Title: Building blocks of W-algebras and duality

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Abstract:

W-algebras are a family of vertex algebras obtained as Hamiltonian reductions of affine vertex algebras parametrized by nilpotent orbits. The W-algebras associated with regular nilpotent orbits enjoy the Feigin-Frenkel duality. More recently, Gaiotto and Rap\v{c}\'ak generalize this result to hook-type W-algebras with the triality for vertex algebras at the corner. In this talk, I will present the correspondence of representation categories for the hook-type W-superalgebras and how to gain general W-algebras in type A from hook-type W-algebras. The talk is based on joint works with my collaborators.

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Building blocks of $\mathcal{W}$-algebras and duality

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based on joint works with
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What are $\mathcal{W}$-algebras?

- $\mathcal{W}$-algebras are vertex algebras describing the chiral symmetries appearing in 2d CFT.

$$\mathrm{Vir}^c : [L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}c$$

- $\mathcal{W}$-algebras are obtained from affine vertex algebras $V^k(\mathfrak{g})$ through BRST reductions

$$\mathcal{W}^k(\mathfrak{g}, f) \stackrel{\text{def}}{=} H_f(V^k(\mathfrak{g})), \quad f \in \mathcal{N} = \bigsqcup_{\lambda} \mathbb{O}_\lambda$$

- $\mathcal{W}$-algebras are "chiralization" of the finite $\mathcal{W}$-algebras quantizing Slodowy slices

$$\mathcal{W}^k(\mathfrak{g}, f) \stackrel{A_\bullet}{\rightsquigarrow} \mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, f) \stackrel{\mathrm{gr}}{\rightsquigarrow} \mathbb{C}[S_f], \quad S_f = f + \mathfrak{g}^e \subset \mathfrak{g}\,.$$

- They include all the known superconformal algebras

$$\mathrm{sVir}^c_{\mathcal{N}=1} \simeq \mathcal{W}^k(\mathfrak{osp}_{1|2}, f_{\mathrm{reg}}), \quad \mathrm{sVir}^c_{\mathcal{N}=2} \simeq \mathcal{W}^k(\mathfrak{sl}_{2|1}, f_{\mathrm{reg}}), \cdots,$$

but their representation theory is not known in general (in particular for $\mathcal{W}$-superalgebras).

Today's talk

- Historically, $\mathcal{W}$-algebras were introduced for regular nilpotent elements [Feigin–Frenkel]. They enjoy the duality

$$\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) \simeq \mathcal{W}^{{}^L k}({}^L\mathfrak{g}, f_{\mathrm{reg}}),$$

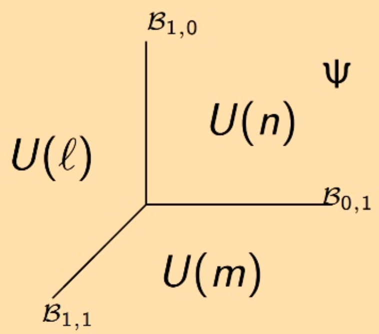
which quantizes the description of the center of affine Kac–Moody Lie algebras

$$\mathcal{Z}(\widehat{\mathfrak{g}}_{-h^\vee}) \simeq \mathcal{W}^{-h^\vee}(\mathfrak{g}, f_{\mathrm{reg}}) \simeq \mathcal{W}^{\infty}({}^L\mathfrak{g}, f_{\mathrm{reg}}) \simeq \mathbb{C}[\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times)]$$

Ex.

$$\mathcal{Z}(\widehat{\mathfrak{sl}}(2)_{-2}) \simeq \mathbb{C}[\partial_z^2 + S(z)].$$

- Suitable generalization of this duality was found by Gaiotto–Rapčák.



I will explain
(1) the duality for hook-type $\mathcal{W}$-s.algebras
(2) its application to their representation theory.
(3) How to gain the other $\mathcal{W}$-algebras (type A)

Examples of $\mathcal{W}$-algebras

Ex.1 $\mathcal{W}^k(\mathfrak{sl}_2, f_2)$: Generator: $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

$$L(z)L(w) \sim \frac{\frac{1}{2}c_2(k)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}, \quad c_2(k) = 13 - 6\left(k+2+\frac{1}{k+2}\right)$$

$$\mathcal{W}^k(\mathfrak{sl}_2, f_{\mathrm{prin}}) \simeq \mathcal{W}^\ell(\mathfrak{sl}_2, f_{\mathrm{prin}}), \quad (k+2)(\ell+2) = 1$$

