Title: Causality and positivity in causally complex operational probabilistic theories

Speakers: Lucien Hardy

Series: Quantum Foundations

Date: November 16, 2023 - 2:00 PM

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Abstract: In the usual operational picture, operations are represented by boxes having inputs and outputs. Further, we usually consider the causally simple case where the inputs are prior to the outputs for each such operation. In this talk (motivated by an attempt to formulate an operational probabilistic field theory) I will consider what I call the "causally complex" situation. Operations are represented by circles. These circles have wires going in and out. Each such wire can represent an input and an output. Further, each operation will have a causal diagram associated with it. The causal structure can be more complicated than the simple case. These circles can be joined together to create new operations. I will discuss conditions on these causally complex operations so that we have positivity (probabilities are non-negative) and causality (to be understood in a time symmetric manner). I will also discuss how these properties compose when we join causally complex operations. Causally complex operations are related to objects in the causaloid formalism as well as to quantum combs.

Zoom link https://pitp.zoom.us/j/99425886198?pwd=ODR0VVFzQUJHeER4OVJ2cEo3cVdDQT09

Causality and positivity in causally complex operational probabilistic theories

Lucien Hardy

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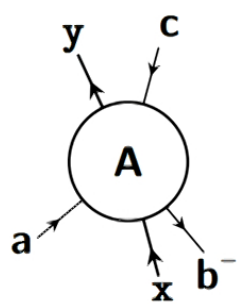

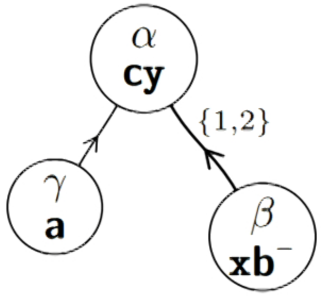

27 frameworks

$$(\mathsf{TF}, \mathsf{TB}, \mathsf{TS}) \times (\mathsf{S}, \mathsf{C}, \mathsf{F}) \times (\mathsf{OPT}, \mathsf{OCT}, \mathsf{OQT})$$

written as txo where

$$t = \begin{cases} \mathsf{TF} & \text{time forward – prob(outcomes|incomes)} \\ \mathsf{TB} & \text{time backward – prob(incomes|outcomes)} \\ \mathsf{TS} & \text{time symmetric – prob(incomes, outcomes)} \end{cases}$$

$$x = \begin{cases} \mathsf{S} & \text{causally Simple (or Square)} \\ \mathsf{C} & \text{causally Complex (or Circle)} \\ \mathsf{F} & \text{Field} \end{cases}$$

27 frameworks

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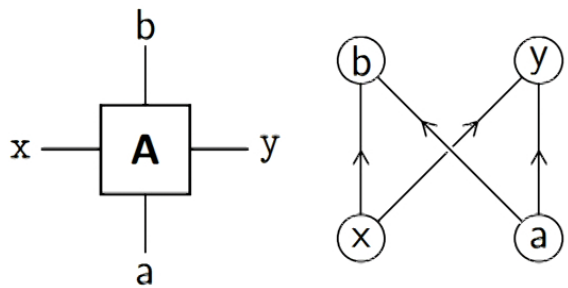
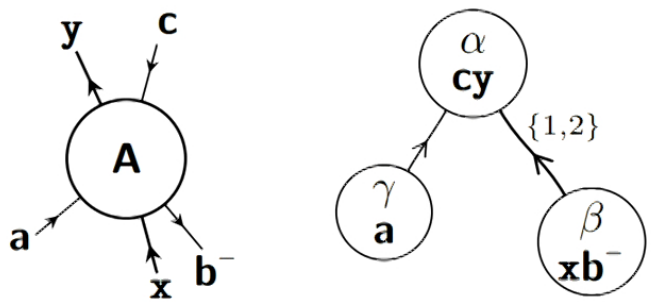
This seminar: TSSOPT, TSCOPT, (and maybe TSSOQT, TSCOQT).

