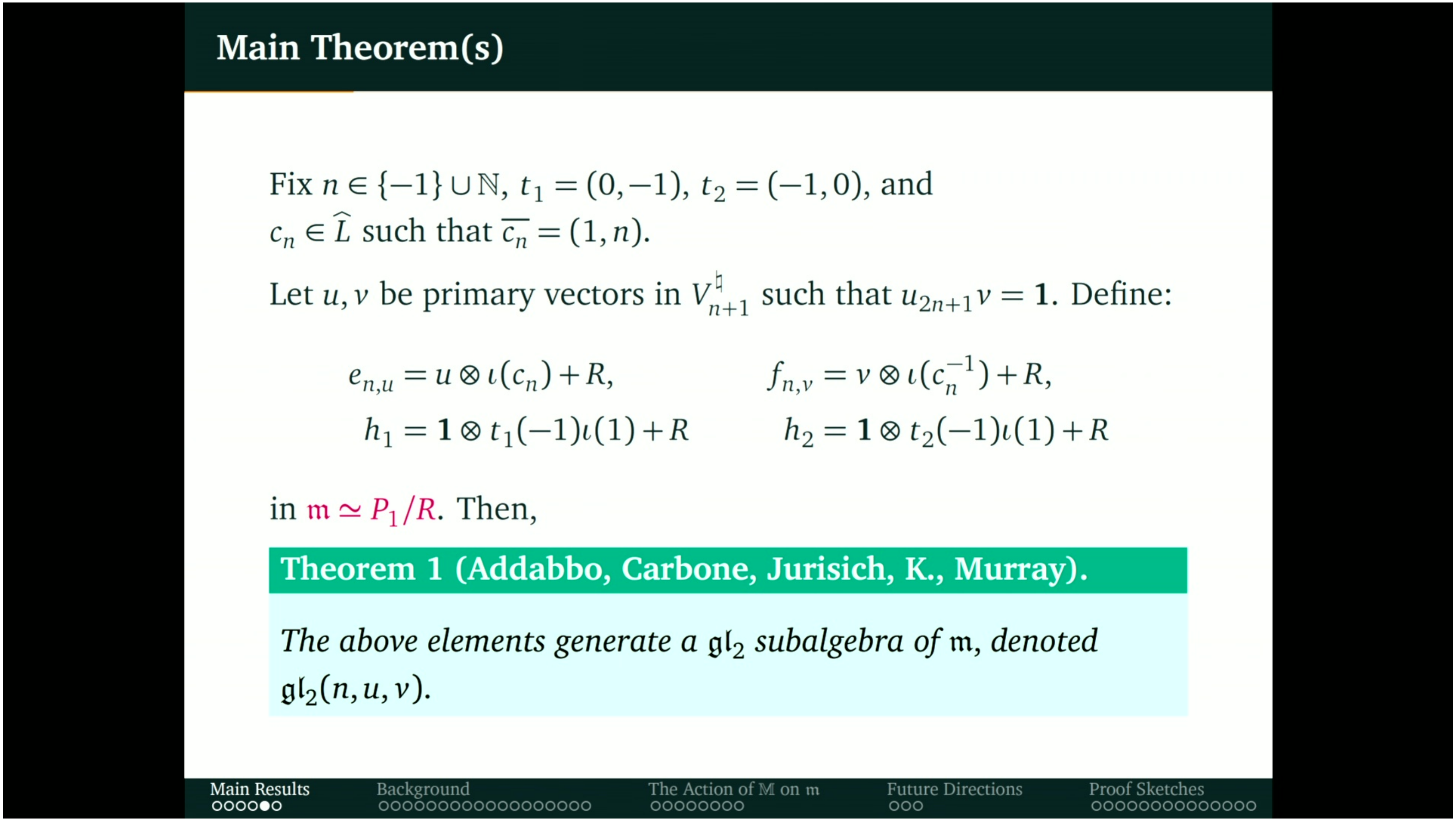
A drawback of this approach is that it doesn't reflect the action of $\mathbb{M}$ on $\mathfrak{m} \simeq P_1/R$.



Our Goal

Our goal is to understand the relationship between the two constructions of $\mathfrak{m}$ better, specifically in light of the $\mathbb{M}$-action.

Our approach: Give explicit elements of the vertex algebra that correspond to generators of these $\mathfrak{gl}_2$ subalgebras.



Main Theorem(s)

Fix $n \in \{-1\} \cup \mathbb{N}$, $t_1 = (0,-1)$, $t_2 = (-1,0)$, and $c_n \in \widehat{L}$ such that $\overline{c_n} = (1,n)$.

Let u, v be primary vectors in $V^{\natural}_{n+1}$ such that $u_{2n+1}v = \mathbf{1}$. Define:

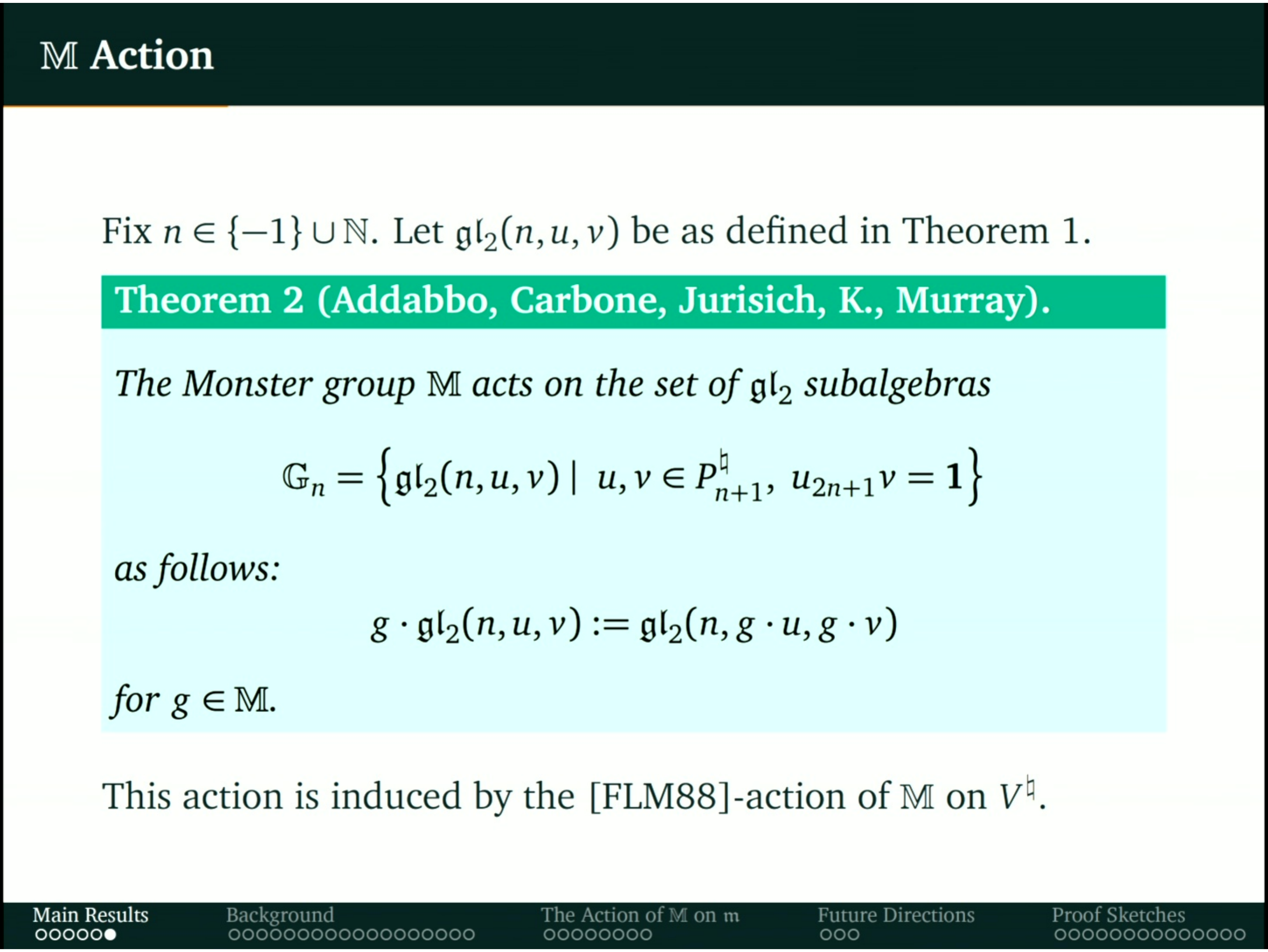
$$e_{n,u} = u \otimes \iota(c_n) + R, \qquad f_{n,v} = v \otimes \iota(c_n^{-1}) + R,$$

$$h_1 = \mathbf{1} \otimes t_1(-1)\iota(1) + R \qquad h_2 = \mathbf{1} \otimes t_2(-1)\iota(1) + R$$

in $\mathfrak{m} \simeq P_1/R$. Then,

Theorem 1 (Addabbo, Carbone, Jurisich, K., Murray).

The above elements generate a $\mathfrak{gl}_2$ *subalgebra of* $\mathfrak{m}$*, denoted* $\mathfrak{gl}_2(n,u,v)$.



$\mathbb{M}$ Action

Fix $n \in \{-1\} \cup \mathbb{N}$. Let $\mathfrak{gl}_2(n,u,v)$ be as defined in Theorem 1.

Theorem 2 (Addabbo, Carbone, Jurisich, K., Murray).

The Monster group $\mathbb{M}$ acts on the set of $\mathfrak{gl}_2$ subalgebras

$$\mathbb{G}_n = \left\{ \mathfrak{gl}_2(n,u,v) \mid \ u,v \in P^{\natural}_{n+1},\ u_{2n+1}v = \mathbf{1} \right\}$$

as follows:

$$g \cdot \mathfrak{gl}_2(n,u,v) := \mathfrak{gl}_2(n, g \cdot u, g \cdot v)$$

for $g \in \mathbb{M}$.

This action is induced by the [FLM88]-action of $\mathbb{M}$ on $V^{\natural}$.

Background

Coincidence?

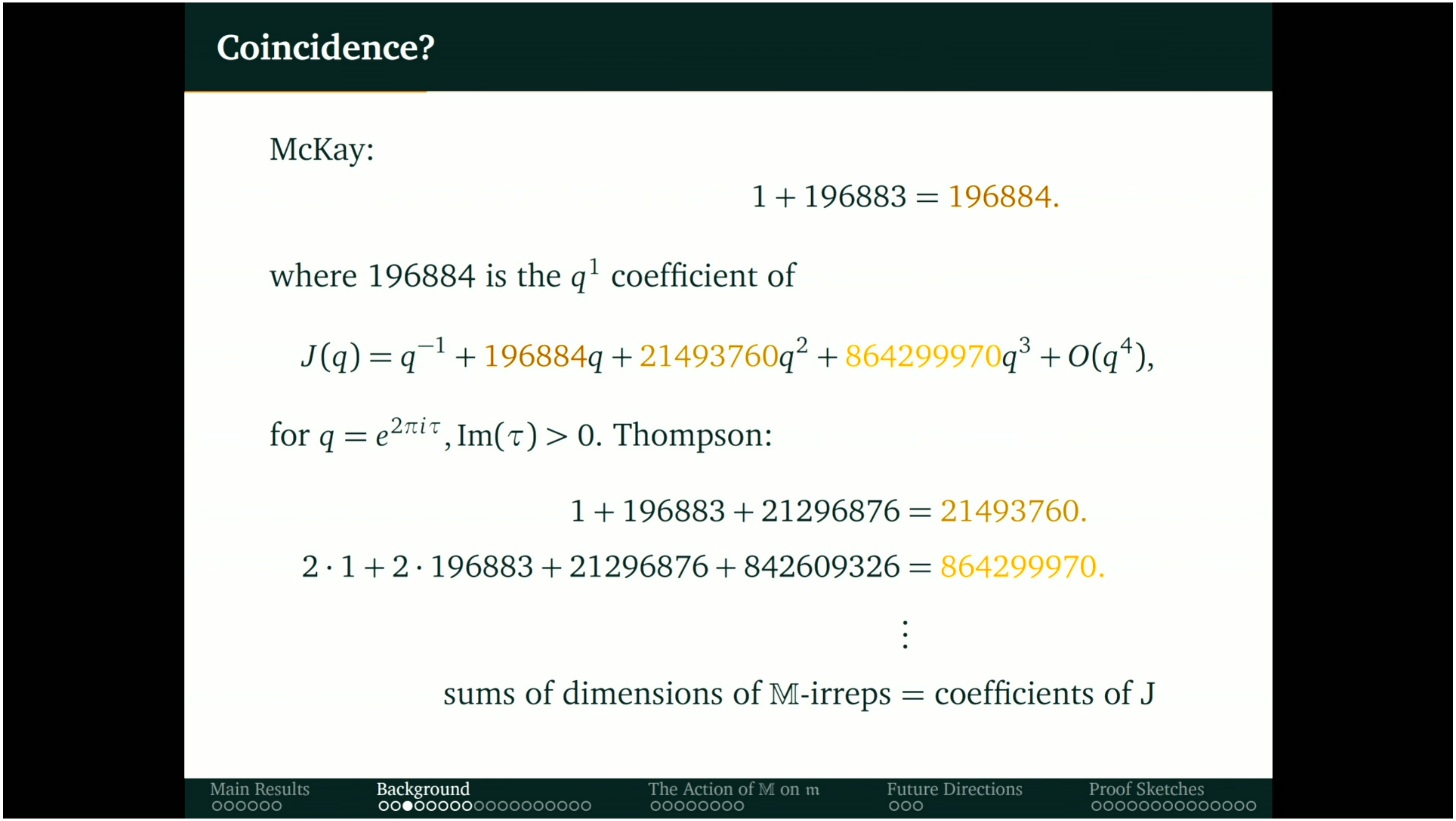
McKay:

$$1 + 196883 = 196884$$

where 196884 is the q^1 coefficient of

$$J(q) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + O(q^4),$$

for $q = e^{2\pi i \tau}, \mathrm{Im}(\tau) > 0$.



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for $q = e^{2\pi i\tau}, \mathrm{Im}(\tau) > 0$. Thompson:

$$1 + 196883 + 21296876 = 21493760.$$

$$2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326 = 864299970.$$

$$\vdots$$

sums of dimensions of $\mathbb{M}$-irreps = coefficients of J

Coincidence? I think not.

Dimensions of $\mathbb{M}$-irreps are the entries in the first column of the character table.

Look at the second column instead:

$$1+4371=4372$$

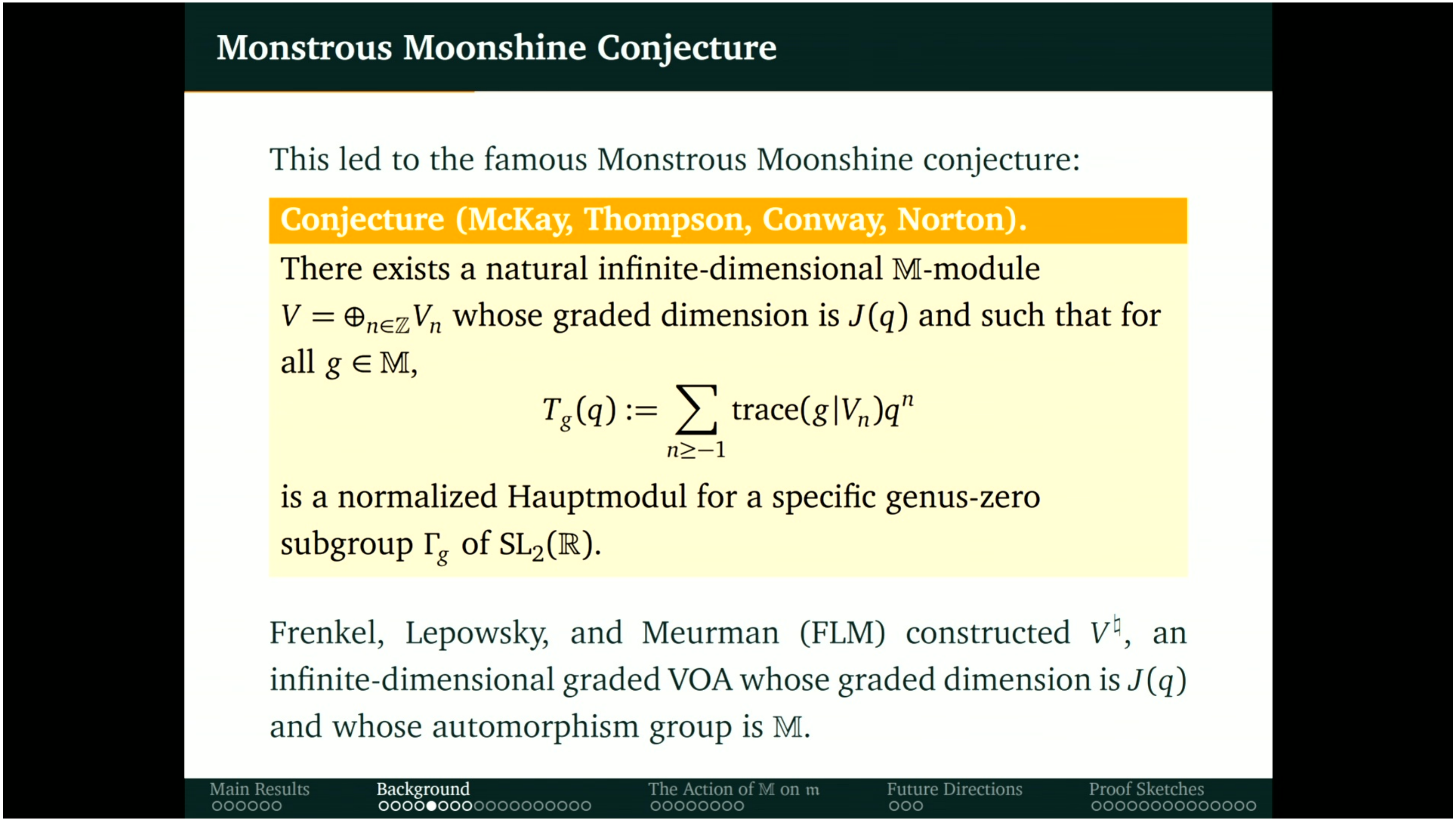
$$1+4371+91884=96256$$

$$2\cdot 1+2\cdot 4371+91884+1139374=1240002$$

$$\vdots$$

Traces of element of order 2 on $\mathbb{M}$-irreps $=$?