Ex.2 $\mathcal{W}^k(\mathfrak{sl}_3, f_3)$: Generators $L(z)$, $W_3(z)$,

$$L(z)L(w) \sim \frac{\frac{1}{2}c_3(k)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}, \quad c_3(k) = 50 - 24\left(k+3+\frac{1}{k+3}\right)$$

$$L(z)W_3(w) \sim \frac{3W_3(w)}{(z-w)^2} + \frac{\partial_w W_3(w)}{(z-w)}$$

$$W_3(z)W_3(w) \sim \frac{-4(3k+4)(5k+12)}{(z-w)^6} + \frac{-12(3k+4)(5k+12)L(w)}{(z-w)^4} + \frac{-6(3k+4)(5k+12)\partial_w L(w)}{(z-w)^3}$$
$$+ \frac{-27(k+2)^2\partial_w^2 L(w) + 24(k+3):L(w)^2:}{(z-w)^2} + \frac{-2(3k^2+14k+18)\partial_w^3 L(w) + 24(k+3):L(w)\partial_w L(w):}{(z-w)}$$

$$\mathcal{W}^k(\mathfrak{sl}_3, f_{\mathrm{reg}}) \simeq \mathcal{W}^\ell(\mathfrak{sl}_3, f_{\mathrm{reg}}), \quad (k+3)(\ell+3) = 1$$

Feigin–Frenkel duality

Theorem [Feigin–Frenkel]

There is an isomorphism of vertex algebras

$$\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) \simeq \mathcal{W}^\ell({}^L\mathfrak{g}, f_{\mathrm{reg}}), \quad r(k+h^\vee)(\ell + {}^L h^\vee) = 1.$$

- For a proof, we use the free field realization

$$\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) \simeq \bigcap \mathrm{Ker} \int Y(e^{-\frac{1}{k+h^\vee}\alpha_i}, z)dz \subset \pi_{\mathfrak{h}}^{k+h^\vee}.$$

The map $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) \hookrightarrow \pi_{\mathfrak{h}}^{k+h^\vee}$ is called the Miura map

$$\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) \hookrightarrow \pi_{\mathfrak{h}}^{k+h^\vee} \xRightarrow{A_\bullet} U(\mathfrak{g}) \supset \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h})$$

$$\begin{array}{ccccccc}
\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}}) & & \xrightarrow[\text{Feigin–Frenkel duality}]{\simeq} & & & & \mathcal{W}^\ell({}^L\mathfrak{g}, f_{\mathrm{reg}}) \\
\downarrow A_\bullet & & & & & & \downarrow A_\bullet \\
\mathcal{Z}(\mathfrak{g}) & \xrightarrow{\simeq} & U(\mathfrak{h})^W & \xrightarrow{\simeq} & U({}^L\mathfrak{h})^W & \xleftarrow{\simeq} & \mathcal{Z}({}^L\mathfrak{g})
\end{array}$$

More on type A

$$\mathcal{W}^k(\mathfrak{gl}_n, f_{\mathrm{reg}}) \simeq \bigcap \mathrm{Ker} \int Y(e^{-\frac{1}{k+h^\vee}\alpha_i}, z)dz \subset \pi^{k+h^\vee}_{\mathfrak{h}(\mathfrak{gl}_m)}$$

$$: (\overline{\partial}_z + x_1(z)) \cdots (\overline{\partial}_z + x_n(z)) := \overline{\partial}_z^n + \sum_{i=1}^{n} W_i(z)\overline{\partial}_z^{n-i}, \quad \overline{\partial}_z := \tfrac{1}{k+n-1}\partial_z$$

$$W_i(z) =: P_i(x(z)) : +(\text{correction by } \partial_z^{>0}x(z)) \quad (\Leftarrow U(\mathfrak{h})^W = \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n})$$

$$W_{3(1)}W_i \propto \sum_{\alpha=1}^{n} \frac{\partial P_3}{\partial x_\alpha}\frac{\partial P_i}{\partial x_\alpha} + \cdots \propto W_{i+1} + (\text{correction by } \partial^{>0}W_{\leq i})$$

- $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is generated by only two fields $W_2(z)\& W_3(z)$
- ${}^{\exists!}\mathcal{W}_\infty[c,\lambda] = \mathcal{W}(2,3,4,\cdots)$ satisfying such property [Linshaw]

$$W_2(z)W_5(w) \sim \frac{-5(16(c+2)\lambda - 37)W_3(w)}{(z-w)^4} + \cdots$$

- $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{reg}}) \simeq \mathcal{W}^\ell(\mathfrak{sl}_n, f_{\mathrm{reg}})$ can also be derived by the coincidence of (c,λ).