Simple, Complex, and Field

Simple

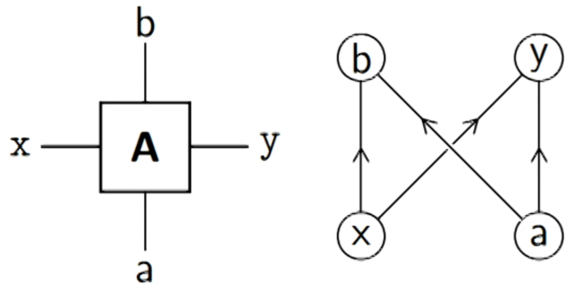


Complex

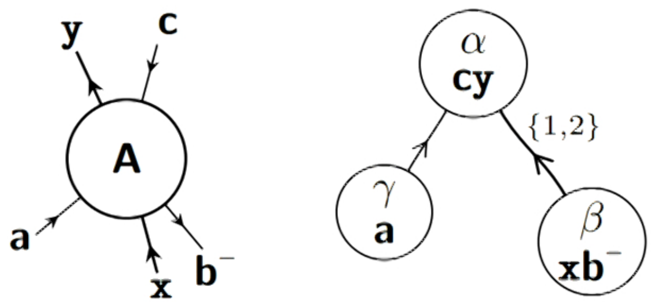


Simple, Complex, and Field

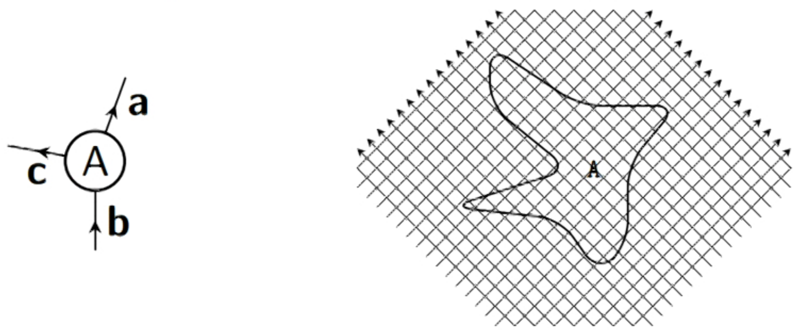
Simple



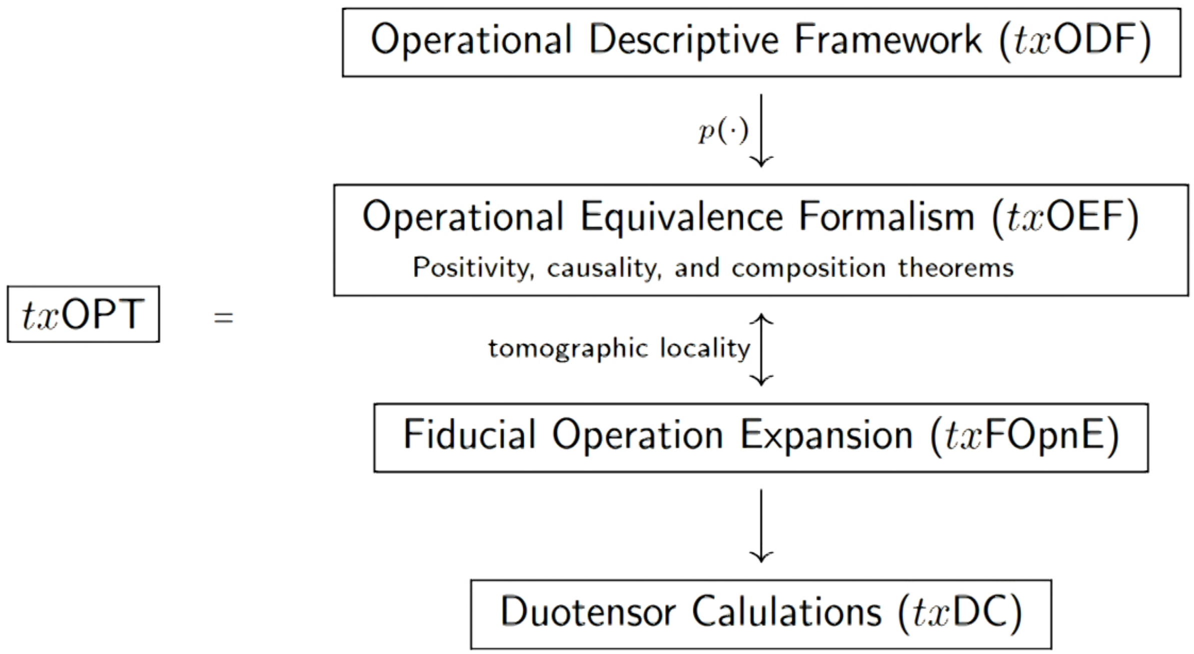
Complex



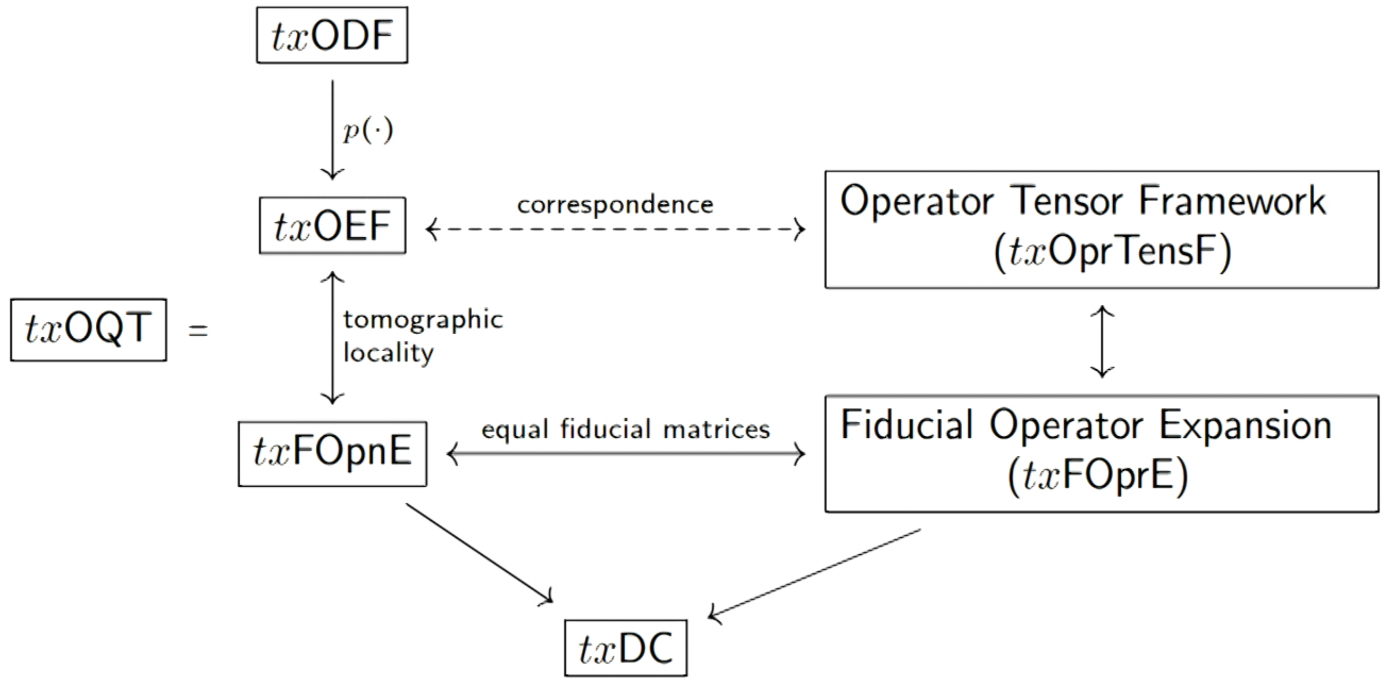
Field



The structure of an OPT



The structure of OQT

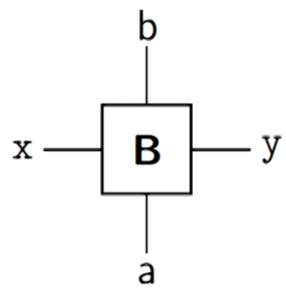


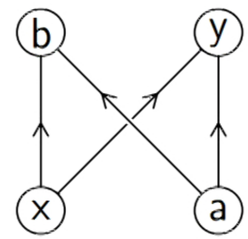
Related work

- This is a natural development of my work from 2001 (5 axioms), 2005-8 (the causaloid approach where higher order objects were defined), 2010-16 (duotensors, operator tensors,...) and 2021 (time