$$T_{2A}(\tau)=q^{-1}+4372q+96256q^{2}+1240002q^{3}+O(q^{4}).$$



Monstrous Moonshine Conjecture

This led to the famous Monstrous Moonshine conjecture:

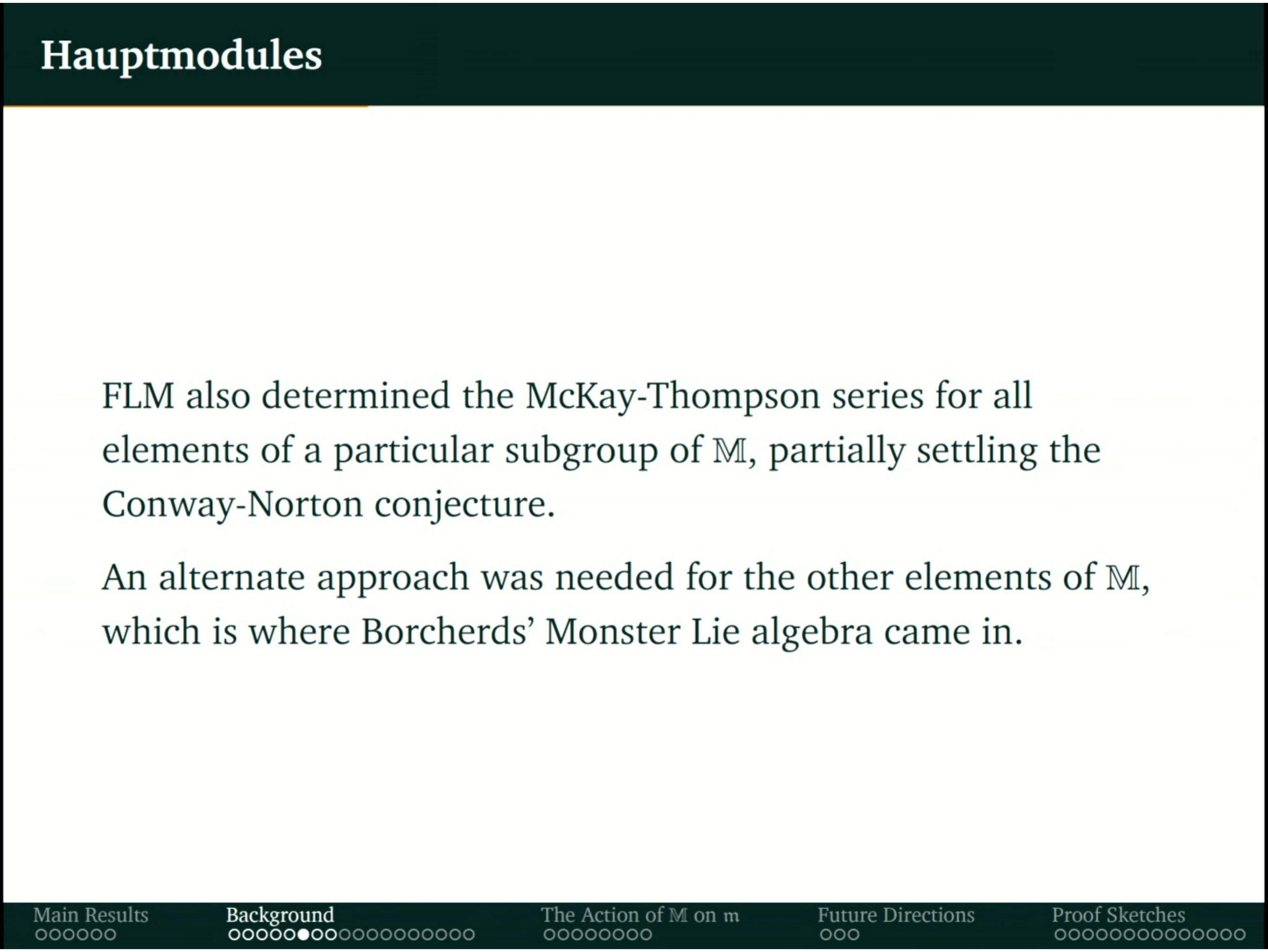
Conjecture (McKay, Thompson, Conway, Norton).

There exists a natural infinite-dimensional $\mathbb{M}$-module $V = \oplus_{n\in\mathbb{Z}} V_n$ whose graded dimension is $J(q)$ and such that for all $g \in \mathbb{M}$,

$$T_g(q) := \sum_{n\geq -1} \mathrm{trace}(g|V_n)q^n$$

is a normalized Hauptmodul for a specific genus-zero subgroup Γ_g of $\mathrm{SL}_2(\mathbb{R})$.

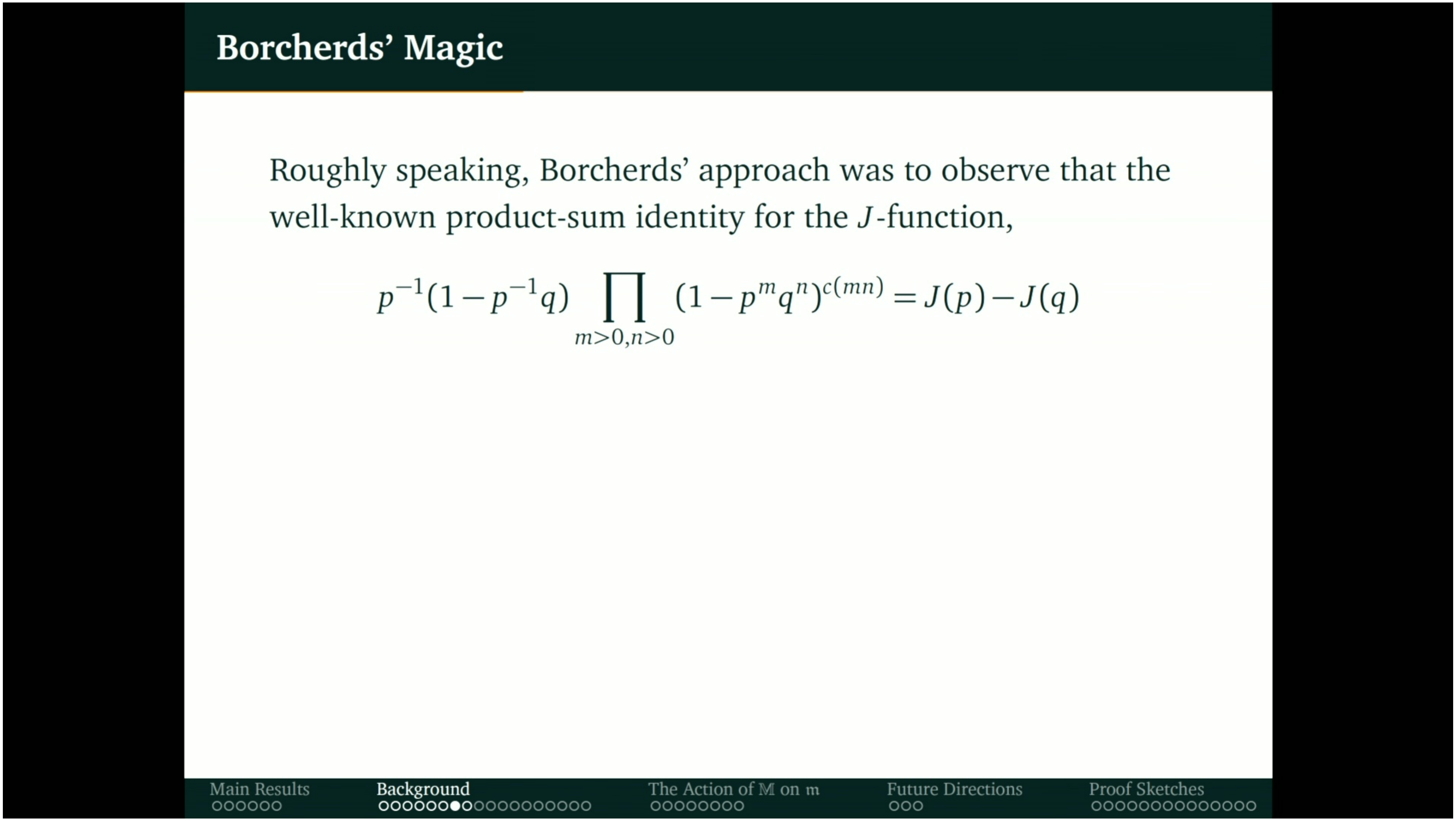
Frenkel, Lepowsky, and Meurman (FLM) constructed $V^\natural$, an infinite-dimensional graded VOA whose graded dimension is $J(q)$ and whose automorphism group is $\mathbb{M}$.



Hauptmodules

FLM also determined the McKay-Thompson series for all elements of a particular subgroup of $\mathbb{M}$, partially settling the Conway-Norton conjecture.

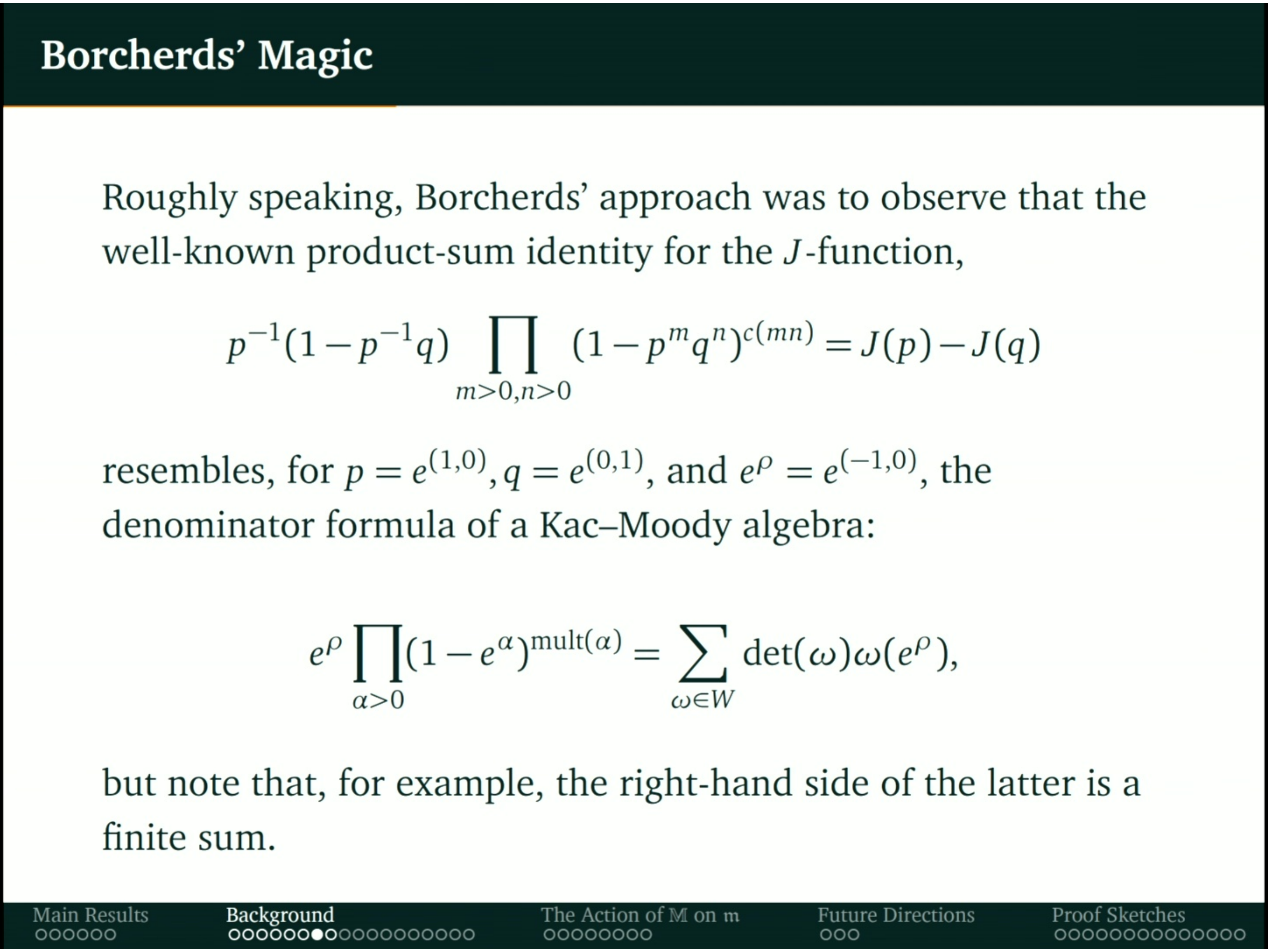
An alternate approach was needed for the other elements of $\mathbb{M}$, which is where Borcherds' Monster Lie algebra came in.



Borcherds' Magic

Roughly speaking, Borcherds' approach was to observe that the well-known product-sum identity for the J-function,

$$p^{-1}(1-p^{-1}q)\prod_{m>0,n>0}(1-p^m q^n)^{c(mn)} = J(p) - J(q)$$



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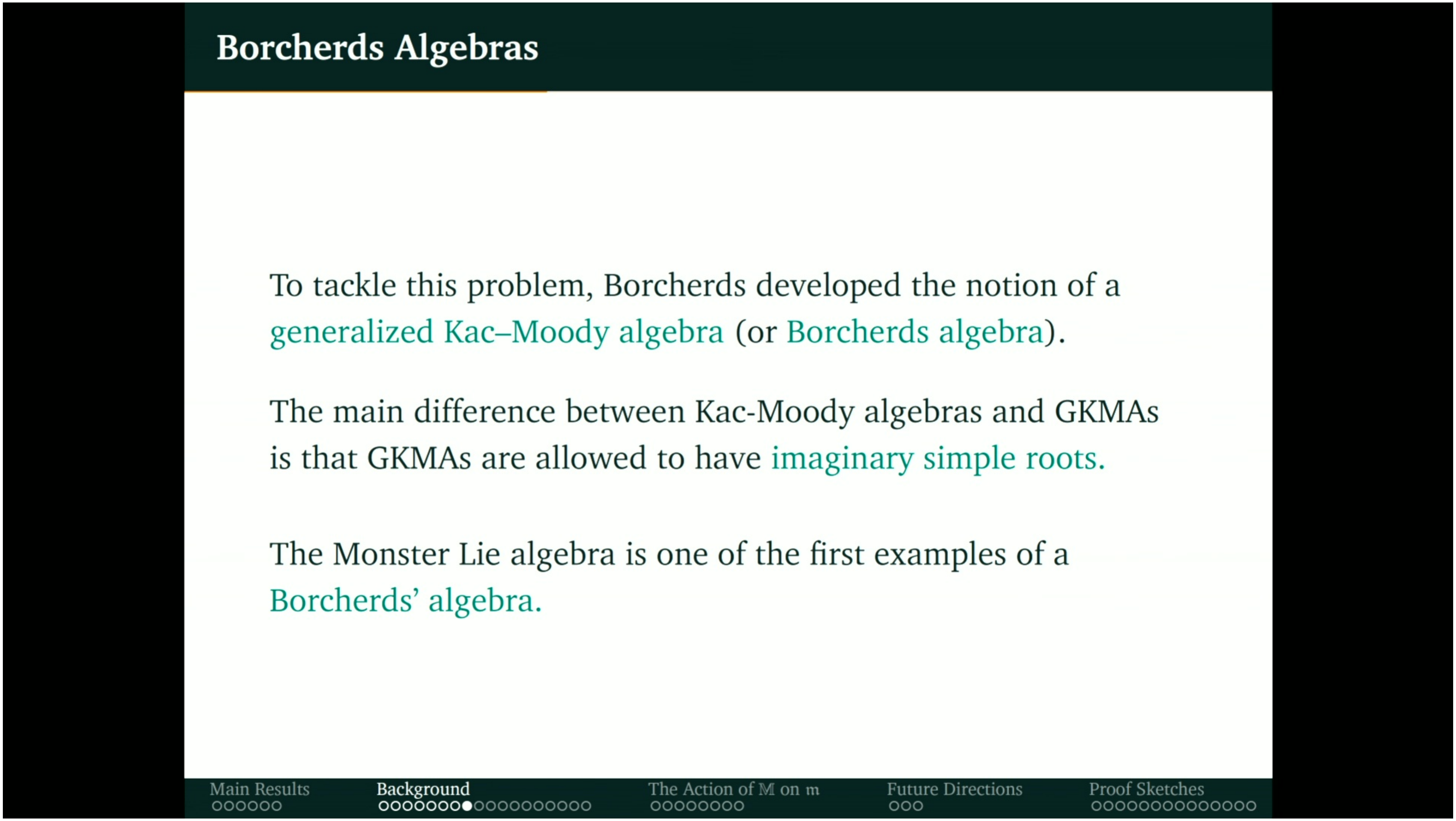
resembles, for $p = e^{(1,0)}, q = e^{(0,1)}$, and $e^{\rho} = e^{(-1,0)}$, the denominator formula of a Kac–Moody algebra:

$$e^{\rho}\prod_{\alpha>0}(1-e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{\omega\in W}\det(\omega)\omega(e^{\rho}),$$

but note that, for example, the right-hand side of the latter is a finite sum.

