Title: Can one region of space encode another?

Speakers: Charlie Cummings

Series: Quantum Fields and Strings

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Abstract: Using a novel version of the gravitational path integral for compact spatial regions at a moment of time symmetry, I argue that a region of space can encode a larger one. In particular, I show that the entanglement entropy of a region of space equals the area of the boundary of the smallest region that contains it. The key insight is to include the effects of the gravitational edge modes associated with the region in the path integral. This result is consistent with a recent conjecture by Bousso and Penington.

Zoom link https://pitp.zoom.us/j/93301151464?pwd=Z2t5QlpUQ3hoaEkwQlFZS2tITGpEQT09

Terrestrial Holography

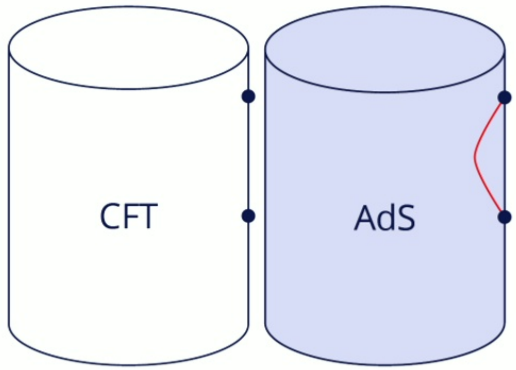
Can one region of space encode another?

Charlie Cummings

University of Pennsylvania

November 28, 2023

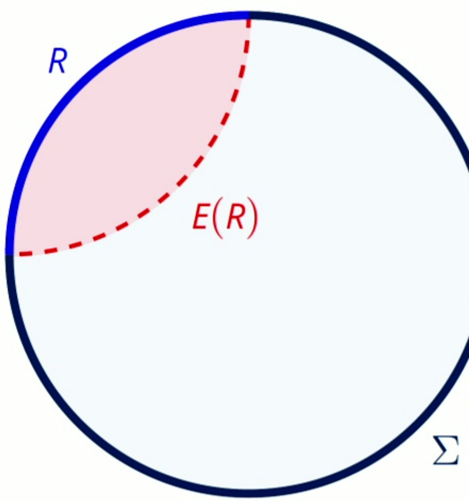
Holography



Boundary	Bulk
$\mathcal{H}_{CFT}$	$\mathcal{H}_{AdS}$
$\langle Z_{CFT} \rangle$	Z_{AdS}
Subregion R	Ent. Wedge $E(R)$
$S(\rho_R)$	$S_{gen} = \frac{A}{4G_N} + S_{out}$

Figure 1: Part of the bulk-to-boundary dictionary

The RT Formula



$$S(\rho_R) = \left[\frac{A(\chi)}{4G_N} + S_{out}(E(R))\right]$$

$$\mathcal{O} \in E(R) \rightarrow \widetilde{\mathcal{O}} \in R$$

- Satisfies crucial consistency checks:
  - Nesting:
    $$R \subset R' \rightarrow E(R) \subset E(R')$$
  - No-Cloning:
    $$R \cap R' = \emptyset \rightarrow E(R) \cap E(R') = \emptyset$$
  - Strong Subadditivity:
    $$S_{gen}(A \cup B) + S_{gen}(B \cup C) \geq S_{gen}(B) + S_{gen}(A \cup B \cup C)$$

Motivation for Bulk→Bulk Holography

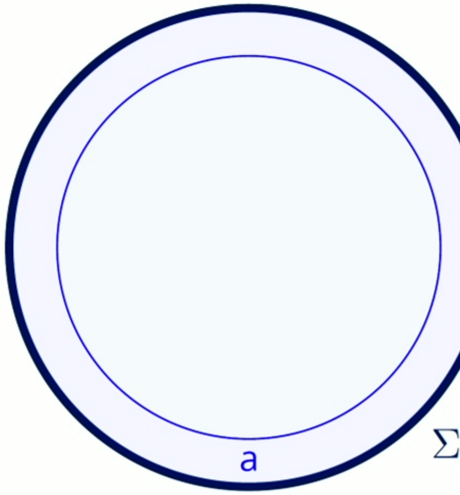
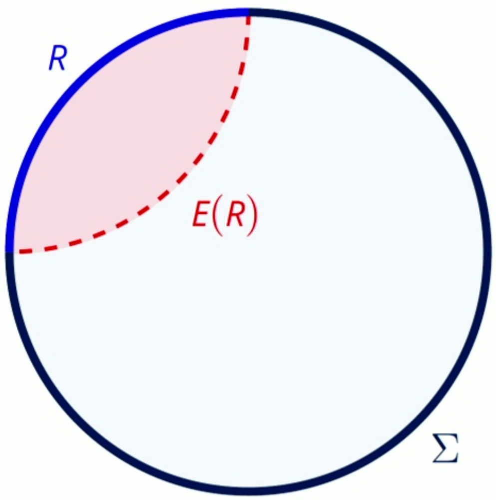