Duality in representations

- The BRST reduction is indeed a functor

$$H_f\colon V^k(\mathfrak{g})\text{-Mod} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}.$$

$$\mathrm{KL}_k(\mathfrak{g}) := (V_k(\mathfrak{g})\text{-Mod})^G$$

is a braided tensor category [Kazhdan–Lusztig] and the functor on it is exact in general [Arakawa]. If $f = f_{\mathrm{reg}}$, then

$$\mathrm{KL}_k(\mathfrak{g}) \xrightarrow{H_f} \mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}})\text{-Mod} \simeq \mathcal{W}^{{}^L k}({}^L\mathfrak{g}, f_{\mathrm{reg}})\text{-Mod} \xleftarrow{H_f} \mathrm{KL}_{{}^L k}({}^L\mathfrak{g}).$$

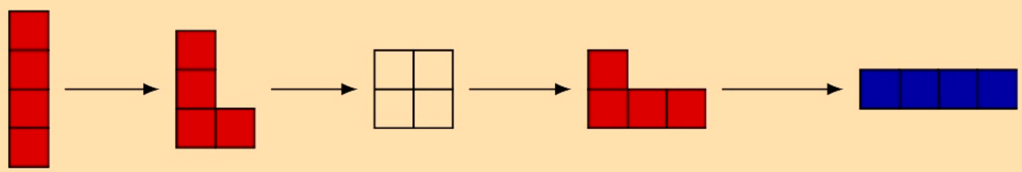
Theorem

(1) [Arakawa] The simple $\mathcal{W}$-algebra $\mathcal{W}_{p,q}(\mathfrak{sl}_n, f_{\mathrm{reg}})$ at level $k_{p,q} = -n + (p+n)/(q+n)$ with $\gcd(p+n, q+n) = 1$ is rational and C_2-cofinite. (The category of f.g. modules is MTC.)

(2) [Frenkel–Kac–Wakimoto, Arakawa–van Ekeren, Creutzig]

$$\mathcal{K}(\mathcal{W}_{p,q}(\mathfrak{sl}_n, f_{\mathrm{reg}})) \simeq \mathcal{K}(L_p(\mathfrak{sl}_n)) \underset{\mathbb{Z}[\mathbb{Z}_n]}{\otimes} \mathcal{K}(L_q(\mathfrak{sl}_n))$$

Dualities beyond $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}})$



Coset construction of $\mathrm{sVir}^c_{\mathcal{N}=2} = \mathcal{W}^\ell(\mathfrak{sl}_{2|1}, f_{\mathrm{reg}})$

$\mathcal{W}^\ell(\mathfrak{sl}_{2|1}, f_{\mathrm{reg}})$ generated by $L(z)$, $J(z)$ (even) $G^\pm(z)$ (odd)

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}, \qquad J(z)J(w) \sim \frac{c/3}{(z-w)^2},$$
$$G^+(z)G^-(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2L(w)+\partial J(w)}{z-w}, \quad \text{etc}$$

Theorem [Di Vecchia–Petersen–Yu–Zheng, Kazama–Suzuki]

(1) For $(k+2)(\ell+1) = 1$, by setting $\Delta = \frac{1}{2}h - x$,

$$\begin{array}{ccc} \mathcal{W}^\ell(\mathfrak{sl}_{2|1}, f_{\mathrm{reg}}) & \xrightarrow{\simeq} & \mathrm{Com}\left(\pi^\Delta, V^k(\mathfrak{sl}_2) \otimes V_{\mathbb{Z}}\right) \\ G^+,\ G^- & \mapsto & \sqrt{\frac{2}{k+2}}e \otimes |1\rangle,\ \sqrt{\frac{2}{k+2}}f \otimes |\text{-}1\rangle \end{array}$$

(2) When the Heisenberg subalgebras are simple,

$$\mathrm{Com}\left(\pi^h, V^k(\mathfrak{sl}_2)\right) \simeq \mathrm{Com}\left(\pi^J, \mathcal{W}^\ell(\mathfrak{sl}_{2|1}, f_{\mathrm{reg}})\right).$$

Feigin–Semikhatov conjecture

Feigin–Semikhatov proposed a generalization for the pair

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \leftrightarrow \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}), \quad f_{\mathrm{sub}} \leftrightarrow f_{n-1,1}.$$

Theorem [CGN]

(1) For $(k+n)(\ell+n-1)=1$, there are isomorphisms of vertex s.algebras

$$\mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}) \xrightarrow{\simeq} \mathrm{Com}(\pi^{\mathrm{diag}}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}),$$

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \xrightarrow{\simeq} \mathrm{Com}(\pi^{\mathrm{diag}}, \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}) \otimes V_{\sqrt{-1}\mathbb{Z}}).$$

(2) When the Heisenberg subalgebras are simple,

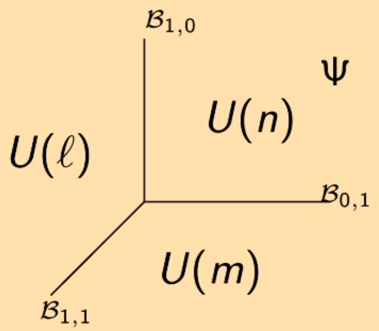
$$\mathrm{Com}\left(\pi, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi, \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}})\right).$$

(3) (1)-(2) also hold for the pair $\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ and $\mathcal{W}^\ell(\mathfrak{osp}_{2|2n}, f_{\mathrm{reg}})$.