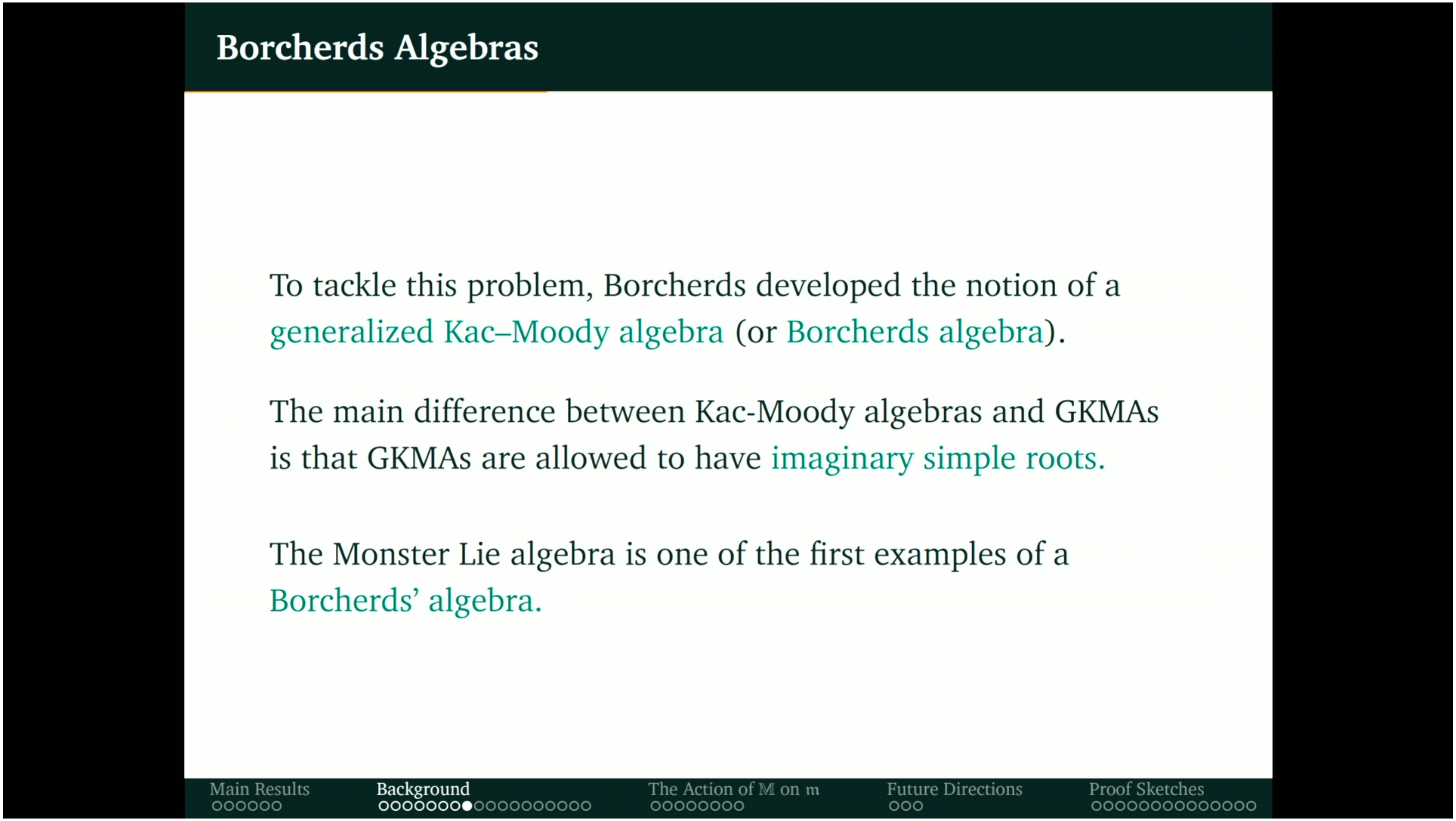


Borcherds Algebras

To tackle this problem, Borcherds developed the notion of a generalized Kac–Moody algebra (or Borcherds algebra).

The main difference between Kac-Moody algebras and GKMAs is that GKMAs are allowed to have imaginary simple roots.

The Monster Lie algebra is one of the first examples of a Borcherds' algebra.



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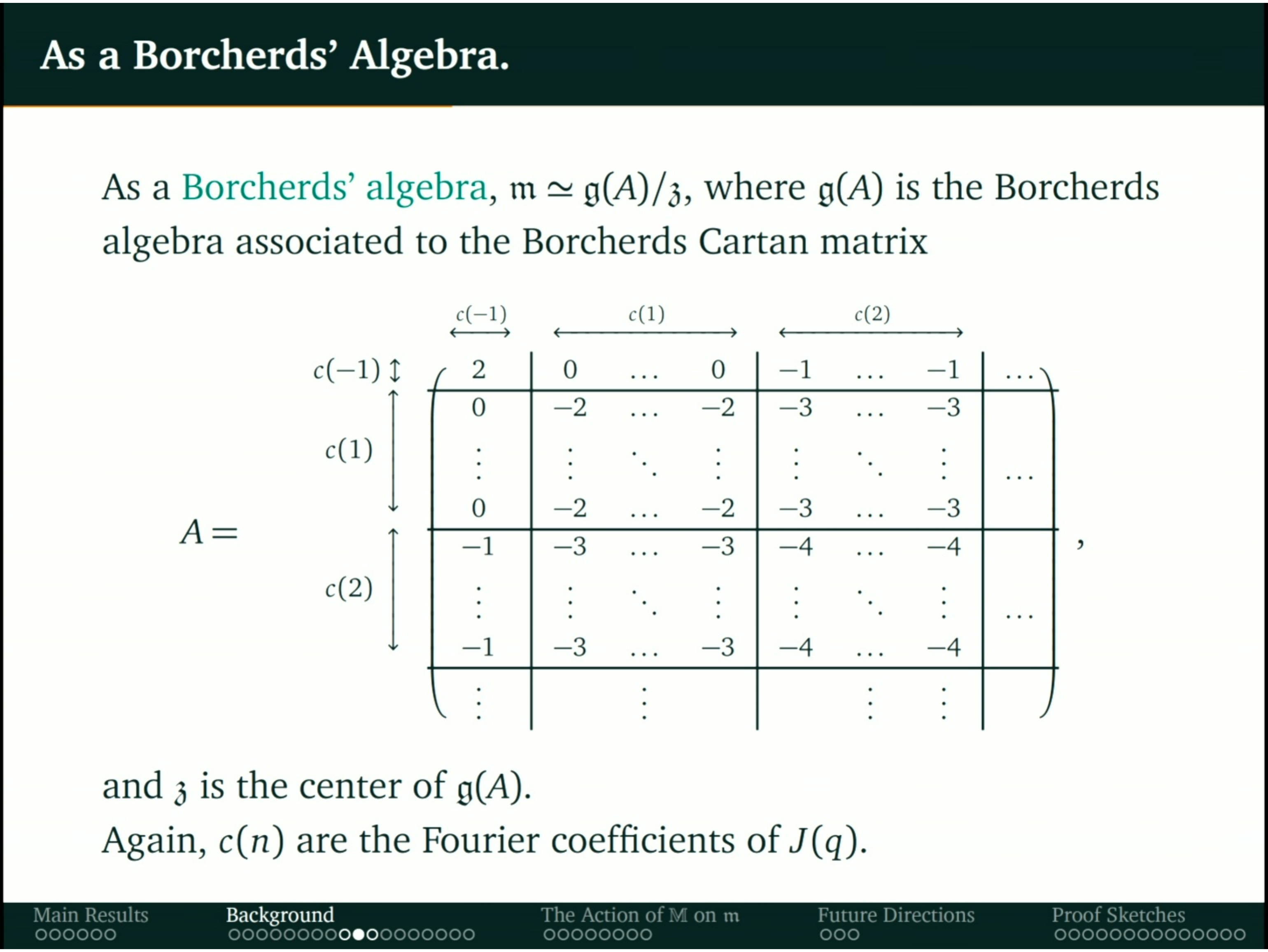
Background

The Borcherds' algebra construction



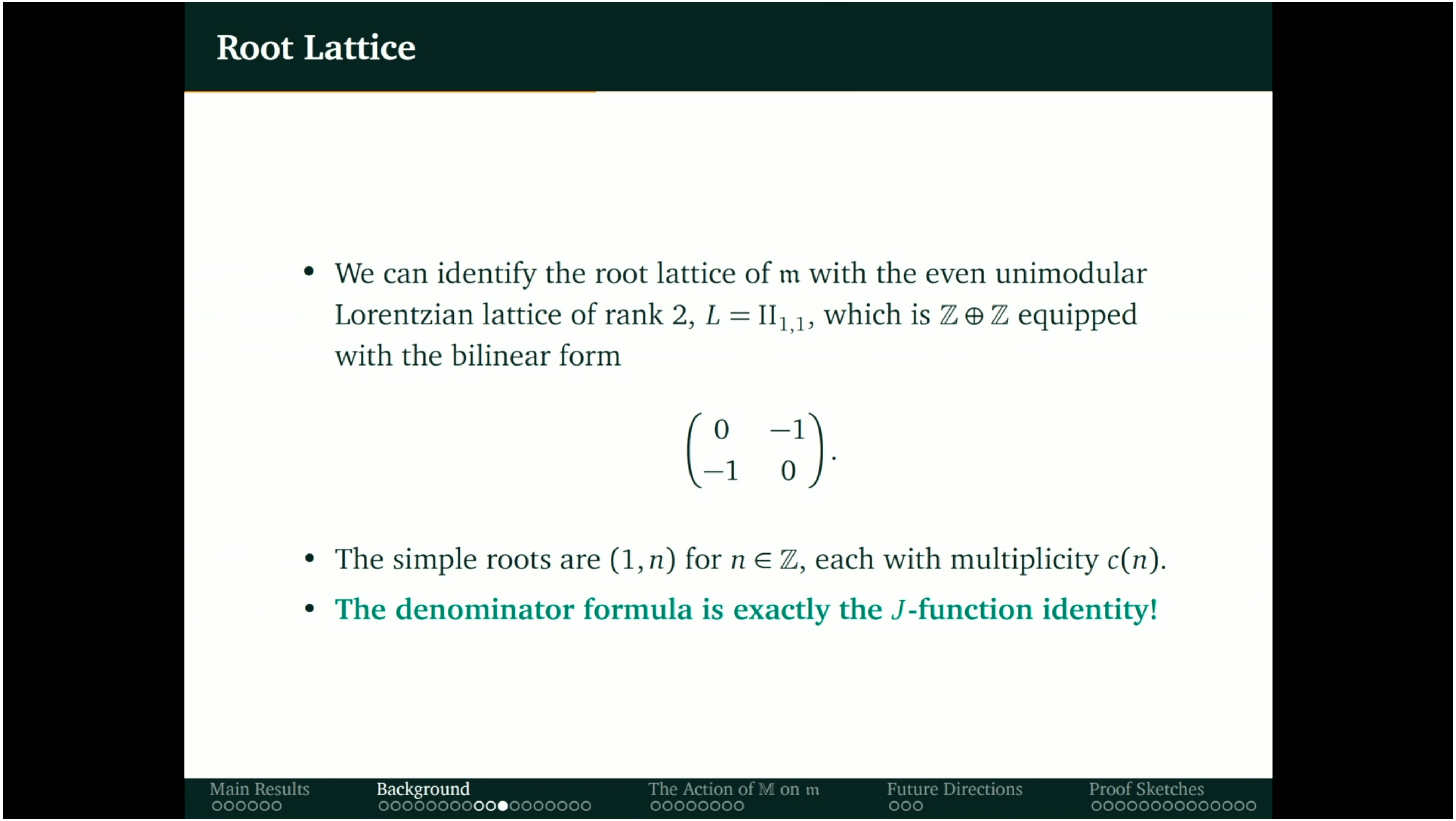
As a Borcherds' Algebra.

As a Borcherds' algebra, $\mathfrak{m} \simeq \mathfrak{g}(A)/\mathfrak{z}$, where $\mathfrak{g}(A)$ is the Borcherds algebra associated to the Borcherds Cartan matrix

$$A = \begin{array}{c|c|ccc|ccc|c}
 & c(-1) & & c(1) & & & c(2) & & \\
c(-1) & 2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\
\hline
 & 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
c(1) & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
 & 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\hline
 & -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
c(2) & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
 & -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\hline
 & \vdots & & \vdots & & & \vdots & \vdots &
\end{array},$$

and $\mathfrak{z}$ is the center of $\mathfrak{g}(A)$.

Again, $c(n)$ are the Fourier coefficients of $J(q)$.



Root Lattice

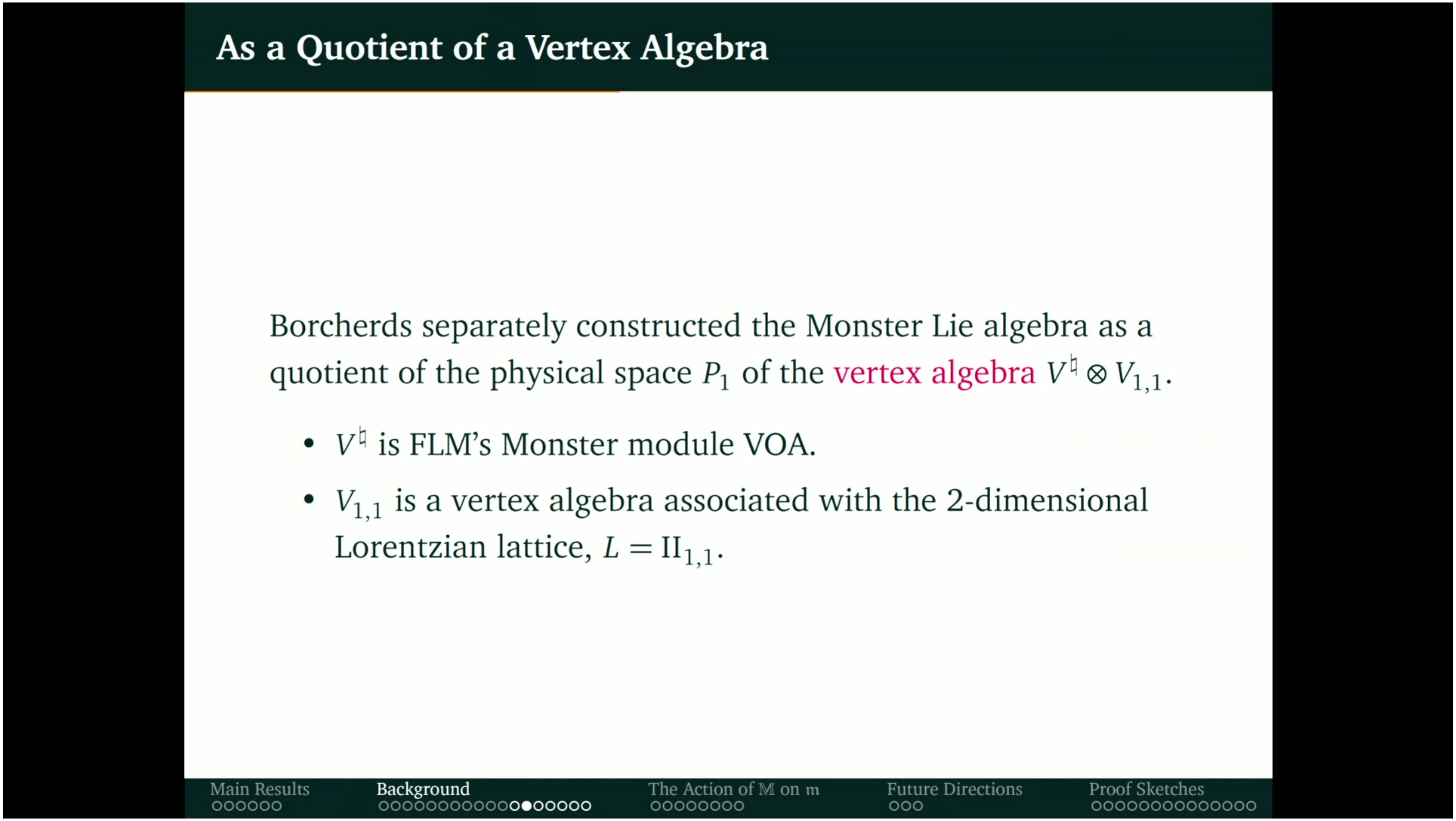
- We can identify the root lattice of $\mathfrak{m}$ with the even unimodular Lorentzian lattice of rank 2, $L = \mathrm{II}_{1,1}$, which is $\mathbb{Z} \oplus \mathbb{Z}$ equipped with the bilinear form

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

- The simple roots are $(1, n)$ for $n \in \mathbb{Z}$, each with multiplicity $c(n)$.
- **The denominator formula is exactly the J-function identity!**

Background

The vertex algebra construction



As a Quotient of a Vertex Algebra

Borcherds separately constructed the Monster Lie algebra as a quotient of the physical space P_1 of the vertex algebra $V^\natural \otimes V_{1,1}$.

- $V^\natural$ is FLM's Monster module VOA.
- $V_{1,1}$ is a vertex algebra associated with the 2-dimensional Lorentzian lattice, $L = \mathrm{II}_{1,1}$.

What is… a Vertex Algebra?