Figure 2: Subregion of AdS

The RT Formula



$$S(\rho_R) = \left[\frac{A(\chi)}{4G_N} + S_{out}(E(R))\right]$$

$$\mathcal{O} \in E(R) \to \widetilde{\mathcal{O}} \in R$$

- Derivation of RT: LM [1304.4926]
- Derivation of encoding: JLMS [1512.06431]
- Derivation of both: tensor networks [1503.06237]
- Covariant generalizations: HRT/QES [0705.0016,1408.3203]
- Beyond AdS?

Motivation for Bulk→Bulk Holography

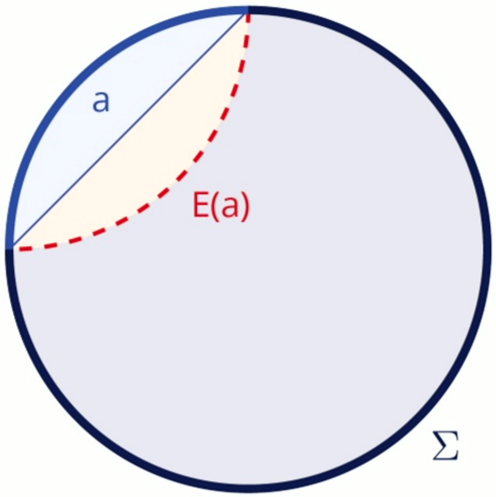
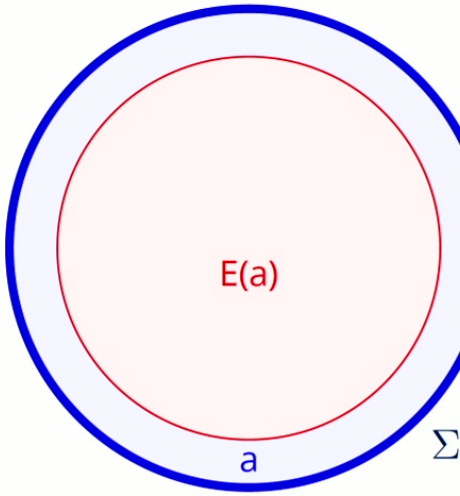


Figure 3: Smaller Subregion of AdS

Motivation for Bulk→Bulk Holography



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Motivation for Bulk→Bulk Holography