Gaiotto–Rapčák's vertex algebras at the corner

Vertex algebras (2d CFT) might appear at boundaries of higher dimenional QFTs. E.g. 2d WZW model appearing at the boundary of 3d Chern–Simons theory.

Gaiotto–Rapčák observed that affine cosets of W-superalgebras can appear at 2d junction of 4d GL-twisted $\mathcal{N}=4$ super Yang–Mills theory with various boundary conditions



$$\mathrm{Com}\left(V(\mathfrak{gl}_{m|\ell}), \mathcal{W}^{\Psi}(\mathfrak{sl}_{n|\ell}, f_{n-m,1^m|1^\ell})\right)$$

$\{(\mathcal{B}_{p,q},\Psi)\}_{(p,q)\in\mathbb{Z}^2/\{\pm1\}}$:half BPS interfaces $\curvearrowleft PSL_2(\mathbb{Z})$

The matrix $X = TS = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ just rotates the Y-junction and thus preserves the VOA at the corner. This gives trialities of vertex algebras.

Duality in hook-type $\mathcal{W}$-superalgebras

Theorem [Creutzig–Linshaw]

For $k, \ell \in \mathbb{C}\backslash\mathbb{Q}$ satisfying $r_{X,Y}(k + h^\vee_{X^+})(\ell + h^\vee_{X^-}) = 1$,

$$\mathrm{Com}(V^{k'}(\mathfrak{g}_X), \mathcal{W}^k_{X^+}(n,m)) \simeq \mathrm{Com}(V^{\ell'}(\mathfrak{g}_X), \mathcal{W}^\ell_{X^-}(n,m)).$$

Examples of dual pairs

- Type A

$$\mathcal{W}^k_{A^+}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,1^m}), \quad \mathcal{W}^\ell_{A^-}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m|m}, f_{n+m|1^m})$$

$$\mathcal{W}^k_{A^+}(n,m) = \mathcal{W}(\underbrace{1^{m^2}}_{\mathfrak{gl}_m}, \underbrace{2, 3, \cdots, n}_{\text{piece of coset}}, \underbrace{(\tfrac{n+1}{2})^{2m}}_{\mathbb{C}^m \oplus \overline{\mathbb{C}}^m}),$$

$$\mathcal{W}^\ell_{A^-}(n,m) = \mathcal{W}(\underbrace{1^{m^2}}_{\mathfrak{gl}_m}, \underbrace{2, 3, \cdots, n+m}_{\text{piece of coset}}, \underbrace{(\tfrac{n+m+1}{2})^{2m}}_{\mathbb{C}^m \oplus \overline{\mathbb{C}}^m}).$$

- Type D

$$\mathcal{W}^k_{D^+}(n,m) = \mathcal{W}^k(\mathfrak{so}_{2(n+m)+1}, f_{\mathfrak{so}_{2n+1}}), \quad \mathcal{W}^\ell_{D^-}(n,m) = \mathcal{W}^k(\mathfrak{osp}_{2m|2(n+m)}, f_{\mathfrak{sp}_{2(n+m)}})$$

Reconstruction of $\mathcal{W}$-superalgebras

In the Feigin–Semikhatov case, we recovered the whole $\mathcal{W}$-s.algebras

$$\mathcal{W}^{\ell}(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}) \xrightarrow{\simeq} \mathrm{Com}(\pi^{\mathrm{diag}}, \mathcal{W}^{k}(\mathfrak{sl}_{n}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}),$$

$$\mathcal{W}^{k}(\mathfrak{sl}_{n}, f_{\mathrm{sub}}) \xrightarrow{\simeq} \mathrm{Com}(\pi^{\mathrm{diag}}, \mathcal{W}^{\ell}(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}) \otimes V_{\sqrt{-1}\mathbb{Z}}).$$

- Can we generalize this reconstruction?

$$\mathcal{W}^{k}(\mathfrak{sl}_{n}, f_{n-1,1}) \simeq \bigoplus_{a\in\mathbb{Z}} \mathscr{C}^{k}_{+}(a) \otimes \pi^{h^{+}}_{a}, \quad \mathcal{W}^{\ell}(\mathfrak{sl}_{n|1}, f_{n|1}) \simeq \bigoplus_{a\in\mathbb{Z}} \mathscr{C}^{\ell}_{-}(a) \otimes \pi^{h^{-}}_{a},$$