symmetry). It is motivated by Quantum Gravity considerations but that will not be evident in this talk.
- This work has, at its core, ideas from the Quantum Combs work of Chiribella, D'Ariano, and Perinotti (2008,9).
- The work is strongly influenced by the diagrammatic line of thinking (Penrose, Seilinger, Coecke, ...) neatly summarised in the book by Coecke and Kissenger.
- The ideas on time symmetry are influenced by the work of Di Biagio, Dona, and Rovelli (2020).
- I will assume definite causal structure throughout but there are strong connections with the work of CDP and Oreshkov, Costa, Brukner (2012) and subsequent work.
- There is a strong synergy with the work of Oeckl - his general boundary formalism (2003) and positive formalism (2012).

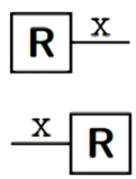
Simple - the elements

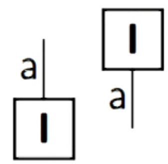
Simple operation



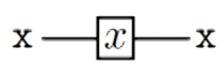


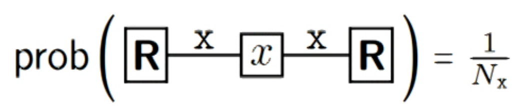
Unique deterministic terminals





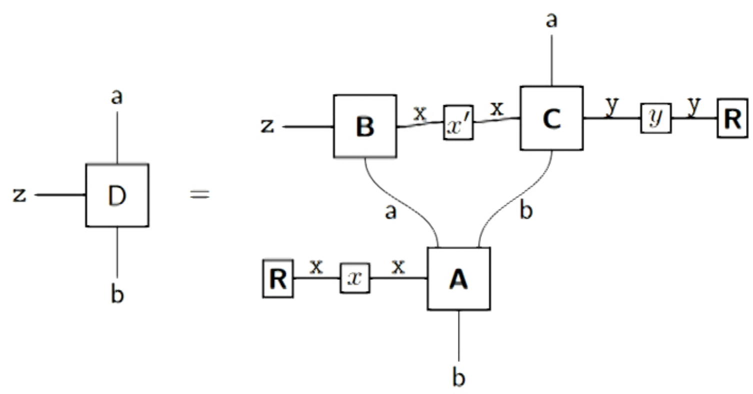
Readout boxes





Deterministic and nondeterministic operations

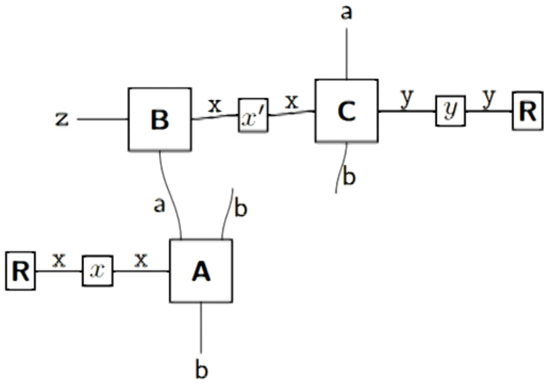
We can make readouts implicit.



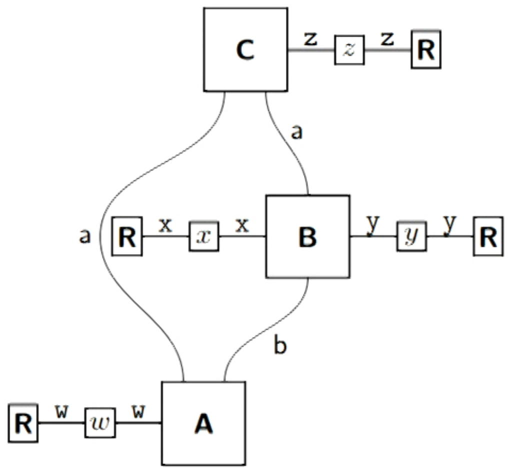
Then we have a nondeterministic operation - denoted by non-bold font (e.g. D).