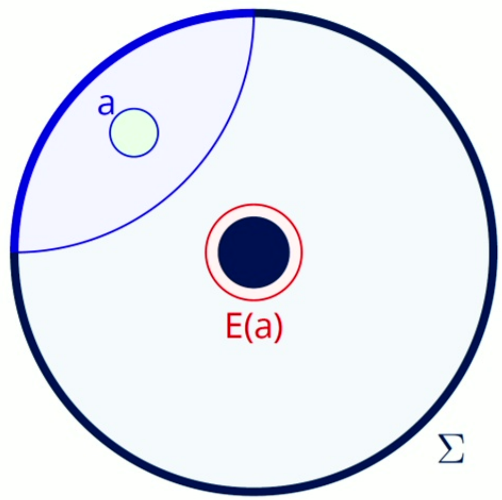


Figure 4: Evaporating Black Hole

Motivation for Bulk→Bulk Holography

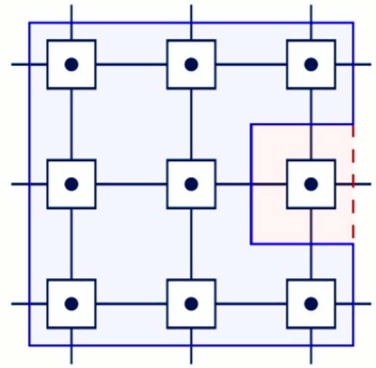
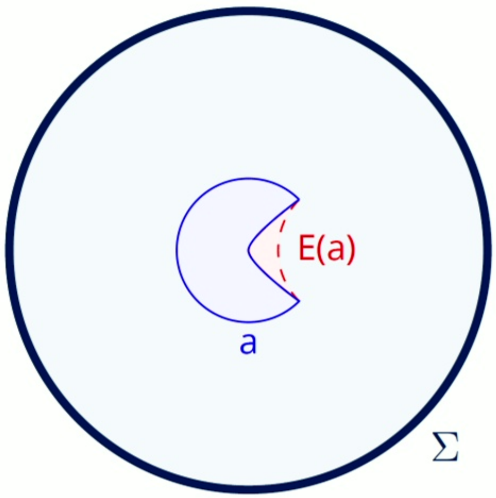


Figure 5: Tensor Network

Bousso-Penington Proposal



- $E(a)$ defined by: 2208.04993
  - $a \subset E(a)$
  - $E(a)$ minimizes $S_{gen} = \frac{A(\partial E(a))}{4G_N} + S_{out}(E(a))$
- Satisfies crucial consistency checks:
  - Reduces to RT if a is a boundary region
  - Nesting:

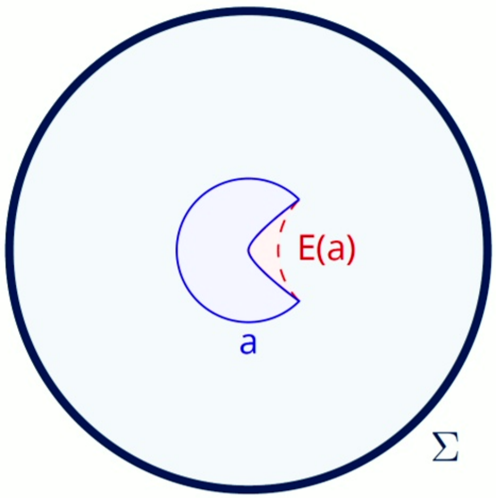
$$a \subset b \to E(a) \subset E(b)$$

  - No-Cloning:

$$a \cap E(b), b \cap E(a) = \emptyset \to E(a) \cap E(b) = \emptyset$$

  - Strong Subadditivity

Bousso-Penington Proposal



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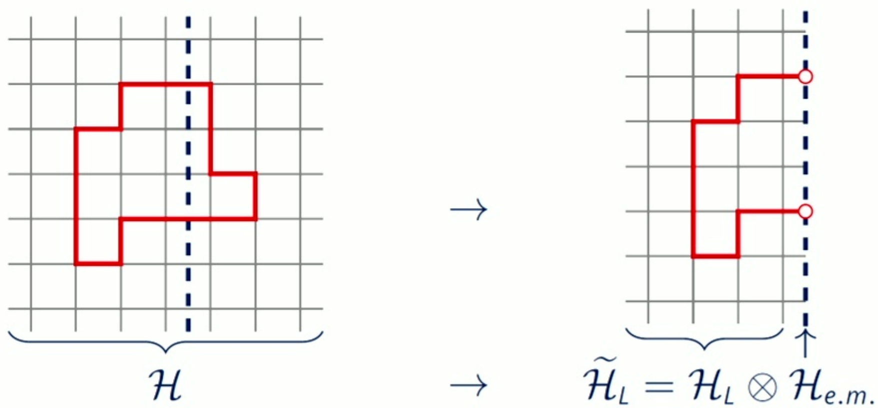
  - Strong Subadditivity
- Derivation of this formula: This talk
- The tool: gravitational edge modes.