- "$\pi^{h^{+}}_{a} \leftrightarrows \pi^{h^{-}}_{a}$" is realized by the **relative semi-infinite cohomology**:

$$H^{n}_{\mathrm{rel}}\left(\widehat{\mathfrak{gl}}_{1}, \pi^{H}_{a} \otimes \pi^{\sqrt{-1}H}_{b}\right) \simeq \delta_{n,0}\delta_{a+b,0}\mathbb{C}[|a\rangle \otimes |b\rangle].$$

  - By setting

$$\mathscr{D} := \bigoplus_{a\in\mathbb{Z}} \pi^{\sqrt{-1}h^{+}}_{-a} \otimes \pi^{h^{-}}_{a} \simeq V_{\mathbb{Z}} \otimes \pi_{\mathbb{Z}}$$

$$\mathcal{W}^{\ell}(\mathfrak{sl}_{n|1}, f_{n|1}) \simeq H^{0}_{\mathrm{rel}}\left(\widehat{\mathfrak{gl}}_{1}, \mathcal{W}^{k}(\mathfrak{sl}_{n}, f_{n-1,1}) \otimes \mathscr{D}\right).$$

In general, for a pair e.g.

$$\mathcal{W}^k_{A^+}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,1^m}) \quad \mathcal{W}^\ell_{A^-}(n,m) = \mathcal{W}^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|1^m})$$

- Gluing objects are **shifted chiral differential operators** [Moriwaki]

$$\mathscr{D}^{\mathrm{ch}}_{G,\kappa}[\theta] \simeq \bigoplus_{\lambda \in P_+} \mathbb{V}^\kappa_{\mathfrak{g}}(\lambda) \otimes \mathbb{V}^{\check{\kappa}}_{\mathfrak{g}}(\lambda^\dagger), \quad \frac{1}{\kappa+h^\vee} + \frac{1}{\check{\kappa}+h^\vee} = r\theta, \ (\theta \in \mathbb{Z})$$

  $\mathscr{D}^{\mathrm{ch}}_{G,\alpha}[0]$ is the chiralization of global differential operators $\mathscr{D}_G$.
- Gluing procedure by

$$H^n_{\mathrm{rel}}\left(\widehat{\mathfrak{g}}, \pi^H_a \otimes \mathbb{V}^\kappa(\lambda) \otimes \mathbb{V}^{\check{\kappa}}(\mu)\right) \simeq \delta_{n,0}\delta_{\lambda,\mu^\dagger}\mathbb{C}[\mathrm{id}_{L(\lambda)}], \ \frac{1}{\kappa+h^\vee} + \frac{1}{\check{\kappa}+h^\vee} = 0.$$

Theorem [CLNS]

For $k, \ell \in \mathbb{C}\backslash\mathbb{Q}$ under duality relation, we have

$$\mathcal{W}^\ell_{X^-}(n,m) \simeq H^0_{\mathrm{rel}}\left(\widehat{\mathfrak{g}}_X, \mathcal{W}^k_{X^+}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G,\alpha'}[1]\right),$$

$$\mathcal{W}^k_{X^+}(n,m) \simeq H^0_{\mathrm{rel}}\left(\widehat{\mathfrak{g}}_X, \mathcal{W}^\ell_{X^-}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G,\alpha'}[-1]\right).$$

Correspondence of module categories

The gluing procedure can be applied also for modules and intertwining operators (=data which gives the tensor products) by using $\mathscr{D}^{\mathrm{ch}}_{G,\kappa}[\theta]$ and their modules in suitable categories.

$$\mathrm{KL}^k_{A^+} := \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,1^m})\text{-mod}^{GL_m}$$
$$\mathrm{KL}^\ell_{A^-} := \mathcal{W}^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|1^m})\text{-mod}^{GL_m}.$$

Theorem

(1) [CGNS] Feigin–Semikhatov case in type A.
(2) [FN] Feigin–Semikhatov case in type D.
(3) [CLN] In general for irrational levels.

$$\mathrm{KL}^{k,\lambda^+}_{X^+} \rightleftarrows \mathrm{KL}^{\ell,\lambda^-}_{X^-} \ (\simeq) \qquad \mathcal{I}_{X^+}\binom{M_3^+}{M_1^+\ M_2^+} \simeq \mathcal{I}_{X^-}\binom{M_3^-}{M_1^-\ M_2^-}$$

Remarks

- Useful to study $\mathcal{W}$-s.algebras (widely open in general)

$$\mathcal{W}_\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{reg}}), \quad \text{[CGNS]} \qquad \mathcal{W}_\ell(\mathfrak{osp}_{2|2n}, f_{\mathrm{reg}}) \quad \text{[FN]}$$