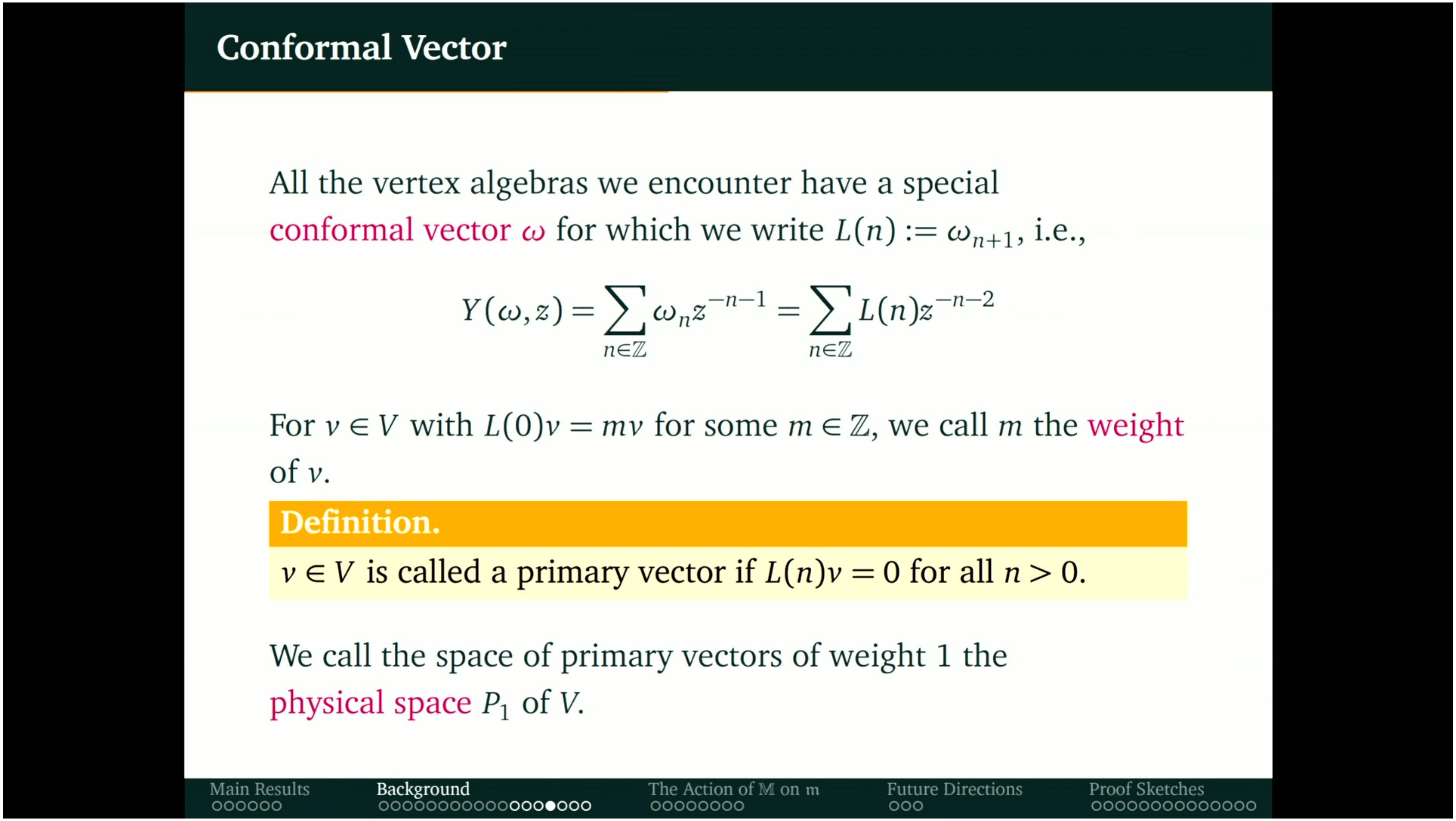
An *algebra* is a vector space V with a linear map

$$\begin{aligned} V &\to \mathrm{End}(V) \\ u &\mapsto (v \mapsto u \cdot v) \end{aligned}$$

A *vertex algebra* is a vector space V with a linear map

$$\begin{aligned} Y(\cdot, z)\colon V &\to \mathrm{End}(V)[[z, z^{-1}]] \\ u &\mapsto Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1} \end{aligned}$$

that is, for each $u, v \in V$ we have bilinear products $u_n v$ for each $n \in \mathbb{Z}$ that satisfy some compatibility axioms.



Conformal Vector

All the vertex algebras we encounter have a special conformal vector ω for which we write $L(n) := \omega_{n+1}$, i.e.,

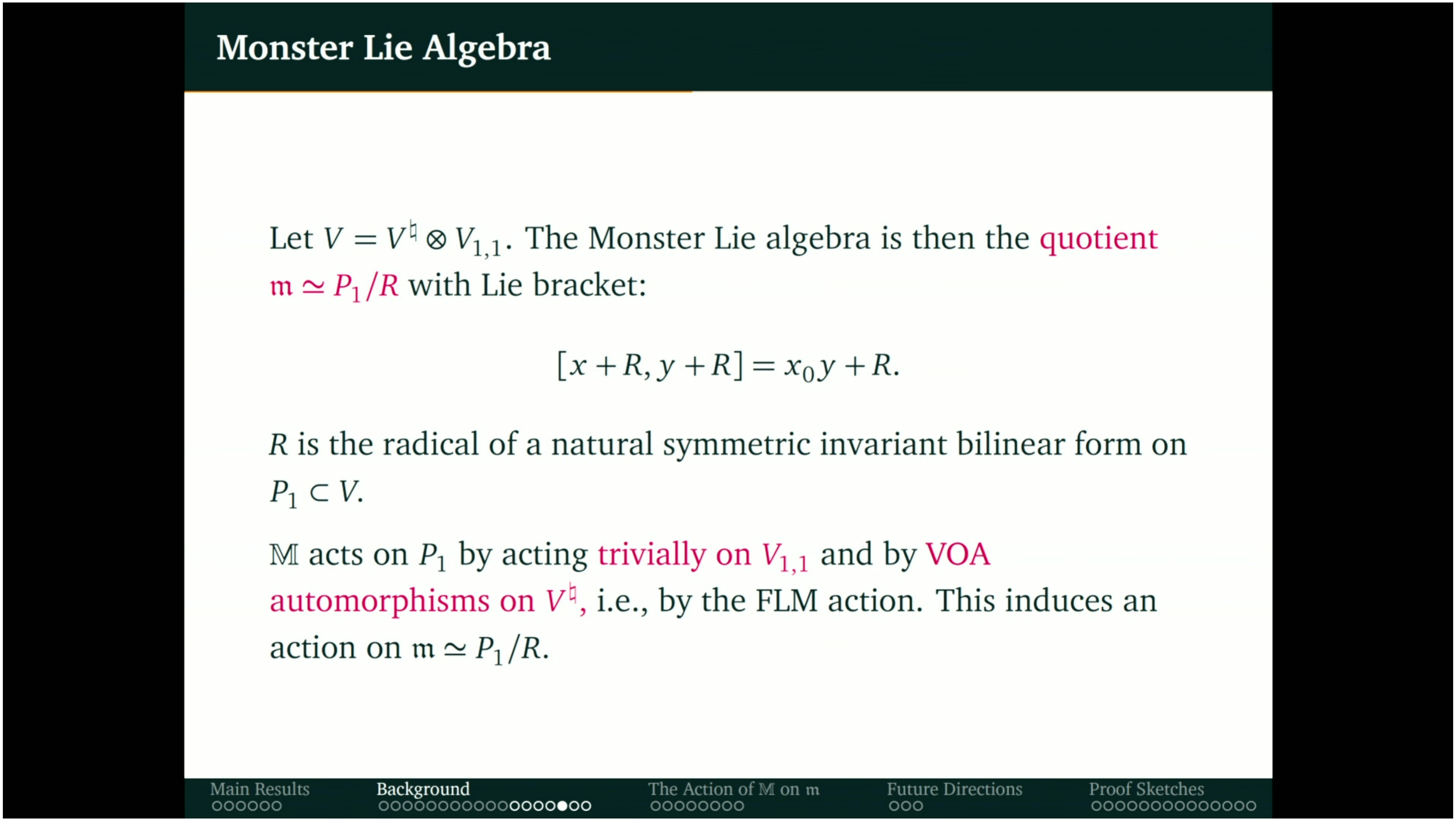
$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

For $v \in V$ with $L(0)v = mv$ for some $m \in \mathbb{Z}$, we call m the weight of v.

Definition.

$v \in V$ is called a primary vector if $L(n)v = 0$ for all $n > 0$.

We call the space of primary vectors of weight 1 the physical space P_1 of V.



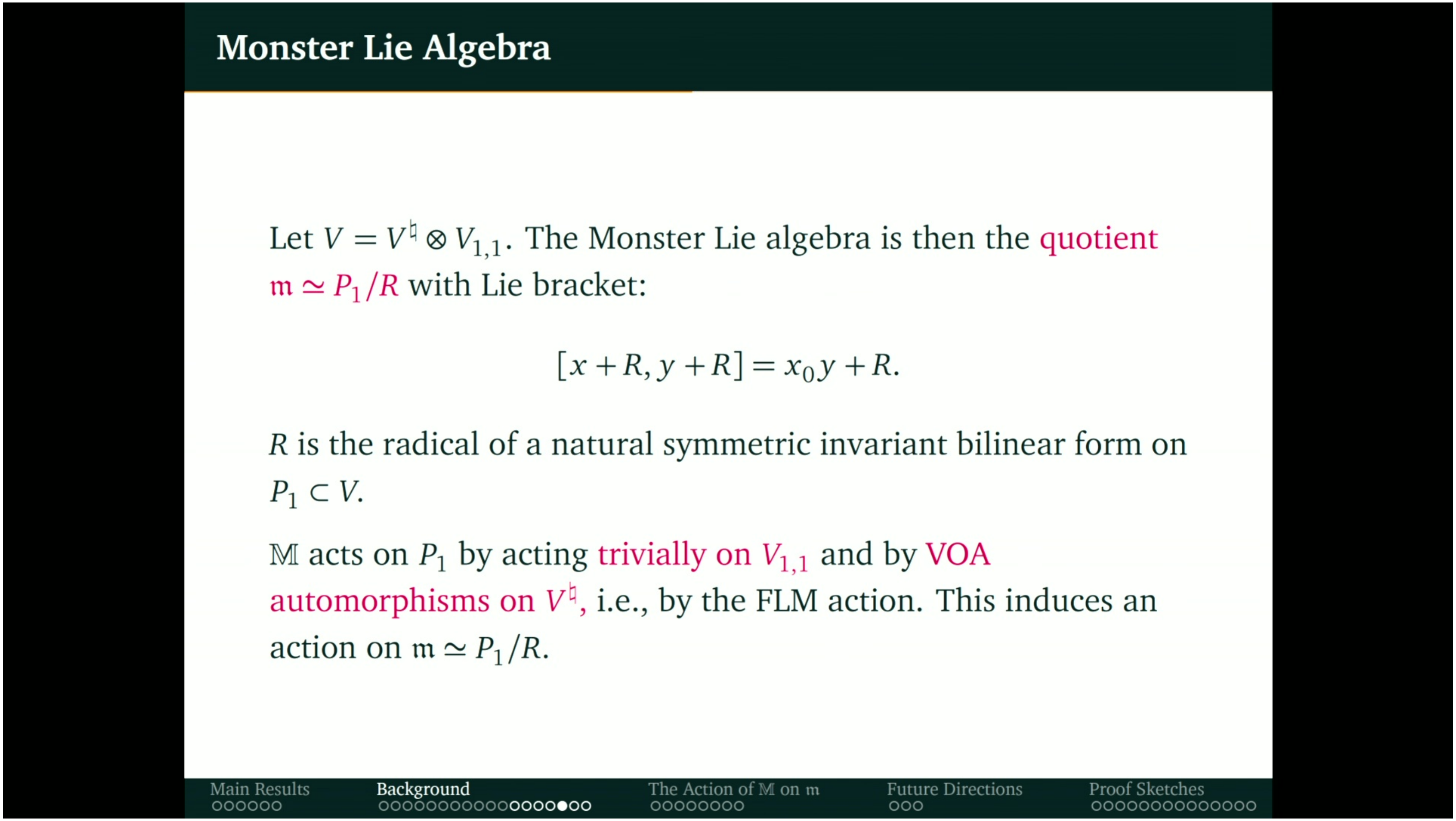
Monster Lie Algebra

Let $V = V^\natural \otimes V_{1,1}$. The Monster Lie algebra is then the quotient $\mathfrak{m} \simeq P_1/R$ with Lie bracket:

$$[x+R, y+R] = x_0 y + R.$$

R is the radical of a natural symmetric invariant bilinear form on $P_1 \subset V$.

$\mathbb{M}$ acts on P_1 by acting trivially on $V_{1,1}$ and by VOA automorphisms on $V^\natural$, i.e., by the FLM action. This induces an action on $\mathfrak{m} \simeq P_1/R$.



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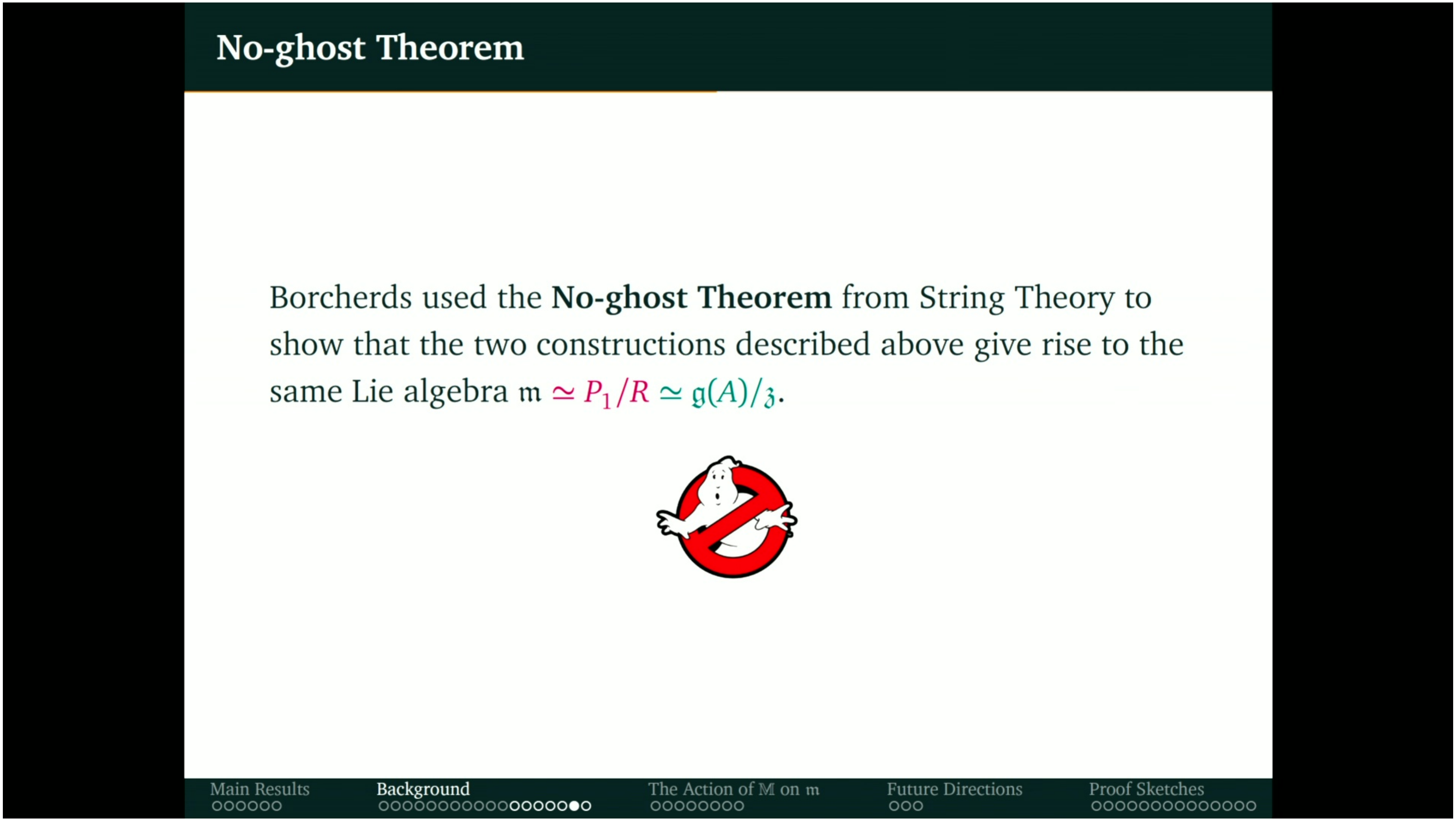
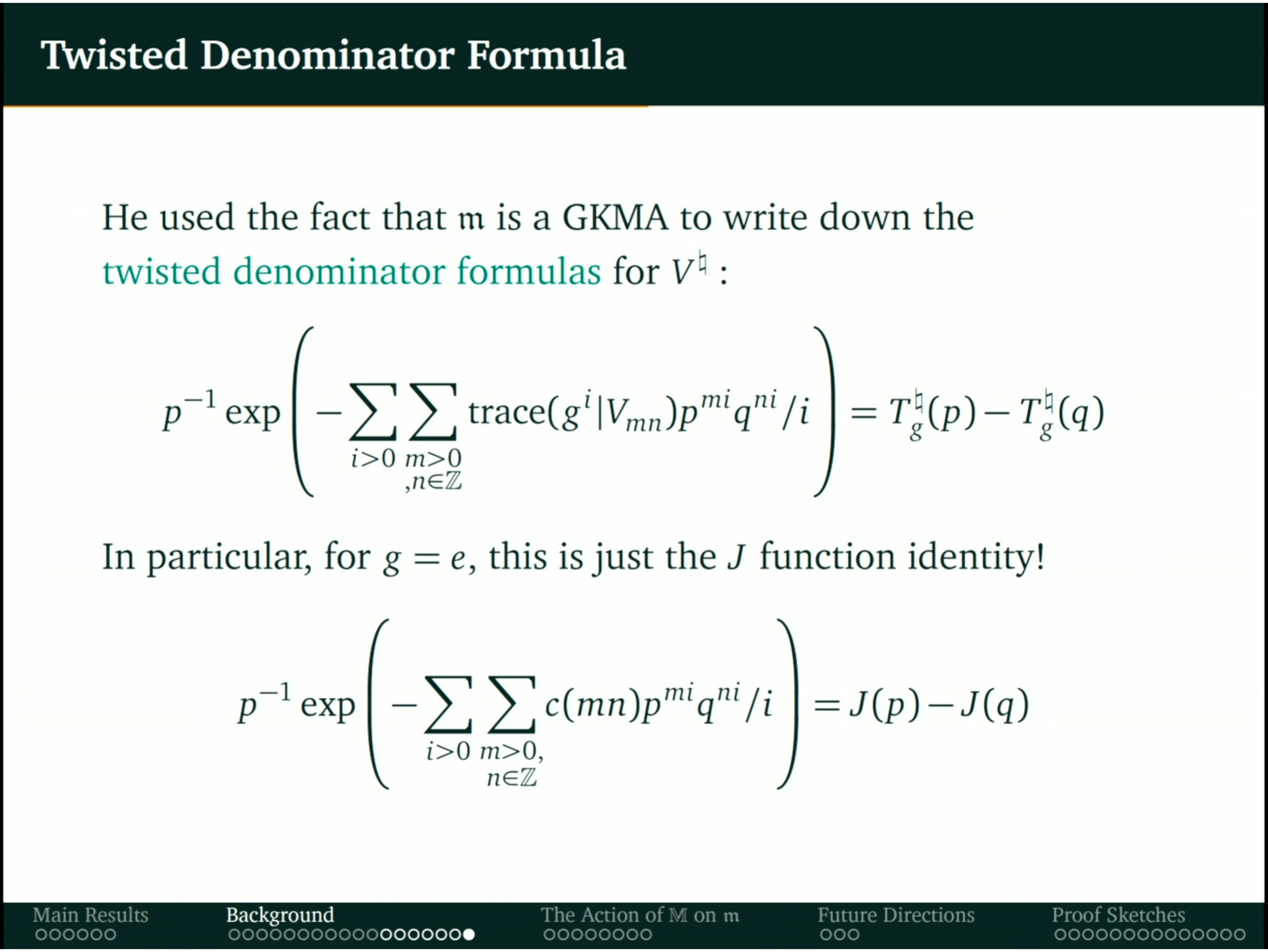


No-ghost Theorem

Borcherds used the **No-ghost Theorem** from String Theory to show that the two constructions described above give rise to the same Lie algebra $\mathfrak{m} \simeq P_1/R \simeq \mathfrak{g}(A)/\mathfrak{z}$.