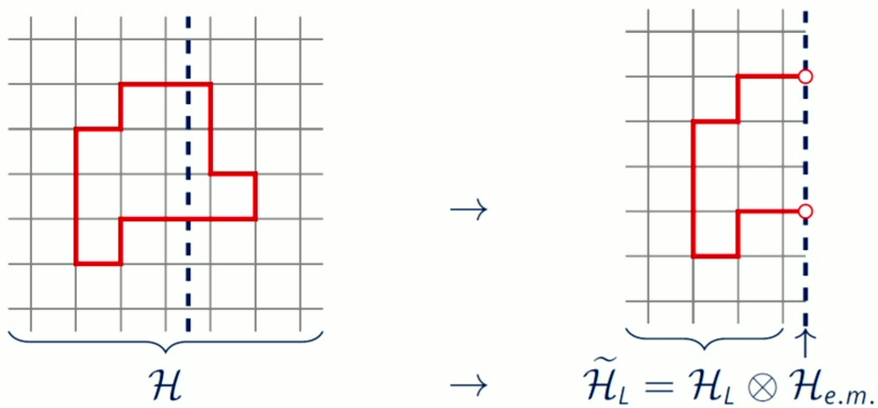
Edge Modes

- Entropy of a subregion requires bipartition $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$.
- Two reasons this could fail:
  - UV issues (removing a cut-off, type III algebra, infinite entanglement between $L/R, \cdots$)
  - IR issues (gauge symmetry, Gauss' law, constraint correlating $L/R, \cdots$)
- Viewpoint of this talk: Tame UV divergences with cutoffs and remove at the end.
- This leaves IR issues with defining the entropy: requires edge modes to resolve.

Example: $U(1)$ lattice gauge theory [1601.04744]

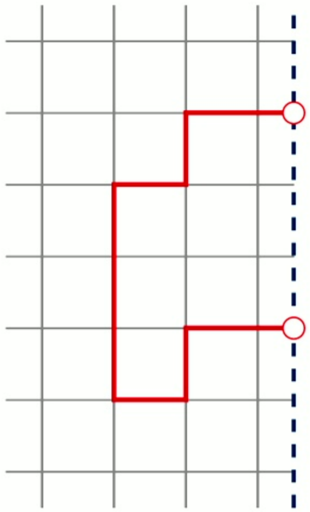


Example: $U(1)$ lattice gauge theory [1601.04744]



- $\mathcal{H}_L \otimes \mathcal{H}_R$ has no Wilson lines straddling the cut: $\mathcal{H}_L \otimes \mathcal{H}_R \subset \mathcal{H}$
- $\widetilde{\mathcal{H}}_L \otimes \widetilde{\mathcal{H}}_R$ has states with non-zero charge: $\mathcal{H} \subset \widetilde{\mathcal{H}}_L \otimes \widetilde{\mathcal{H}}_R$
- $\mathcal{H} = \widetilde{\mathcal{H}}_L \otimes \widetilde{\mathcal{H}}_R|_{Q+\bar{Q}=0}$

The Extended Hilbert Space $\widetilde{\mathcal{H}}_L$



- Edge modes $\sim$ Stueckelberg fields

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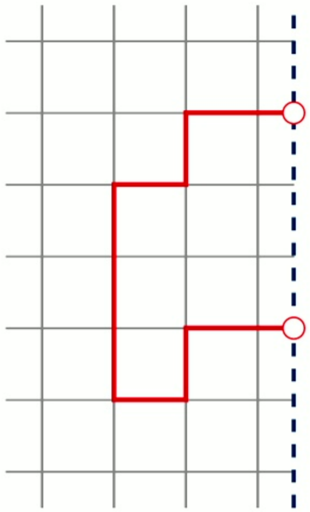
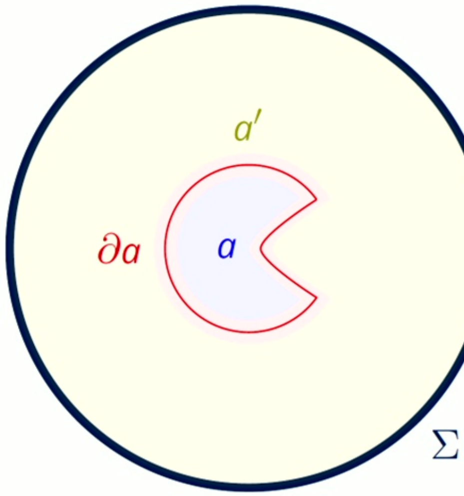