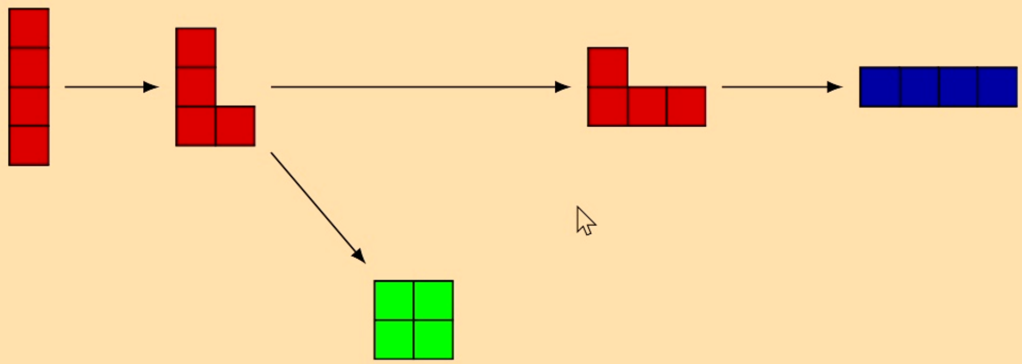
- Character formulas can be derived [FN, CLN]: several interesting functions appear: Appell–Lerch type sums/q-deformation of classical identities (e.g. q-binomial identities)/

$$\mathrm{ch}_{\mathbf{M}^k_{A^+}(\lambda)} \sim s_\lambda(z_1, \cdots, z_m) R^+, \quad \mathrm{ch}_{\mathbf{M}^\ell_{A^-}(\lambda)} \sim \frac{s_\lambda(q, \cdots, q^m)}{\prod(1 + z_i q^{\frac{m}{2}+\lambda_j - j})} R^-$$

Some question and outlook

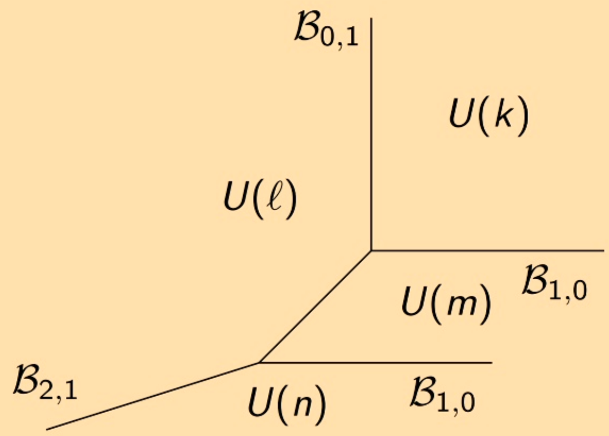
- The Zhu's algebras for hook-type $\mathcal{W}$-algebras are known to have a nice presentation,called infinitesimal Cherednik algebras. Do they also have some dualities?
- The most general triality of Y-algebras seems accessible via the inverse Hamiltonial reduction [Fehily].

Leaving hook-type $\mathcal{W}$-algebras



"Gluing" more boundary conditions results in applying successive Hamiltonian reductions [Procházka-Rapčák].

Example:



$$\mathcal{L}^{\psi}_{k,\ell,m,n} = \mathrm{Com}\left(V^{\Psi^\natural}(\mathfrak{gl}_{n|\ell}), H_{f_{m-n,1^*}}(\mathcal{W}^{\Psi}(\mathfrak{gl}_{k|\ell}, f_{k-m,1^m|1^\ell}))\right)$$

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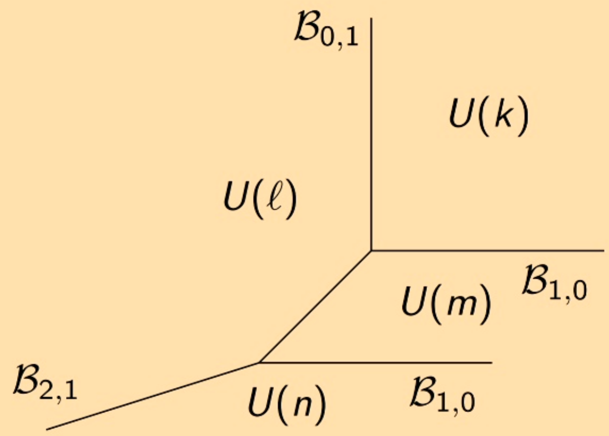


$$\mathcal{L}^{\psi}_{k,\ell,m,n} = \mathrm{Com}\left(V^{\Psi^{\natural}}(\mathfrak{gl}_{n|\ell}), H_{f_{m-n,1^*}}(\mathcal{W}^{\Psi}(\mathfrak{gl}_{k|\ell}, f_{k-m,1^m|1^\ell}))\right)$$

The vertex algebra here might be realized as general $\mathcal{W}$-algebras.