Twisted Denominator Formula

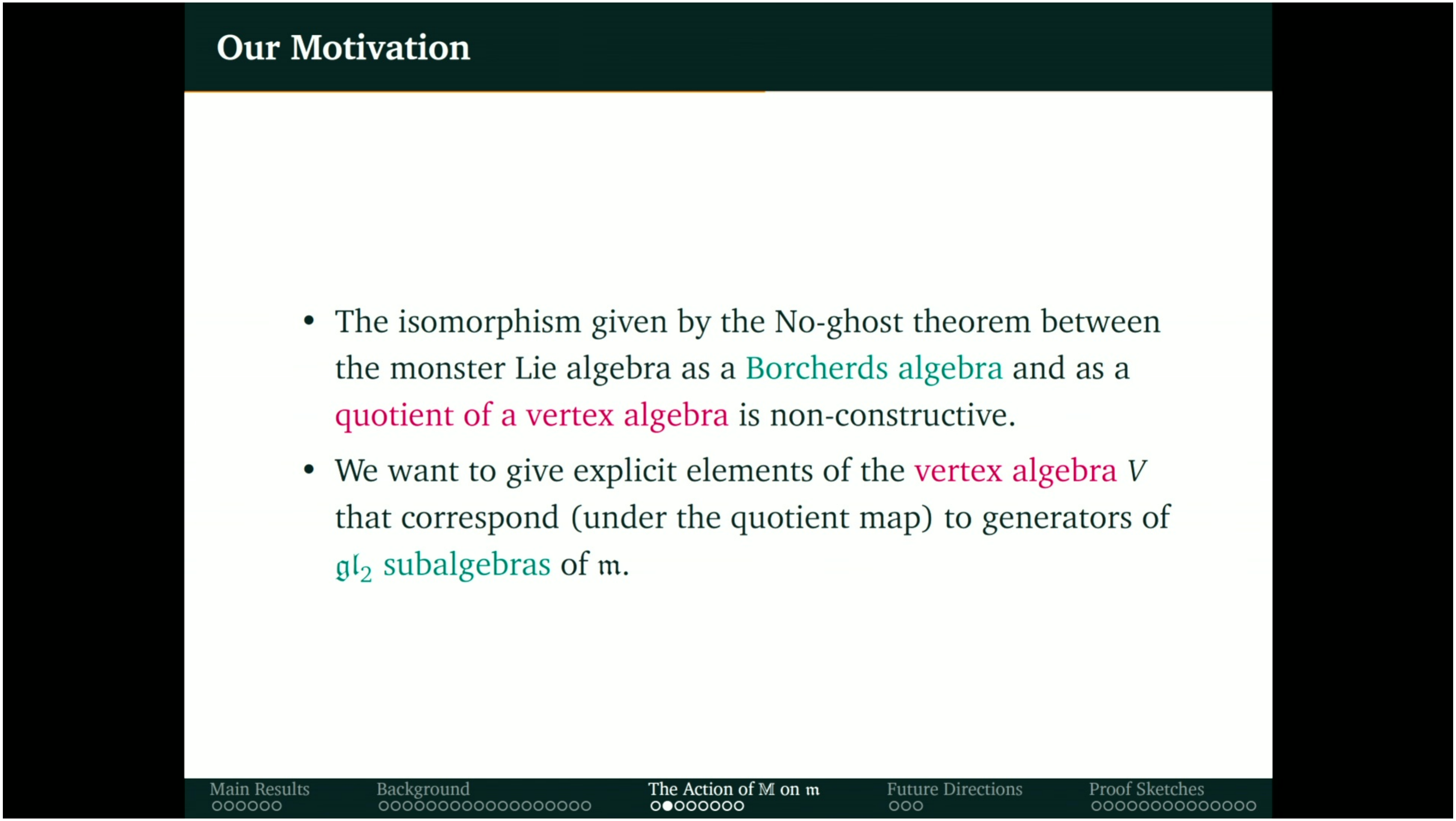
He used the fact that $\mathfrak{m}$ is a GKMA to write down the twisted denominator formulas for $V^\natural$:

$$p^{-1}\exp\left(-\sum_{i>0}\sum_{\substack{m>0\\,n\in\mathbb{Z}}}\mathrm{trace}(g^i|V_{mn})p^{mi}q^{ni}/i\right)=T_g^\natural(p)-T_g^\natural(q)$$

In particular, for $g=e$, this is just the J function identity!

$$p^{-1}\exp\left(-\sum_{i>0}\sum_{\substack{m>0,\\ n\in\mathbb{Z}}}c(mn)p^{mi}q^{ni}/i\right)=J(p)-J(q)$$

The Action of $\mathbb{M}$ on $\mathfrak{m}$



Our Motivation

- The isomorphism given by the No-ghost theorem between the monster Lie algebra as a Borcherds algebra and as a quotient of a vertex algebra is non-constructive.
- We want to give explicit elements of the vertex algebra V that correspond (under the quotient map) to generators of $\mathfrak{gl}_2$ subalgebras of $\mathfrak{m}$.

Action of $\mathbb{M}$

Jurisich, Lepowsky and Wilson described the $\mathfrak{gl}_2$ subalgebra corresponding to the unique real simple root $(1,-1)$ in terms of elements of $\mathfrak{m} \simeq P_1/R$:

Lemma 3 (Jurisich-Lepowsky-Wilson).

Let $a, b \in \hat{L}$ such that $\bar{a} = (1,1)$ and $\bar{b} = (1,-1)$. Then

$$e = \mathbf{1} \otimes \iota(b) + R, \qquad f = \mathbf{1} \otimes \iota(b^{-1}) + R,$$
$$h = \mathbf{1} \otimes \bar{b}(-1)\iota(1) + R, \qquad z = \mathbf{1} \otimes \bar{a}(-1)\iota(1) + R$$

are a basis for a $\mathfrak{gl}_2$ subalgebra $\mathfrak{gl}_2(-1)$ in $\mathfrak{m}$.

The vertex algebraic description of this subalgebra makes it easy to see that $\mathbb{M}$ acts trivially on it.

Our Work

Fix $n \in \{-1\} \cup \mathbb{N}$, $t_1 = (0,-1)$, $t_2 = (-1,0)$, and $c_n \in \widehat{L}$ such that $\overline{c_n} = (1,n)$.

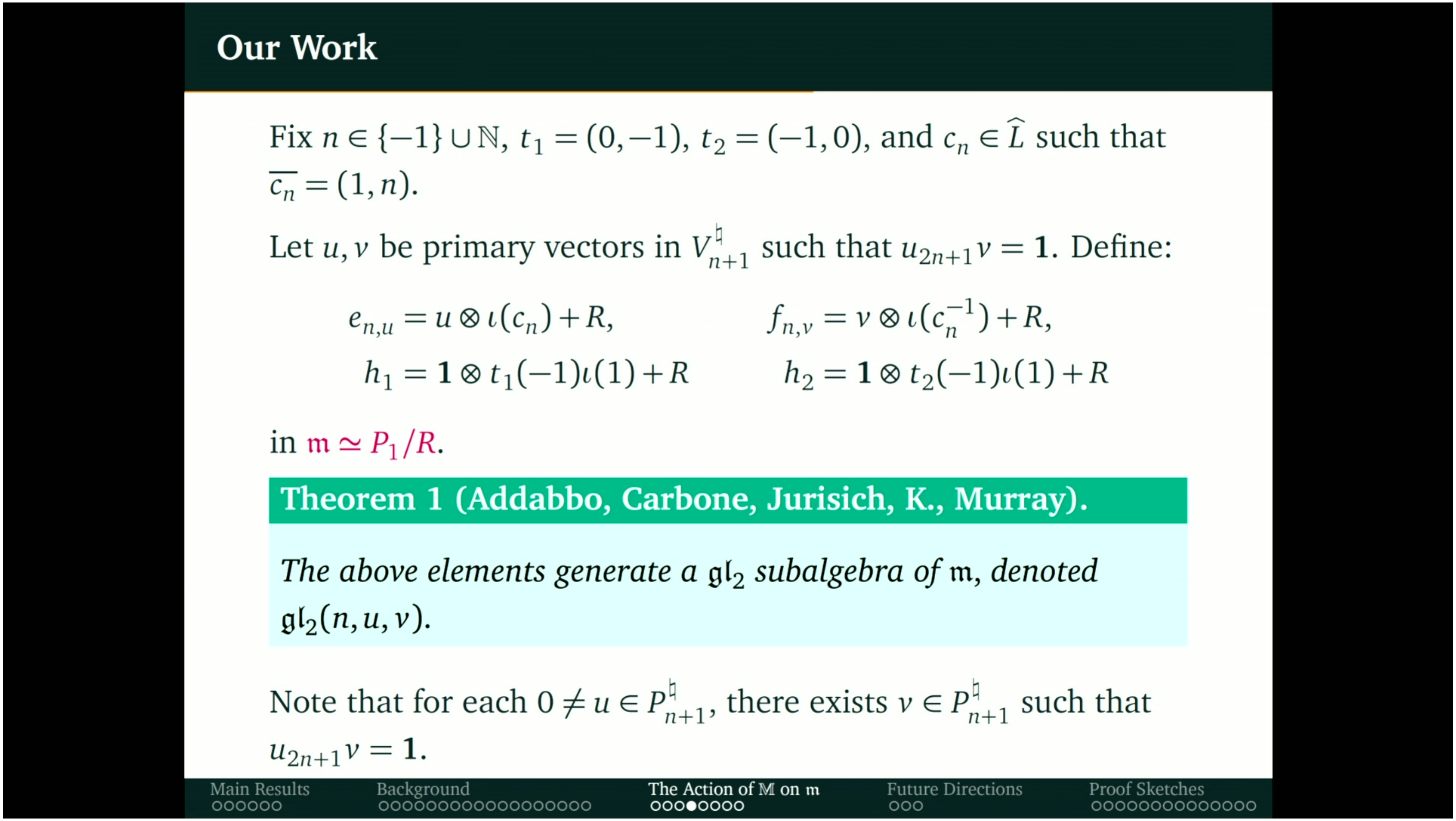
Let u, v be primary vectors in $V^{\natural}_{n+1}$ such that $u_{2n+1}v = \mathbf{1}$. Define:

$$
\begin{aligned}
e_{n,u} &= u \otimes \iota(c_n) + R, & f_{n,v} &= v \otimes \iota(c_n^{-1}) + R,\\
h_1 &= \mathbf{1} \otimes t_1(-1)\iota(1) + R & h_2 &= \mathbf{1} \otimes t_2(-1)\iota(1) + R
\end{aligned}
$$

in $\mathfrak{m} \simeq P_1/R$.

Theorem 1 (Addabbo, Carbone, Jurisich, K., Murray).

The above elements generate a $\mathfrak{gl}_2$ subalgebra of $\mathfrak{m}$, denoted $\mathfrak{gl}_2(n,u,v)$.



Our Work

Fix $n \in \{-1\} \cup \mathbb{N}$, $t_1 = (0,-1)$, $t_2 = (-1,0)$, and $c_n \in \widehat{L}$ such that $\overline{c_n} = (1,n)$.

Let u, v be primary vectors in $V^{\natural}_{n+1}$ such that $u_{2n+1} v = \mathbf{1}$. Define:

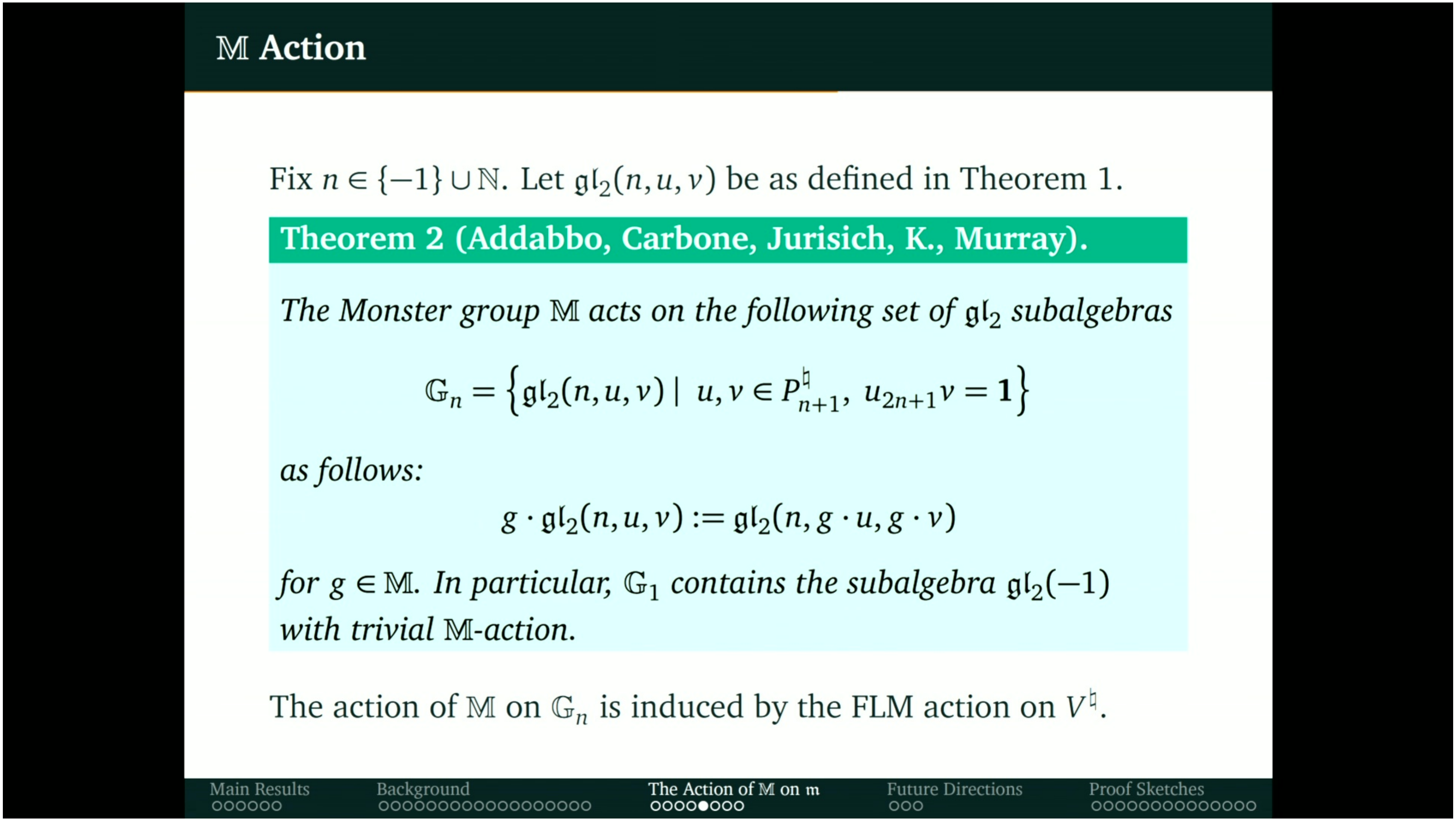
$$\begin{aligned} e_{n,u} &= u \otimes \iota(c_n) + R, & f_{n,v} &= v \otimes \iota(c_n^{-1}) + R, \\ h_1 &= \mathbf{1} \otimes t_1(-1)\iota(1) + R & h_2 &= \mathbf{1} \otimes t_2(-1)\iota(1) + R \end{aligned}$$

in $\mathfrak{m} \simeq P_1/R$.

Theorem 1 (Addabbo, Carbone, Jurisich, K., Murray).

The above elements generate a $\mathfrak{gl}_2$ subalgebra of $\mathfrak{m}$, denoted $\mathfrak{gl}_2(n,u,v)$.

Note that for each $0 \neq u \in P^{\natural}_{n+1}$, there exists $v \in P^{\natural}_{n+1}$ such that $u_{2n+1} v = \mathbf{1}$.



$\mathbb{M}$ Action

Fix $n \in \{-1\} \cup \mathbb{N}$. Let $\mathfrak{gl}_2(n,u,v)$ be as defined in Theorem 1.

Theorem 2 (Addabbo, Carbone, Jurisich, K., Murray).