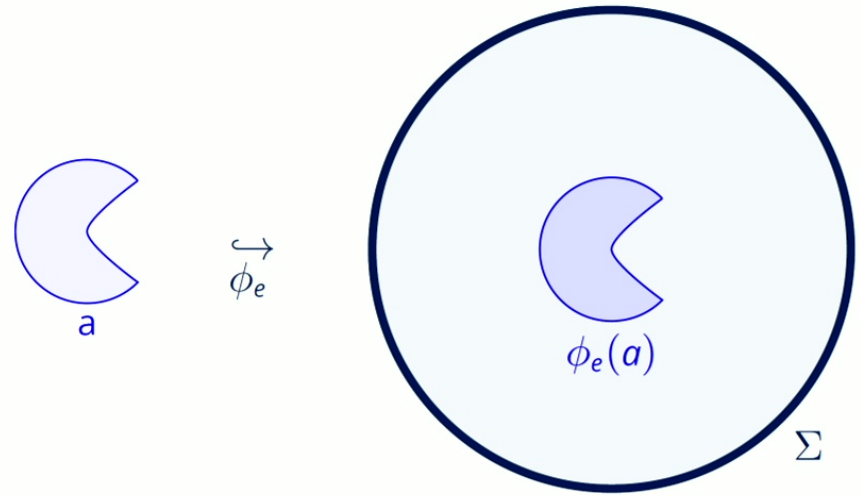
- Edge modes $\sim$ Stueckelberg fields
- Gauge group of $\mathcal{H}$: $U(1)_{LR} = U(1)_L \oplus U(1)_\partial \oplus U(1)_R$
- Gauge sym. group of $\widetilde{\mathcal{H}}_L$: $U(1)_L$
- *Global* sym. group of $\widetilde{\mathcal{H}}_L$: $U(1)_\partial$
  - Implies $\widetilde{\mathcal{H}}_L = \oplus_Q \widetilde{\mathcal{H}}_{L,Q}$
- $\widetilde{\mathcal{H}}_L \otimes \widetilde{\mathcal{H}}_R$ has a global symmetry $U(1)_\partial \oplus U(1)_\partial$
- Entangling product gauges this symmetry: restores full gauge invariance

Edge Modes in Gravity [1601.04744]

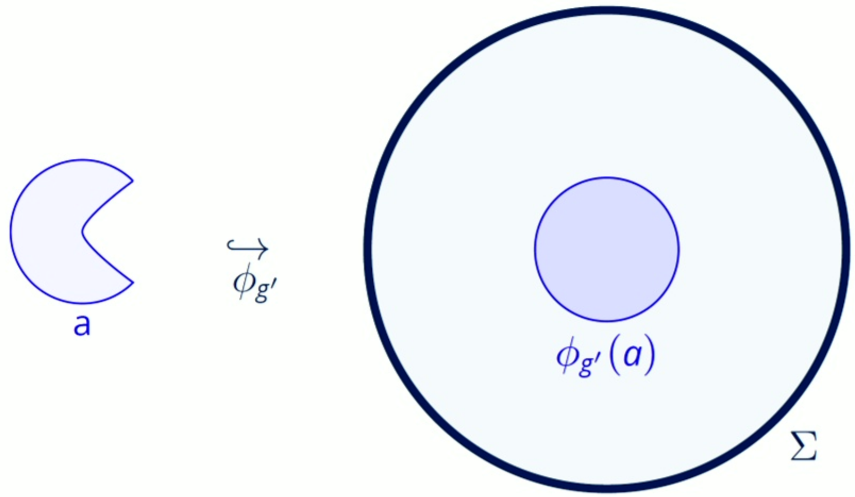
- Gravity is a gauge theory of diffeomorphisms: gravity has edge modes.
- Covariant phase space analysis: gravitational edge modes $\sim$ coordinate systems
- $\mathit{Diff}(\Sigma) = \mathit{Diff}(a) \oplus \mathit{ECS} \oplus \mathit{Diff}(a')$



Edge Modes as Embeddings



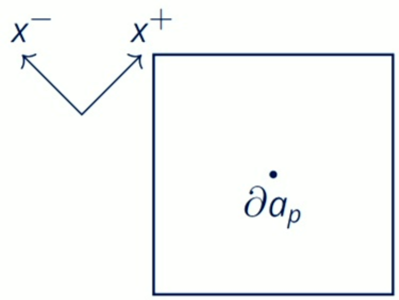
Edge Modes as Embeddings



- Let $g \in ECS$. Then $\phi_g = X_g \circ \phi_e$.
- ECS elements $\leftrightarrow$ embeddings of $a \hookrightarrow \mathcal{M}$.