Conjecture [CFLN, in preparation]

Assume $k \notin \mathbb{Q}$ is a generic level. Let $N \in \mathbb{Z}_{>0}$ and $\lambda = \{\lambda_1, \dots, \lambda_n\}$ to be a partition of N with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

(1) $W^k(\mathfrak{sl}_N, f_\mu)$ can be obtained by the successive reductions:

$$\mathcal{W}^k(\mathfrak{sl}_N, f_\lambda) \simeq H_{\lambda_n} H_{\lambda_{n-1},1^*} \dots H_{\lambda_1,1^*}\left(V^k(\mathfrak{sl}_N)\right).$$

(2) Let $\mu = \lambda \setminus \{\lambda_1\} := \{\lambda_2, \dots, \lambda_n\}$ be a truncated partition of λ.

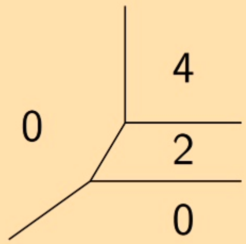
$$\mathrm{Com}\left(V^{k^\sharp}(\mathfrak{gl}_M), \mathcal{W}^k(\mathfrak{sl}_N, f_{\lambda_1,1^M})\right) \otimes \mathcal{W}^{k^\sharp}(\mathfrak{sl}_M, f_\mu) \otimes \pi \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_N, f_\lambda).$$

It implies that $\mathcal{W}^k(\mathfrak{sl}_N, f_\lambda)$ are indeed extensions of tensor products of many Y-algebras:

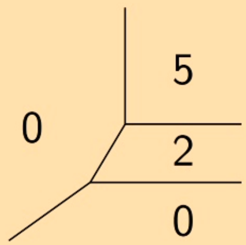
Conjecture says there is a conformal embedding

$$C^k(\mathfrak{sl}_7, f_{3,1^3}) \otimes C^{k+2}(\mathfrak{sl}_4, f_{2,1^2}) \otimes C^{k+3}(\mathfrak{sl}_2, f_2) \otimes \pi^2 \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_7, f_{3,2,2}).$$

The conjectures are true up to $N = 5$ [CFLN, in preparation]. For example,



$$\longleftrightarrow \quad \mathcal{W}^k(\mathfrak{sl}_4, f_{2,2})$$



$$\longleftrightarrow \quad \mathcal{W}^k(\mathfrak{sl}_5, f_{3,2})$$

Inverse Hamiltonian Reduction

Recall that the set of nilpotent orbit is partially ordered.
Idea: If $\mathbb{O}_f \leq \mathbb{O}_{f'}$ (+ conditions) then

$$\mathcal{W}^k(\mathfrak{g}, f) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f') \otimes \text{free fields}.$$

Free fields: Various chiral differential operators $\beta\gamma = \mathscr{D}^{\mathrm{ch}}_{\mathbb{G}_a}$, $\Pi = \mathscr{D}^{\mathrm{ch}}_{\mathbb{G}_m}$,

Ex.

$V^k(\mathfrak{sl}_2) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_2, f_2) \otimes \Pi$ [Semikhatov,Adamović]

$V^k(\mathfrak{sl}_3) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_3, f_{2,1}) \otimes \beta\gamma \otimes \Pi$ [Adamović–Creutzig–Genra]

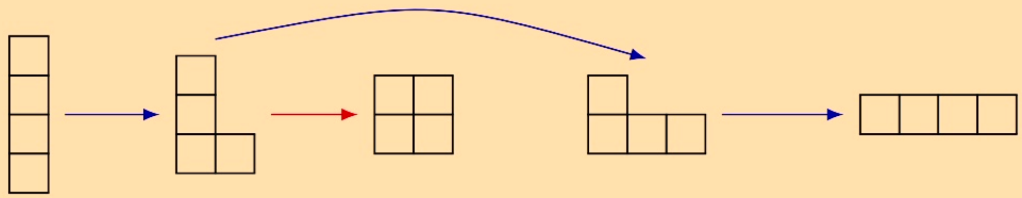
$\mathcal{W}^k(\mathfrak{sl}_3, f_{2,1}) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_3, f_3) \otimes \Pi$ [Adamović–Kawasetsu–Ridout]

Theorem [Fehily]

For $m' \leq m \leq n$,

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{m',1^{n-m'}}) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_n, f_{m,1^{n-m}}) \otimes \beta\gamma^* \otimes \Pi^*$$

Ex. $\mathfrak{sl}_4$:



For almost all levels $k \in \mathbb{C}$, we have a conformal embedding,

$$\begin{aligned} \langle L, W_3 \rangle \otimes \mathcal{W}^{k+1}(\mathfrak{sl}_2, f_2) \otimes \pi &\hookrightarrow \mathcal{W}^k(\mathfrak{sl}_4, f_{2,2}) \\ \langle L, W_3 \rangle \otimes V^{k+1}(\mathfrak{sl}_2) \otimes \pi &\hookrightarrow \mathcal{W}^k(\mathfrak{sl}_4, f_{2,1^2}) \end{aligned}$$

where $\langle L, W_3 \rangle \simeq \mathrm{Com}\left(V^{k+1}(\mathfrak{gl}_2), \mathcal{W}^k(\mathfrak{sl}_4, f_{2,1^2})\right)$ at $k \notin \mathbb{Q}$.

Theorem [CFLN, in preparation]

For almost all level $k \in \mathbb{C}$, we have the conformal embedding

$$\mathcal{W}^k(\mathfrak{sl}_4, f_{2,1^2}) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_4, f_{2,2}) \otimes \Pi^{1/2},$$

and it induces an isomorphism

$$H_{f_2}(\mathcal{W}^k(\mathfrak{sl}_4, f_{2,1^2})) \simeq \mathcal{W}^k(\mathfrak{sl}_4, f_{2,2})$$

Relaxed highest-weight modules [Feigin-Semikhatov-Tipunin]

Originally, the iHR has been studied to obtain categories of $V^k(\mathfrak{g})$ beyond category $\mathcal{O}$ [Adamovic, Ridout,...].

Definition/Theorem [Kac–Wakimoto]

For $k = -n + \frac{p}{q}$ with $p \geq n$, the simple highest weight $V^k(\mathfrak{sl}_n)$-modules $L_k(\lambda)$ which is "$\mathfrak{sl}_n[z^{\pm q}]$-integrable" are called the admissible representations. $\mathrm{SL}_2(\mathbb{Z})$ acts on the space of their (normalized) characters forms.

The modularity of the characters is explained by the rationality of $\mathcal{W}$-algebras.

$$H_f : \mathrm{KL}_k(\mathfrak{sl}_n) \twoheadrightarrow \mathrm{KL}_k^{\mathcal{W}}(\mathfrak{sl}_n) \subset \mathcal{W}_k(\mathfrak{g}, f)\text{-mod}$$

is believed to be a braided tensor functor. To gain the full category, we need to start with a larger category.

$$L_k(\mathfrak{sl}_2) \hookrightarrow \mathcal{W}_k(\mathfrak{sl}_2) \otimes \Pi$$

0-relaxed 1-relaxed 2-relaxed

Theorem [Arakawa-van Ekeren, McRae]

The $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{sl}_n, f_{q^s,r})$ is rational at levels

$$k = -n + \frac{p}{q}, \qquad (p,q) = 1,\ p \geq n,$$

+ classification of simple modules.

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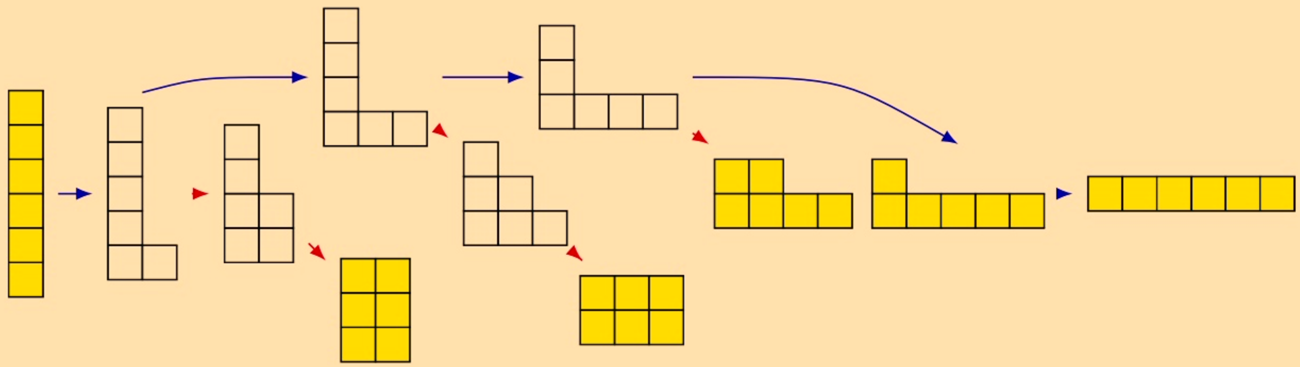

Therefore, our conjecture should lead to the construction of nicely-behaving representation categories of $V^k(\mathfrak{sl}_n)$ for all admissible levels!

Thank you for your attention!