The Monster group $\mathbb{M}$ acts on the following set of $\mathfrak{gl}_2$ subalgebras

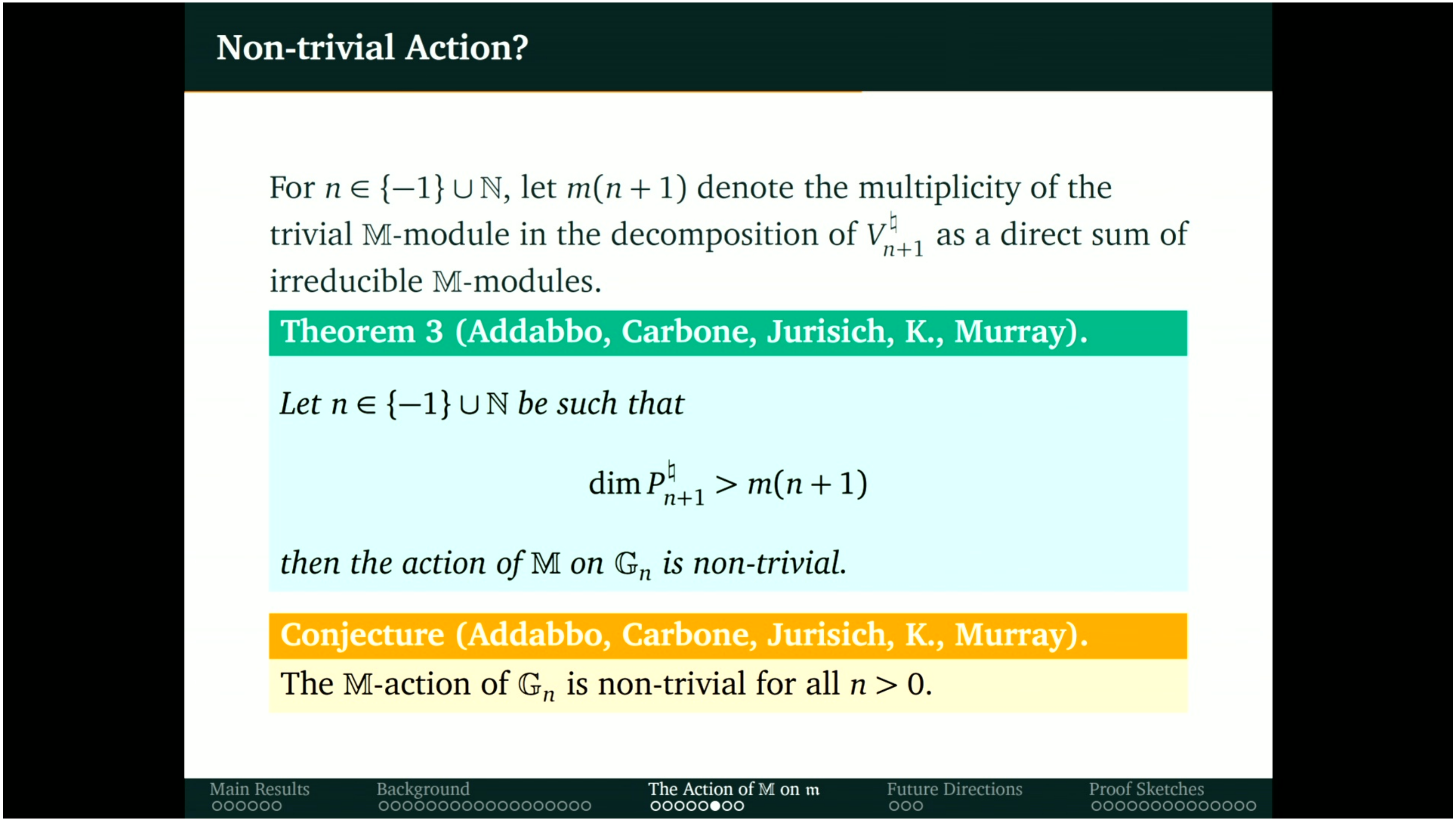
$$\mathbb{G}_n = \left\{ \mathfrak{gl}_2(n,u,v) \mid \ u,v \in P^{\natural}_{n+1},\ u_{2n+1}v = \mathbf{1} \right\}$$

as follows:

$$g \cdot \mathfrak{gl}_2(n,u,v) := \mathfrak{gl}_2(n, g \cdot u, g \cdot v)$$

for $g \in \mathbb{M}$. In particular, $\mathbb{G}_1$ contains the subalgebra $\mathfrak{gl}_2(-1)$ with trivial $\mathbb{M}$-action.

The action of $\mathbb{M}$ on $\mathbb{G}_n$ is induced by the FLM action on $V^{\natural}$.



Non-trivial Action?

For $n \in \{-1\} \cup \mathbb{N}$, let $m(n+1)$ denote the multiplicity of the trivial $\mathbb{M}$-module in the decomposition of $V^{\natural}_{n+1}$ as a direct sum of irreducible $\mathbb{M}$-modules.

Theorem 3 (Addabbo, Carbone, Jurisich, K., Murray).

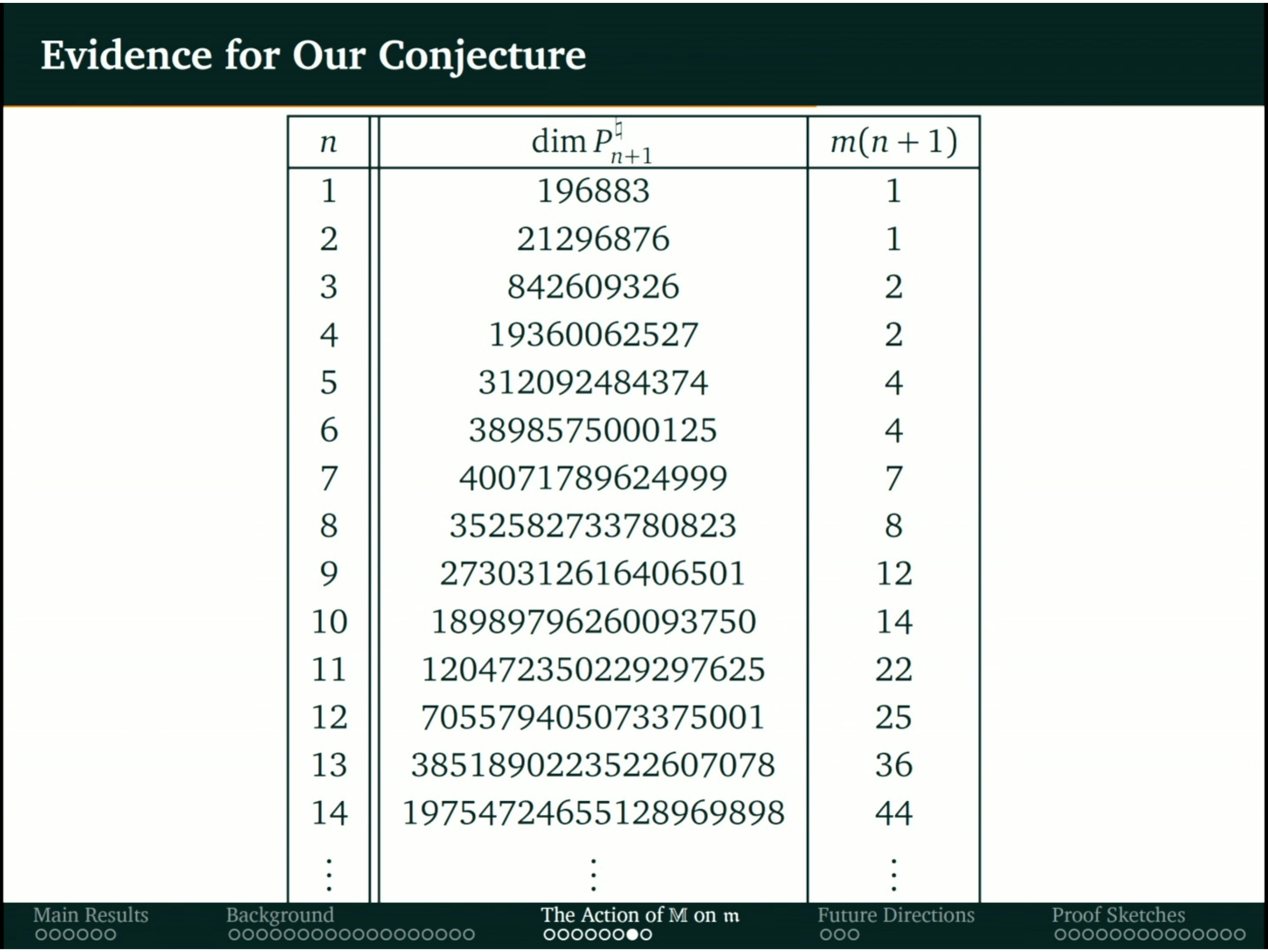
Let $n \in \{-1\} \cup \mathbb{N}$ *be such that*

$$\dim P^{\natural}_{n+1} > m(n+1)$$

then the action of $\mathbb{M}$ *on* $\mathbb{G}_n$ *is non-trivial.*

Conjecture (Addabbo, Carbone, Jurisich, K., Murray).

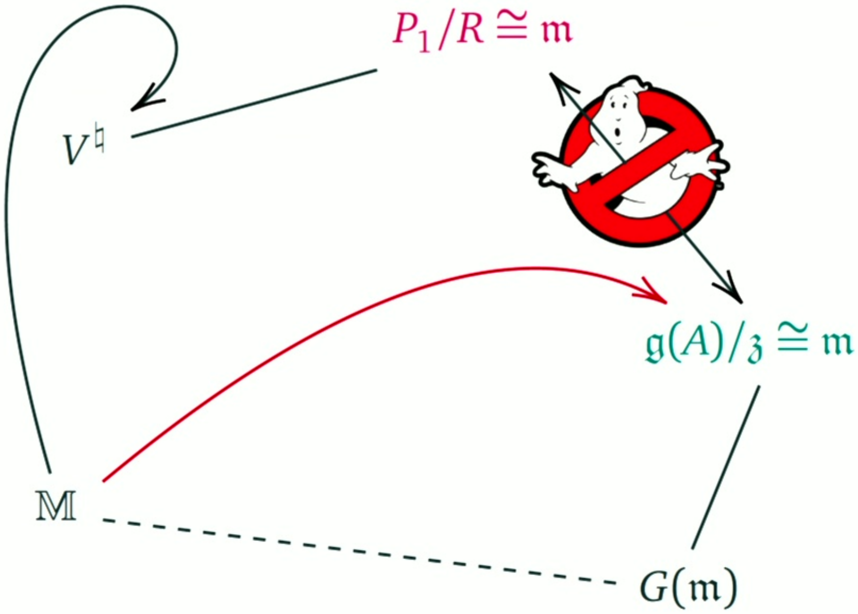
The $\mathbb{M}$-action of $\mathbb{G}_n$ is non-trivial for all $n > 0$.



Evidence for Our Conjecture

n	$\dim P^{\natural}_{n+1}$	$m(n+1)$
1	196883	1
2	21296876	1
3	842609326	2
4	19360062527	2
5	312092484374	4
6	3898575000125	4
7	40071789624999	7
8	352582733780823	8
9	2730312616406501	12
10	18989796260093750	14
11	120472350229297625	22
12	705579405073375001	25
13	3851890223522607078	36
14	19754724655128969898	44
⋮	⋮	⋮

Summary



$P_1/R \cong \mathfrak{m}$

$V^\natural$

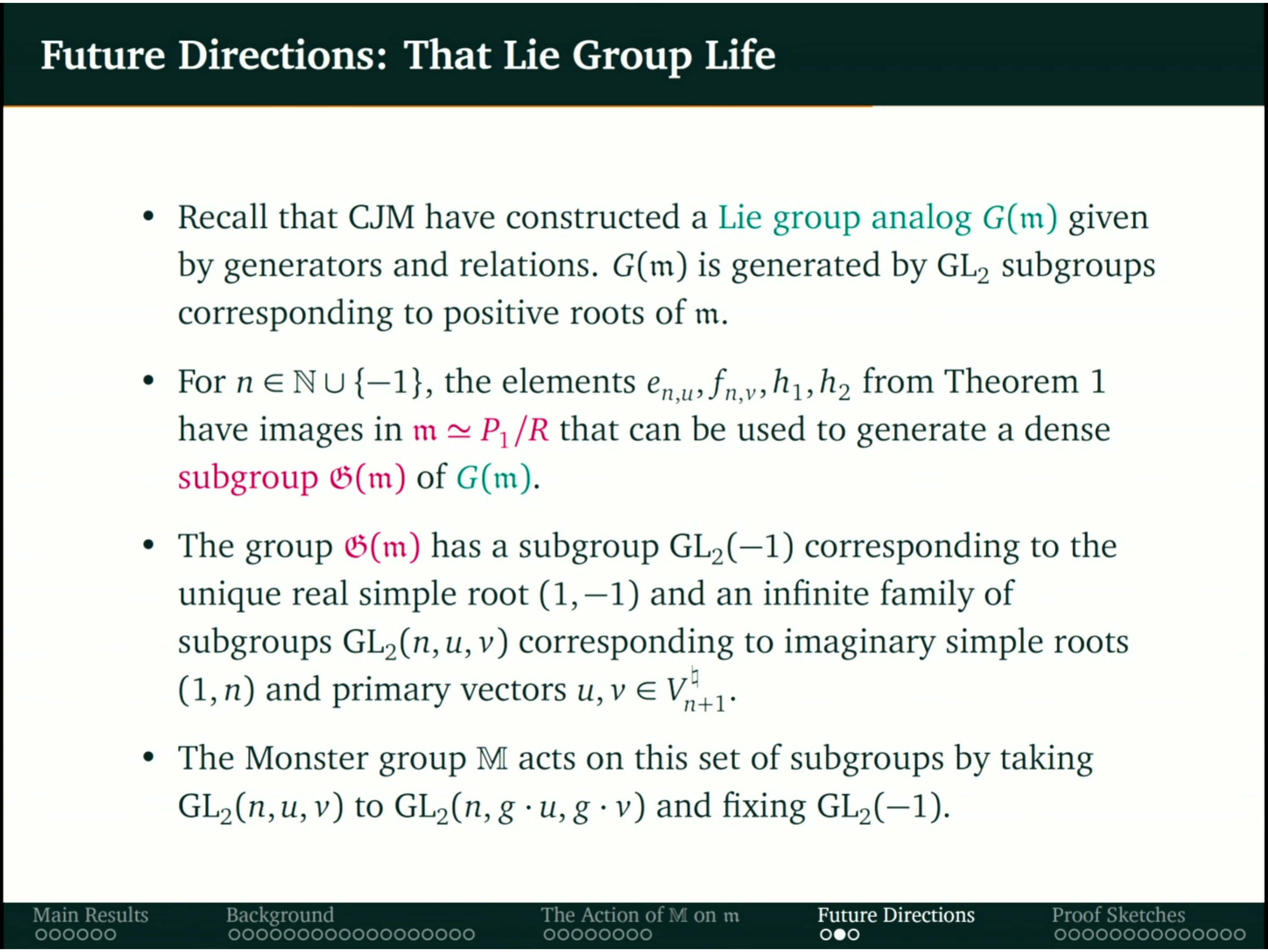
$\mathfrak{g}(A)/\mathfrak{z} \cong \mathfrak{m}$

$\mathbb{M}$

$G(\mathfrak{m})$

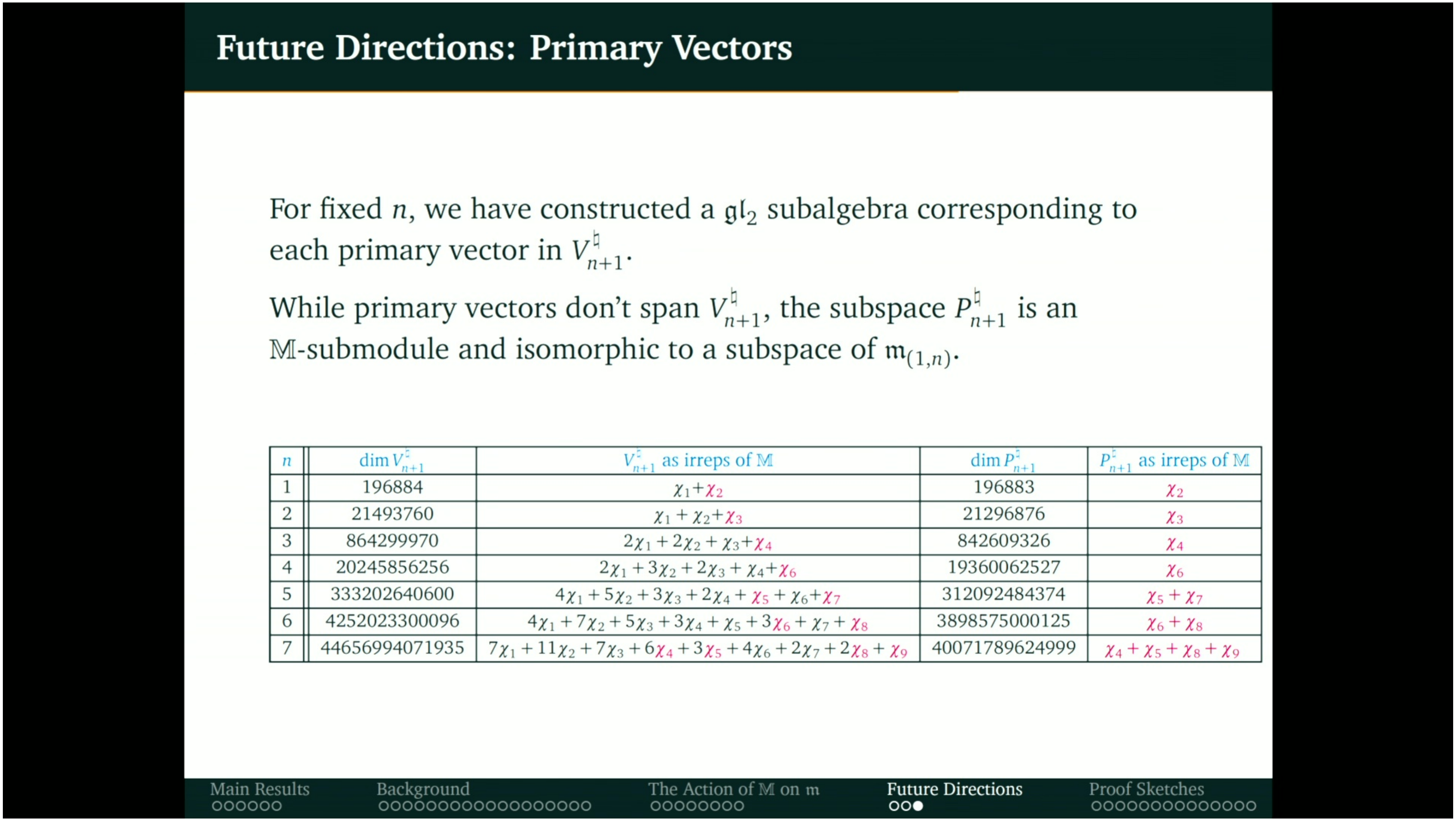
Future Directions

Future Directions



Future Directions: That Lie Group Life

- Recall that CJM have constructed a Lie group analog $G(\mathfrak{m})$ given by generators and relations. $G(\mathfrak{m})$ is generated by GL_2 subgroups corresponding to positive roots of $\mathfrak{m}$.
- For $n \in \mathbb{N} \cup \{-1\}$, the elements $e_{n,u}, f_{n,v}, h_1, h_2$ from Theorem 1 have images in $\mathfrak{m} \simeq P_1/R$ that can be used to generate a dense subgroup $\mathfrak{G}(\mathfrak{m})$ of $G(\mathfrak{m})$.
- The group $\mathfrak{G}(\mathfrak{m})$ has a subgroup $\mathrm{GL}_2(-1)$ corresponding to the unique real simple root $(1,-1)$ and an infinite family of subgroups $\mathrm{GL}_2(n,u,v)$ corresponding to imaginary simple roots $(1,n)$ and primary vectors $u, v \in V^{\natural}_{n+1}$.
- The Monster group $\mathbb{M}$ acts on this set of subgroups by taking $\mathrm{GL}_2(n,u,v)$ to $\mathrm{GL}_2(n, g \cdot u, g \cdot v)$ and fixing $\mathrm{GL}_2(-1)$.