Extended Corner Symmetry [1601.04744]

- Universal form of symmetry group: $\mathfrak{ecs} = \mathfrak{diff}(\partial a) \ltimes (\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2)_{\partial a}$
- ECS = Diffs which preserve "$2 + (D-2)$" decomposition of the metric



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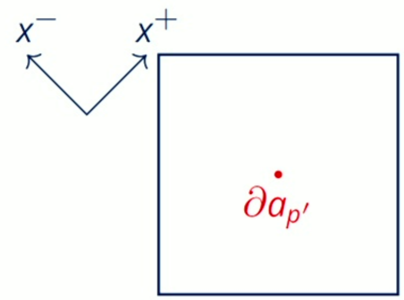
- $\mathfrak{diff}(\partial a)$: coordinate changes of ∂a



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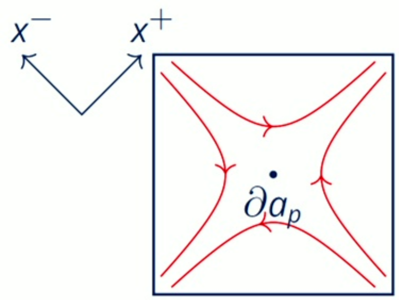
- $\mathfrak{sl}(2,\mathbb{R})$: boosts/rescalings of normal (null) vectors $X^{\pm}$ that fix ∂a



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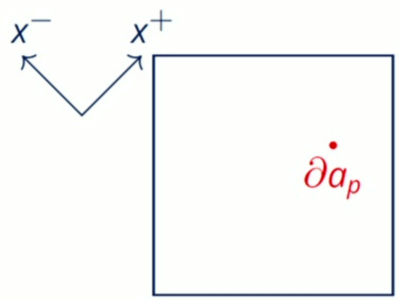
- $\mathfrak{diff}(\partial a)$: coordinate changes of ∂a
- $\mathfrak{sl}(2,\mathbb{R})$: boosts/rescalings of normal (null) vectors $X^{\pm}$ that fix ∂a
- $\mathbb{R}^2$: null translations which do *not* fix ∂a



$$\widetilde{\mathcal{H}}_a = \bigoplus_R \widetilde{\mathcal{H}}_{a,R}$$

The Setup

- $\mathcal{H}_\Sigma$ is the full Hilbert space on all of Σ, with an algebra of operators $\mathcal{A}_\Sigma$.
- Pick a state $|\Psi\rangle$ with a fixed background geometry $g^0_{\mu\nu}$.
- $\mathcal{A}_\Sigma$ includes perturbative gravitons $h_{\mu\nu}$
  - The total metric is $g_{\mu\nu} = g^0_{\mu\nu} + \sqrt{32\pi G_N} h_{\mu\nu}$
- We work perturbatively in G_N and focus on the $G_N \to 0$ limit.

$$\mathcal{H}_\Sigma \to \widetilde{\mathcal{H}}_a = \mathcal{H}_a^{QFT} \otimes L^2(ECS)$$
$$|\Psi\rangle\langle\Psi| \to \rho_a(\chi_a, \phi_g)$$

The Extended Hilbert Space of Gravity

$$\mathcal{H}_\Sigma \to \widetilde{\mathcal{H}}_a = \mathcal{H}_a^{QFT} \otimes L^2(ECS)$$
$$|\Psi\rangle\langle\Psi| \to \rho_a(\chi_a, \phi_g)$$

- $\mathcal{H}_a^{QFT}$: QFT operators, including gravitons $h_{\mu\nu}$.
  - $Diff(a)$ is a gauge symmetry of $\mathcal{H}_a^{QFT}$
- $L^2(ECS)$: Wave functions over the edge modes
  - Interpreted as wave functions over possible embeddings of $a \hookrightarrow \Sigma$.
- Problem: $L^2(ECS)$ "too big".
- Solution: This is because e.g. $\mathfrak{diff}(\partial a) \subset \mathfrak{ecs}$ contains Planckian fluctuations.
- In practice: Cutoff high dimensional representations of ECS.

Decomposing the Entropy

- ECS is a global symmetry for a: $\widetilde{\mathcal{H}}_a = \oplus_R \widetilde{\mathcal{H}}_{a,R}$
- $|\Psi\rangle$ is invariant under *ECS*:

$$\rho_a = \sum_R p_R \rho_R \otimes \frac{1}{d_R} \mathrm{Id}_R$$

- ρ_R: Quantum fields (including gravitons) in a representation R of *ECS*.
- $\frac{1}{d_R}\mathrm{Id}_R$: Maximally mixed states of the edge modes.