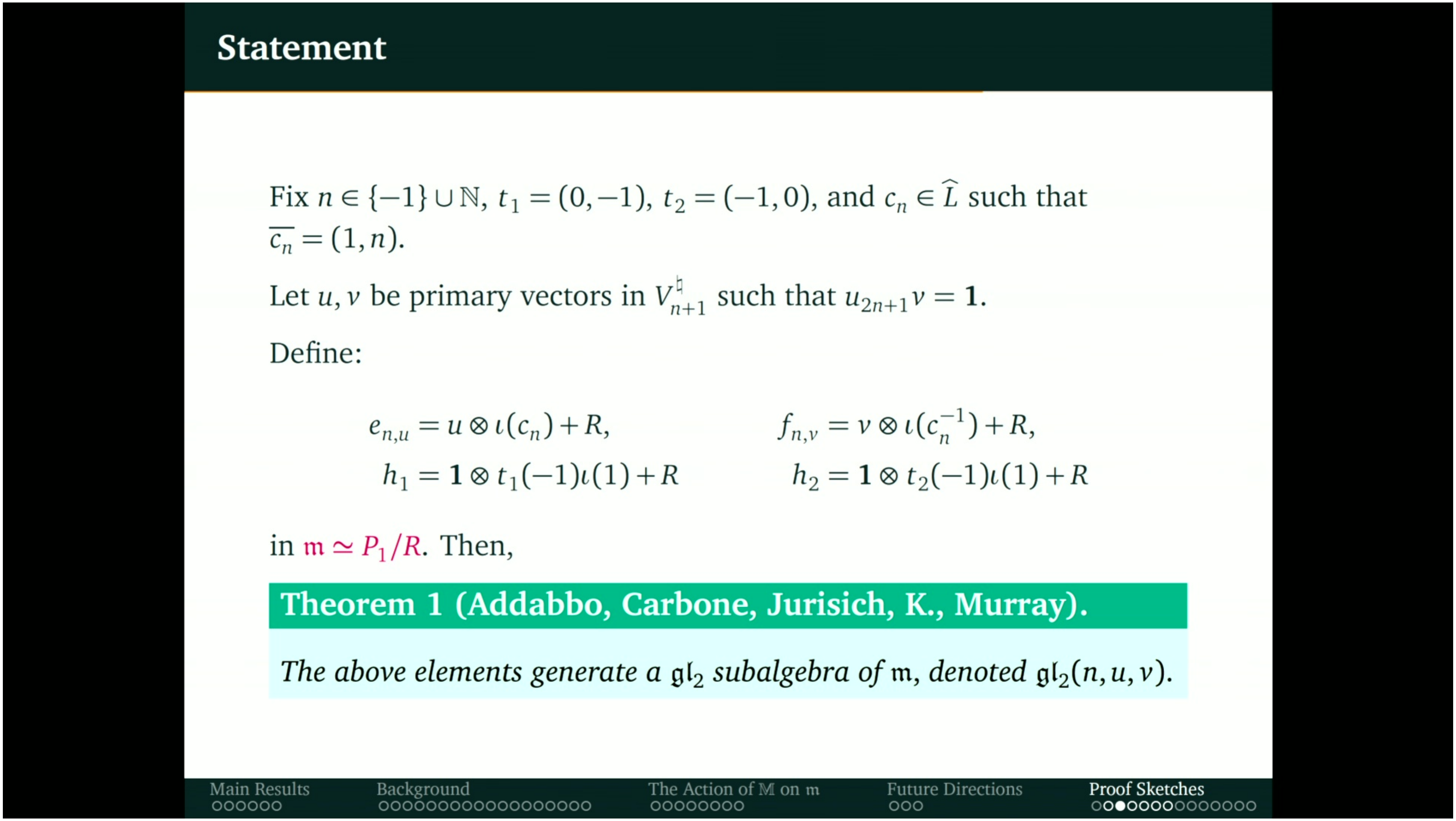
Future Directions: Primary Vectors

For fixed n, we have constructed a $\mathfrak{gl}_2$ subalgebra corresponding to each primary vector in $V^{\natural}_{n+1}$.

While primary vectors don't span $V^{\natural}_{n+1}$, the subspace $P^{\natural}_{n+1}$ is an $\mathbb{M}$-submodule and isomorphic to a subspace of $\mathfrak{m}_{(1,n)}$.

n	$\dim V^{\natural}_{n+1}$	$V^{\natural}_{n+1}$ as irreps of $\mathbb{M}$	$\dim P^{\natural}_{n+1}$	$P^{\natural}_{n+1}$ as irreps of $\mathbb{M}$
1	196884	$\chi_1+\chi_2$	196883	χ_2
2	21493760	$\chi_1+\chi_2+\chi_3$	21296876	χ_3
3	864299970	$2\chi_1+2\chi_2+\chi_3+\chi_4$	842609326	χ_4
4	20245856256	$2\chi_1+3\chi_2+2\chi_3+\chi_4+\chi_6$	19360062527	χ_6
5	333202640600	$4\chi_1+5\chi_2+3\chi_3+2\chi_4+\chi_5+\chi_6+\chi_7$	312092484374	$\chi_5+\chi_7$
6	4252023300096	$4\chi_1+7\chi_2+5\chi_3+3\chi_4+\chi_5+3\chi_6+\chi_7+\chi_8$	3898575000125	$\chi_6+\chi_8$
7	44656994071935	$7\chi_1+11\chi_2+7\chi_3+6\chi_4+3\chi_5+4\chi_6+2\chi_7+2\chi_8+\chi_9$	40071789624999	$\chi_4+\chi_5+\chi_8+\chi_9$

Proof Sketches



Statement

Fix $n \in \{-1\} \cup \mathbb{N}$, $t_1 = (0,-1)$, $t_2 = (-1,0)$, and $c_n \in \widehat{L}$ such that $\overline{c_n} = (1,n)$.

Let u, v be primary vectors in $V^{\natural}_{n+1}$ such that $u_{2n+1}v = \mathbf{1}$.

Define:

$$\begin{aligned} e_{n,u} &= u \otimes \iota(c_n) + R, & f_{n,v} &= v \otimes \iota(c_n^{-1}) + R, \\ h_1 &= \mathbf{1} \otimes t_1(-1)\iota(1) + R & h_2 &= \mathbf{1} \otimes t_2(-1)\iota(1) + R \end{aligned}$$

in $\mathfrak{m} \simeq P_1/R$. Then,

Theorem 1 (Addabbo, Carbone, Jurisich, K., Murray).

The above elements generate a $\mathfrak{gl}_2$ subalgebra of $\mathfrak{m}$, denoted $\mathfrak{gl}_2(n,u,v)$.

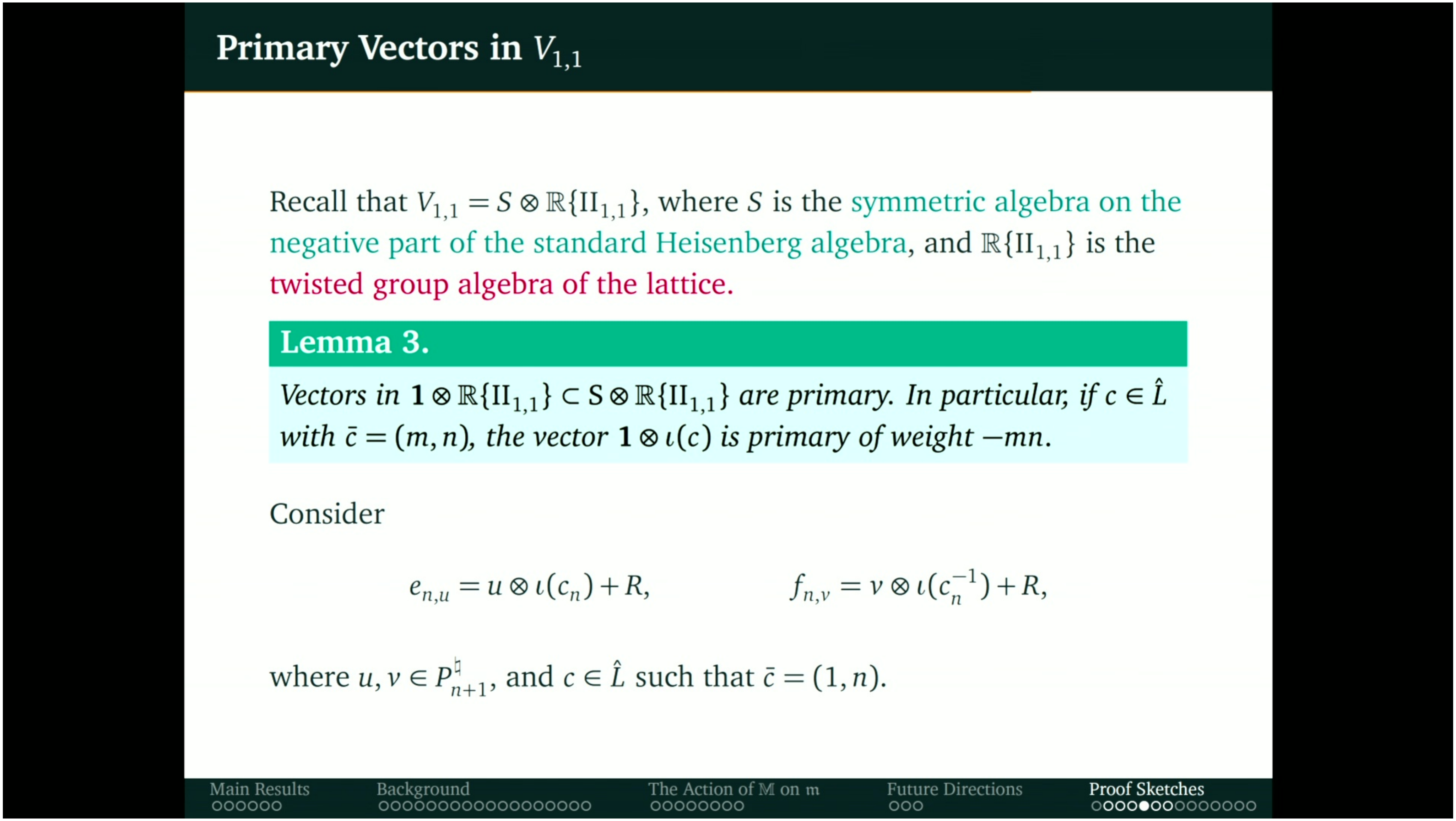
Idea

We are looking for primary vectors of weight 1 in $V^\natural \otimes V_{1,1}$ whose projection under the quotient map generates a copy of $\mathfrak{gl}_2 \subset \mathfrak{m}$.

We make use of the following lemma:

Lemma 2.

If x in $V^\natural$ is primary of weight n and y in $V_{1,1}$ is primary of weight m then $x \otimes y$ is primary in $V^\natural \otimes V_{1,1}$ of weight $n+m$.



Primary Vectors in $V_{1,1}$

Recall that $V_{1,1} = S \otimes \mathbb{R}\{\mathrm{II}_{1,1}\}$, where S is the symmetric algebra on the negative part of the standard Heisenberg algebra, and $\mathbb{R}\{\mathrm{II}_{1,1}\}$ is the twisted group algebra of the lattice.

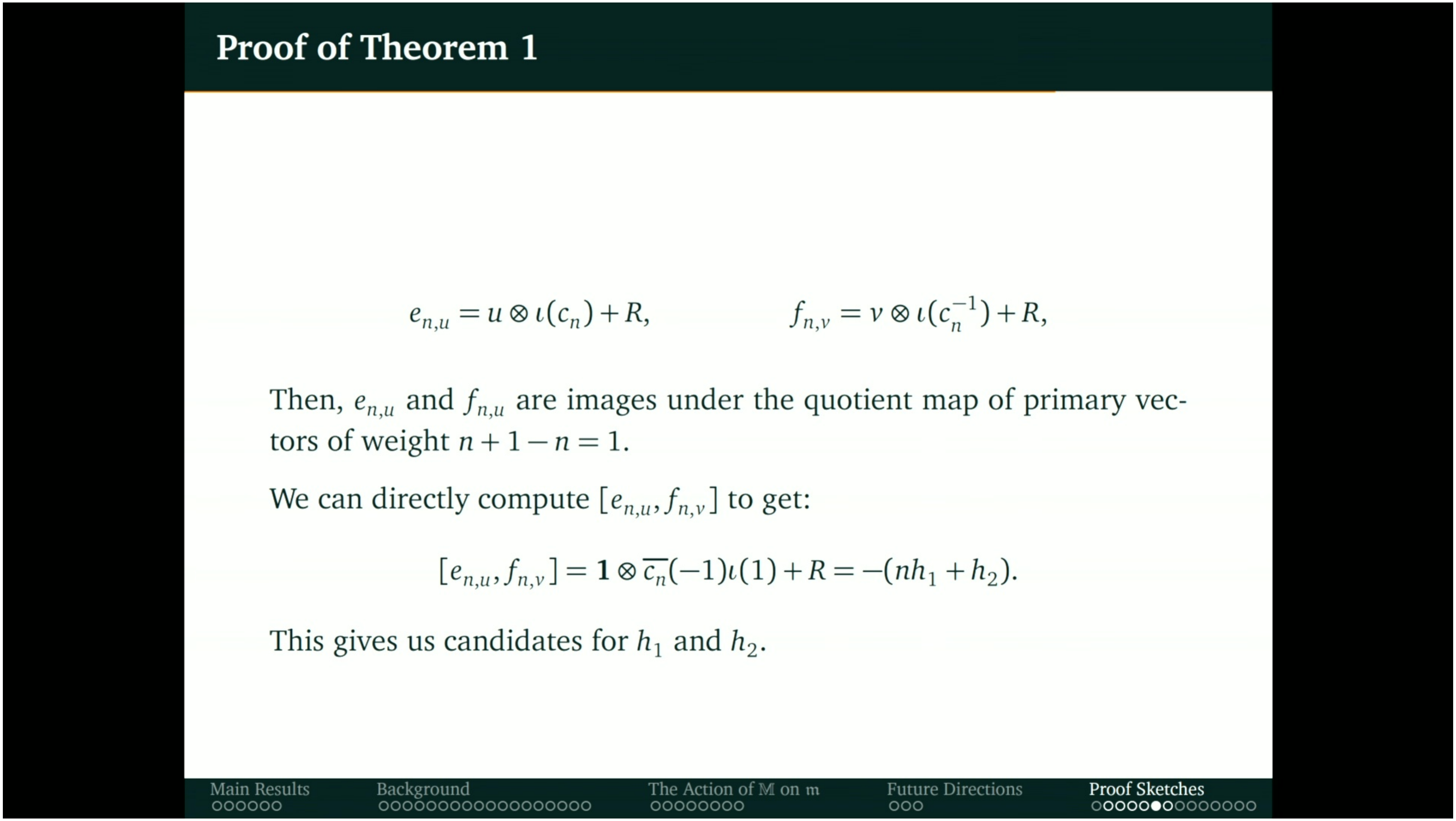
Lemma 3.

Vectors in $\mathbf{1} \otimes \mathbb{R}\{\mathrm{II}_{1,1}\} \subset \mathrm{S} \otimes \mathbb{R}\{\mathrm{II}_{1,1}\}$ *are primary. In particular, if* $c \in \hat{L}$ *with* $\bar{c} = (m,n)$*, the vector* $\mathbf{1} \otimes \iota(c)$ *is primary of weight* $-mn$*.*

Consider

$$e_{n,u} = u \otimes \iota(c_n) + R, \qquad f_{n,v} = v \otimes \iota(c_n^{-1}) + R,$$

where $u, v \in P^{\natural}_{n+1}$, and $c \in \hat{L}$ such that $\bar{c} = (1,n)$.



Proof of Theorem 1

$$e_{n,u} = u \otimes \iota(c_n) + R, \qquad f_{n,v} = v \otimes \iota(c_n^{-1}) + R,$$

Then, $e_{n,u}$ and $f_{n,u}$ are images under the quotient map of primary vectors of weight $n+1-n=1$.