Decomposing the Entropy

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- ρ_R: Quantum fields (including gravitons) in a representation R of *ECS*.
- $\frac{1}{d_R}\mathrm{Id}_R$: Maximally mixed states of the edge modes.

$$S(\rho_a) = -\sum_R p_R \ln p_R + \sum_R p_R \ln d_R - \sum_R p_R \, \mathrm{tr}_{QFT,R}\, \rho_R \ln \rho_R$$

Renormalizing the Entropy

- Let $\Omega = \sum_R d_R \sim \mathcal{Z}(\beta = 0)$
- Renormalize by rescaling $\mathbf{tr}_R \to \Omega^{-1}\,\mathbf{tr}_R$ and $\rho_R \to \Omega\rho_R$ to keep $\mathbf{tr}_{QFT}\,\mathbf{tr}_R\,\rho_R = 1$

$$
\begin{aligned}
S(\rho_a) &= -\sum_R p_R \ln p_R + \sum_R p_R \ln d_R - \ln\Omega - \sum_R p_R\,\mathbf{tr}_R\,\mathbf{tr}_{QFT}\,\rho_R \ln\rho_R \\
&= -\sum_R p_R \ln p_R + \sum_R p_R \ln\left(\frac{d_R}{\Omega}\right) + \sum_R p_R S_R \\
&= -S_{rel}\left(p_R \,\middle|\, \frac{d_R}{\Omega}\right) + \sum_R p_R S_R
\end{aligned}
$$

The Semi-Classical Limit

$$S(\rho_a) = -S_{rel}\left(p_R \,\middle|\, \frac{d_R}{\Omega}\right) + \langle S_R \rangle_R$$

- Semi-classical assumption: Take $p_R = \frac{d_R}{\Omega}$
- Intuition: flat in charge basis $\leftrightarrow$ peaked in embeddings basis
- Alternatively: its a coherent state in the limit $\Omega \to \infty$

$$S(\rho_a) = -\frac{1}{\Omega} \sum_R d_R \,\mathrm{tr}_R \,\mathrm{tr}_{QFT}\, \rho_R \ln \rho_R$$

Removing the Cutoff

$$S(\rho_a) = -\frac{1}{\Omega} \sum_R d_R \, \mathrm{tr}_R \, \mathrm{tr}_{QFT} \, \rho_R \ln \rho_R$$

- Trace over R basis $\longrightarrow$ trace over G basis:

$$S(\rho_a) = \frac{\mathrm{tr}_{ECS}}{\Omega} \left[-\, \mathrm{tr}_{QFT} \, \rho[\phi_g] \ln \rho[\phi_g] \right]$$

Removing the Cutoff

$$S(\rho_a) = -\frac{1}{\Omega}\sum_R d_R\, \mathrm{tr}_R\, \mathrm{tr}_{QFT}\, \rho_R \ln \rho_R$$

- Trace over R basis $\longrightarrow$ trace over G basis:

$$S(\rho_a) = \frac{\mathrm{tr}_{ECS}}{\Omega}\left[-\,\mathrm{tr}_{QFT}\, \rho[\phi_g] \ln \rho[\phi_g]\right]$$

- Naively, $\Omega \to \infty$ means ECS is "too big" to take a trace over.
- Actually very familiar: integral over non-locally compact space $\equiv$ path integral

$$S(\rho_a) \mapsto \int_{ECS} D\phi_g \left[-\,\mathrm{tr}_{QFT}\, \rho[\phi_g] \ln \rho[\phi_g]\right]$$

Evaluating The Entropy

$$S(\rho_a) = \int_{ECS} D\phi_g \left[-\, \mathbf{tr}_{QFT}\, \rho[\phi_g] \ln \rho[\phi_g] \right]$$

- Replica trick + Euclidean Path Integral + χ Saddle point

$$S(\rho_a) = -\partial_n \left[\int_{ECS} D\phi_g \, \mathbf{tr}_{QFT} \left(\rho[\phi_g]^n \right) \right]_{n=1}$$

$$S(\rho_a) = -\partial_n \left[\int_{ECS} D\phi_g \int_n \frac{D\chi}{Diff(\phi_g(a))} e^{-S[\phi_g^*(\chi)]} \right]_{n=1}$$

$$S(\rho_a) \approx -\partial_n \left[\int_{ECS \setminus Diff} D\phi_g e^{-I_n[\phi_g] + nI_1} \right]_{n=1}$$

Evaluating The Entropy

$$S(\rho_a) = -\partial_n \left[\int_{ECS \setminus Diff} D\phi_g e^{-I_n[\phi_g]+nI_1} \right]_{n=1}$$

- ϕ_g Saddle point + Evaluate on-shell action:

$$\begin{aligned} S(\rho_a) &= -\partial_n \left[e^{-I_n[\phi_{min}]+nI_1} \right]_{n=1} \\ &= \partial_n I_n[\phi_{min}]_{n=1} - I_1 \\ &= \min_{\phi \in ECS \setminus Diff} \left[\frac{A[\partial\phi(a)]}{4G_N} + S_{out}[\phi(a)] \right] \end{aligned}$$

Why Outward Deformations?

$$S(\rho_a) = \langle S_R \rangle_R = \min_{\phi \in ECS \setminus Diff} \left[\frac{A[\partial\phi(a)]}{4G_N} + S_{out}[\phi(a)] \right]$$

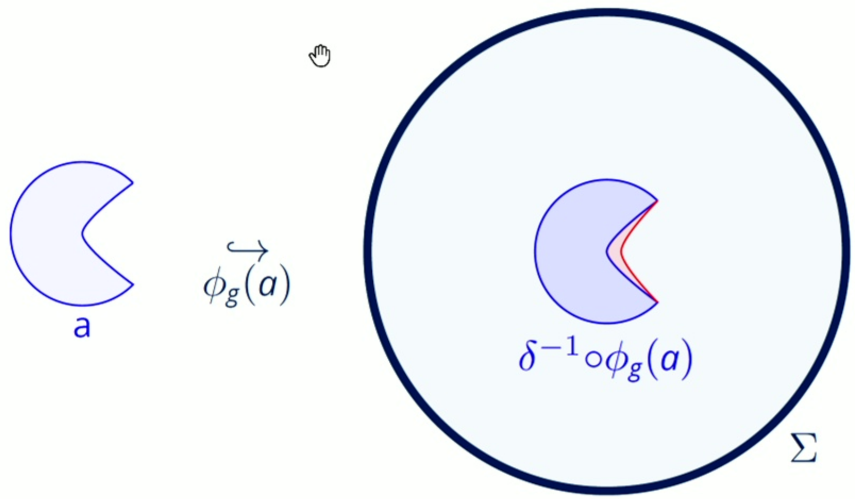
- Can $ECS \setminus Diff =$ All Embeddings? No.
  - $\langle S_R \rangle_R \neq 0$ generically
  - $ECS \setminus Diff =$ All Embeddings implies the RHS always vanishes

The path integral automatically imposes restrictions on allowed ϕ.

Outward Deformations Only

$$Diff(\Sigma) = Diff(\phi_g(a)) \oplus ECS \oplus Diff(\phi_g(a'))$$

- $\widetilde{\mathcal{H}}_a$ has $Diff(\phi_g(a))$ gauge symmetry



Recap

- Goal: Compute entropy of a subregion a around a semi-classical background
- Proposal: use bulk path integral

$$Z(a) = \int \frac{D\phi_g D\chi}{Diff(\phi_g(a))} e^{-S[\phi_g^*(\chi)]}$$

- Saddlepoint approximation:

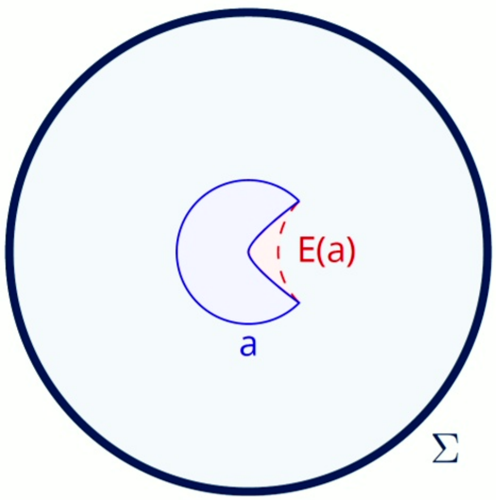
$$S(\rho_a) = \min_{\phi \in out} \left[\frac{A[\partial\phi(a)]}{4G_N} + S_{out}[\phi(a)] \right]$$

- Let ϕ_* be the embedding which attains this minimum:

$$E(a) := \phi_*(a)$$

- Nowhere assumes a is in an AdS spacetime!

Conclusions and Future Work



- $E(a)$ defined by:
  - $a \subset E(a)$
  - $E(a)$ minimizes $S_{gen} = \frac{A(\partial E(a))}{4G_N} + S_{out}(E(a))$
- We proved the Bousso-Penington conjecture for the entropy of a bulk subregion
- Dynamical spacetimes?
- Explicit example? JT?
- Connection to von Neumann algebras and observers?
- Proving encoding like in JLMS?
  - "Terrestrial Holography"