We can directly compute $[e_{n,u}, f_{n,v}]$ to get:

$$[e_{n,u}, f_{n,v}] = \mathbf{1} \otimes \overline{c_n}(-1)\iota(1) + R = -(nh_1 + h_2).$$

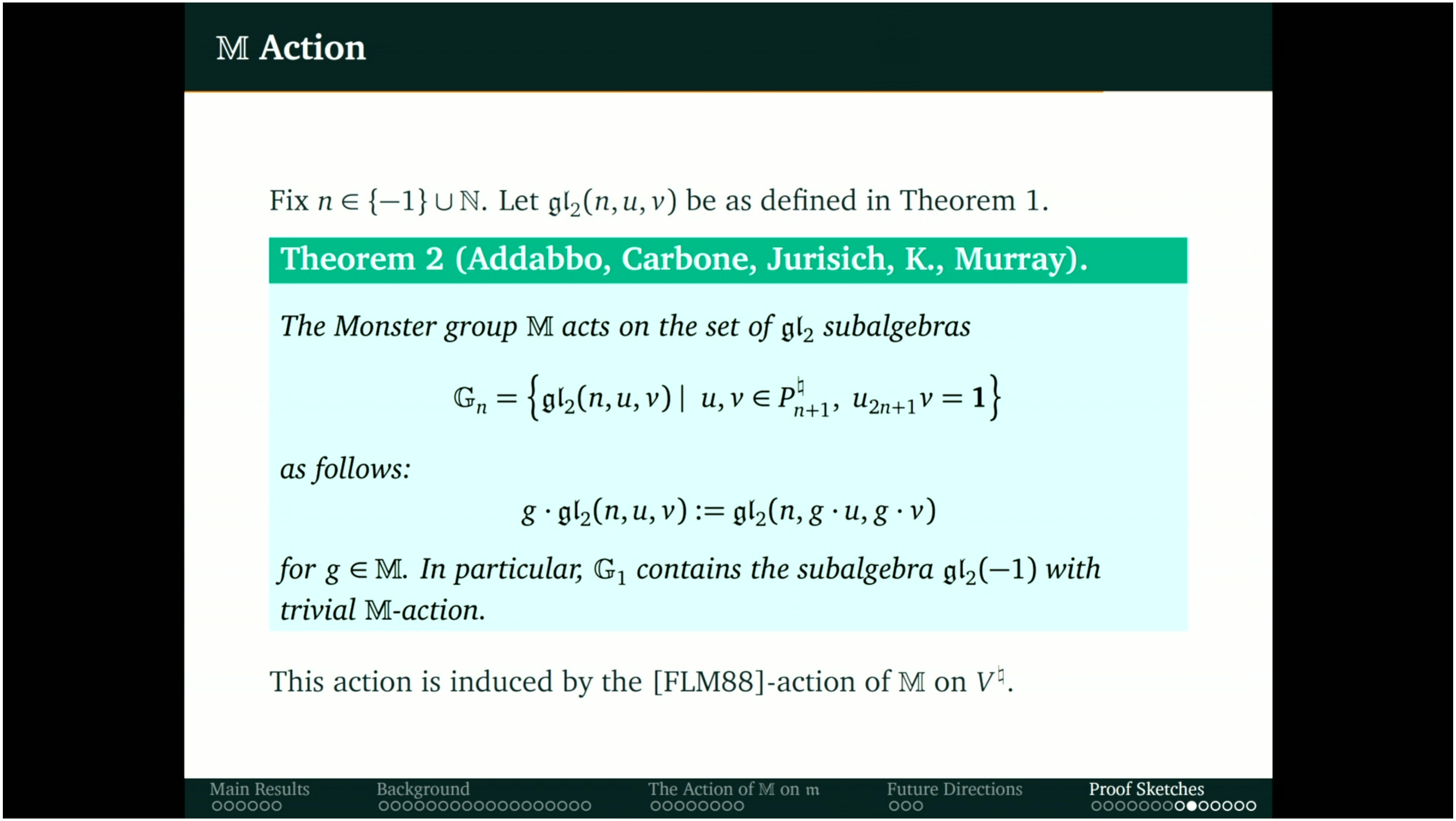
This gives us candidates for h_1 and h_2.

QED

A simple vertex algebra computation then shows that

$$\begin{aligned} [e_{n,u}, f_{n,v}] &= -(nh_1 + h_2) & [h_1, e_{n,u}] &= e_{n,u} \\ [h_2, e_{n,u}] &= ne_{n,u} & [h_1, f_{n,v}] &= -f_{n,v} \\ [h_2, f_{n,v}] &= -nf_{n,v} & [h_1, h_2] &= 0. \end{aligned}$$

that is, the elements $\{e_{n,u}, f_{n,v}, h_1, h_2\}$ satisfy (a subset of) the defining relations of $\mathfrak{m}$, and generate a lie algebra isomorphic to $\mathfrak{gl}_2$.



$\mathbb{M}$ Action

Fix $n \in \{-1\} \cup \mathbb{N}$. Let $\mathfrak{gl}_2(n,u,v)$ be as defined in Theorem 1.

Theorem 2 (Addabbo, Carbone, Jurisich, K., Murray).

The Monster group $\mathbb{M}$ acts on the set of $\mathfrak{gl}_2$ subalgebras

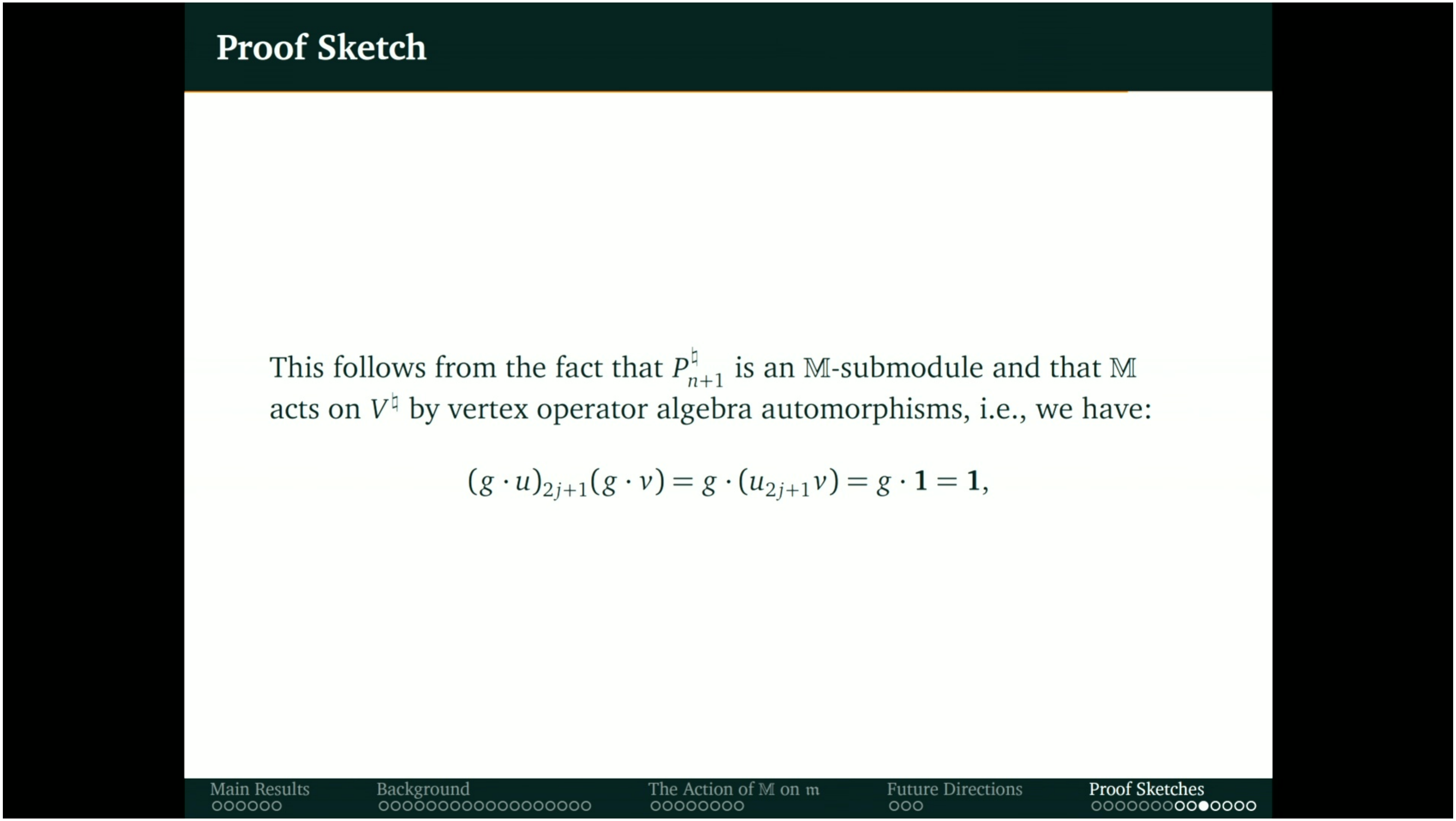
$$\mathbb{G}_n = \left\{ \mathfrak{gl}_2(n,u,v) \mid \ u,v \in P^{\natural}_{n+1},\ u_{2n+1}v = \mathbf{1} \right\}$$

as follows:

$$g \cdot \mathfrak{gl}_2(n,u,v) := \mathfrak{gl}_2(n, g\cdot u, g \cdot v)$$

for $g \in \mathbb{M}$. In particular, $\mathbb{G}_1$ contains the subalgebra $\mathfrak{gl}_2(-1)$ with trivial $\mathbb{M}$-action.

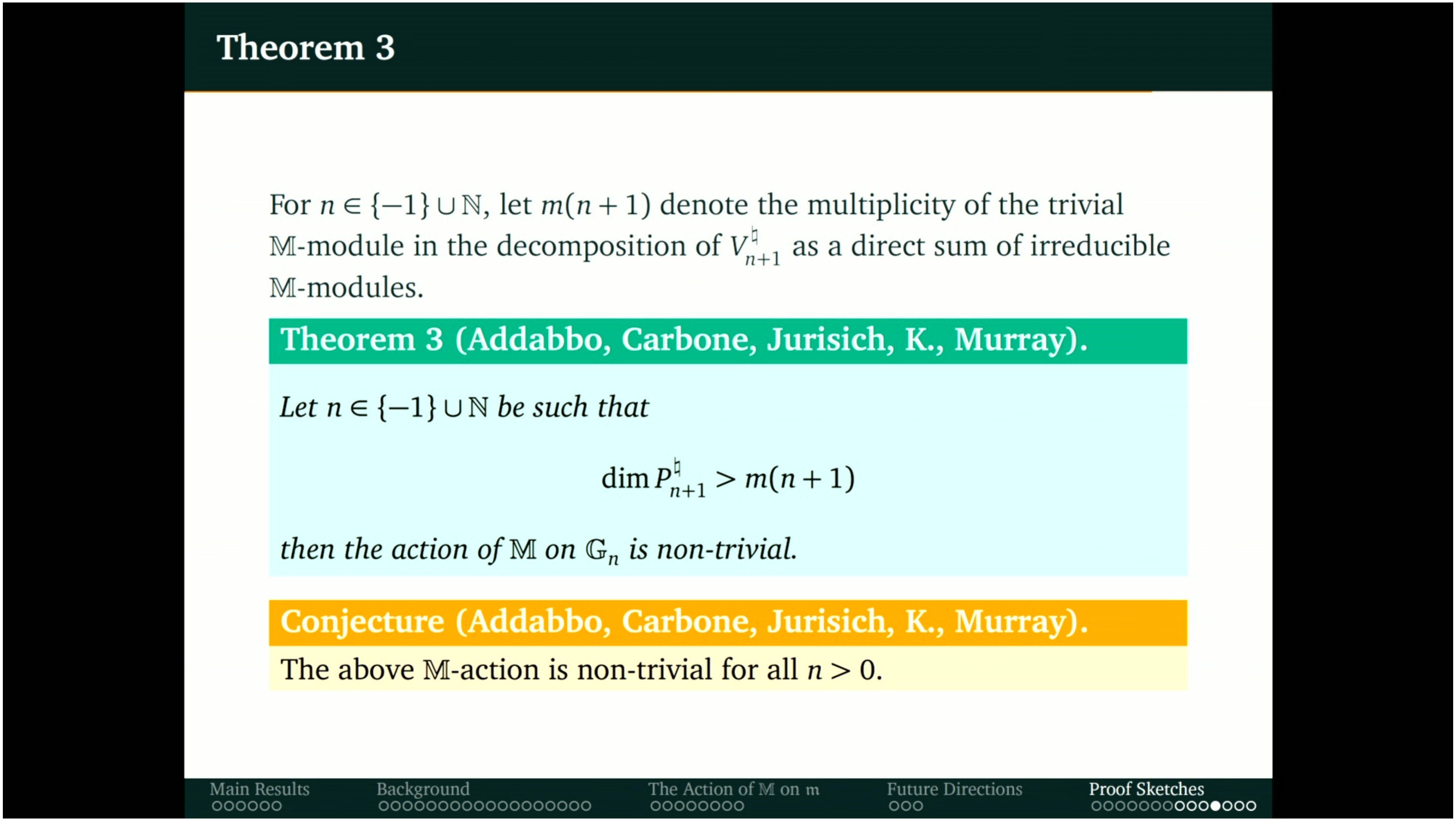
This action is induced by the [FLM88]-action of $\mathbb{M}$ on $V^{\natural}$.



Proof Sketch

This follows from the fact that $P^\natural_{n+1}$ is an $\mathbb{M}$-submodule and that $\mathbb{M}$ acts on $V^\natural$ by vertex operator algebra automorphisms, i.e., we have:

$$(g \cdot u)_{2j+1}(g \cdot v) = g \cdot (u_{2j+1}v) = g \cdot \mathbf{1} = \mathbf{1},$$



Theorem 3

For $n \in \{-1\} \cup \mathbb{N}$, let $m(n+1)$ denote the multiplicity of the trivial $\mathbb{M}$-module in the decomposition of $V^{\natural}_{n+1}$ as a direct sum of irreducible $\mathbb{M}$-modules.

Theorem 3 (Addabbo, Carbone, Jurisich, K., Murray).

Let $n \in \{-1\} \cup \mathbb{N}$ *be such that*

$$\dim P^{\natural}_{n+1} > m(n+1)$$

then the action of $\mathbb{M}$ *on* $\mathbb{G}_n$ *is non-trivial.*

Conjecture (Addabbo, Carbone, Jurisich, K., Murray).

The above $\mathbb{M}$-action is non-trivial for all $n > 0$.

Proof of Theorem 3

Lemma 4.

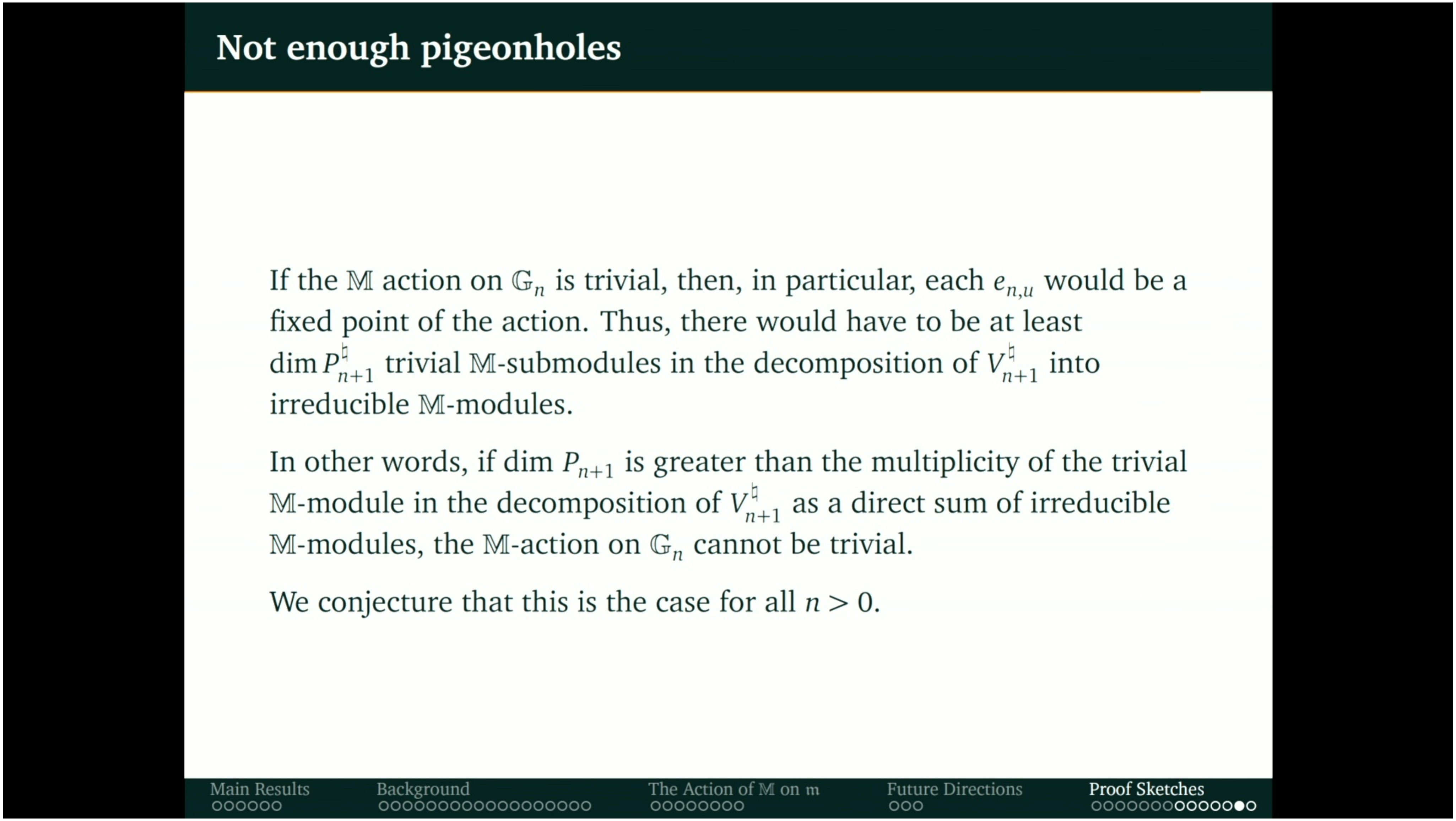
For fixed $n > 0$ and fixed $c_n \in \widehat{L}$ such that $\bar{c_n} = (1, n)$,

$$E_n := \left\{ e_{n,u} = u \otimes \iota(c_n) + R \mid u \in P^{\natural}_{n+1} \right\} \subseteq \mathfrak{m}_{(1,n)}$$

is an $\mathbb{M}$-submodule of $\mathfrak{m}_{(1,n)}$, isomorphic to $P^{\natural}_{n+1}$.

Let $e_{n,u} \in E_n$. Then, $e_{n,u} \in \mathfrak{m}_{(1,n)}$, and by the No-ghost Theorem, $\mathfrak{m}_{(1,n)} \cong V^{\natural}_{n+1}$ as $\mathbb{M}$-modules.

Recall that each $e_{n,u} \in E_n$ is a generator of $\mathfrak{gl}_2(n, u, v) \in \mathbb{G}_n$.



Not enough pigeonholes

If the $\mathbb{M}$ action on $\mathbb{G}_n$ is trivial, then, in particular, each $e_{n,u}$ would be a fixed point of the action. Thus, there would have to be at least $\dim P^\natural_{n+1}$ trivial $\mathbb{M}$-submodules in the decomposition of $V^\natural_{n+1}$ into irreducible $\mathbb{M}$-modules.

In other words, if $\dim P_{n+1}$ is greater than the multiplicity of the trivial $\mathbb{M}$-module in the decomposition of $V^\natural_{n+1}$ as a direct sum of irreducible $\mathbb{M}$-modules, the $\mathbb{M}$-action on $\mathbb{G}_n$ cannot be trivial.

We conjecture that this is the case for all $n > 0$.

Thank you for your attention!