If there are no implicit readouts have a deterministic operation - denoted by bold font (e.g. **B**).

Simple Networks and Circuits



Simple network



Simple circuit

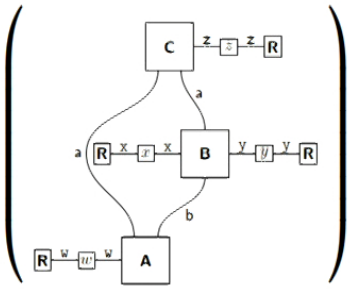
Must be DAGs.

We can complete a network into a circuit by a *complement network*.

Probability assumption and equivalence

Probability assumption. *Every circuit has a probability associated with it that depends only on the specification of that circuit.*

prob



$= \text{prob}(w, x, y, z | \text{circuit spec})$

Equivalence *Two networks are equivalent (denoted by ≡) if they have the same probability when completed into a circuit by the same complement network for all such complement networks. This means that the two networks have to have the same causal structure.*

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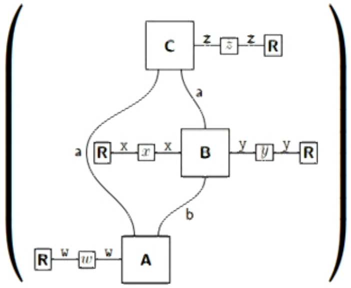

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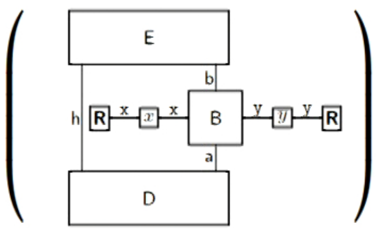
This notion of equivalence can be linearly extended (using the $p(\cdot)$ function) to allow us to expand in terms of fiducial elements.

Physicality conditions

We impose two conditions on simple operations

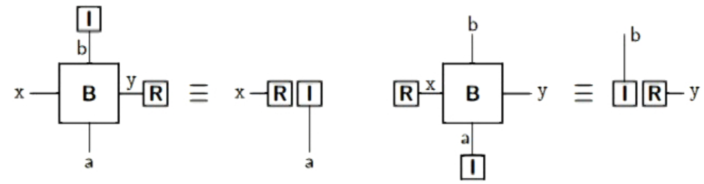
Tester positivity

$0 \leq$ prob



for all pure D and E. This guarantees that circuits have non-negative probability.

Simple double causality



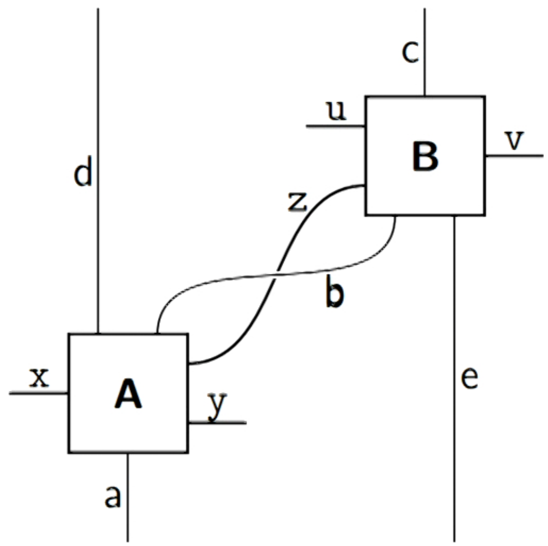
Guarantees causality and subunity. Nondeterministic operations have $\lesssim$ rather than $\equiv$.

Operations satisfying tester positivity and simple double causality are called *physical*

Simple forward causality means cannot signal from future to past without conditioning on something in the future.

Simple backward causality means cannot signal from past to future without conditioning on something in the past.

Physicality under composition

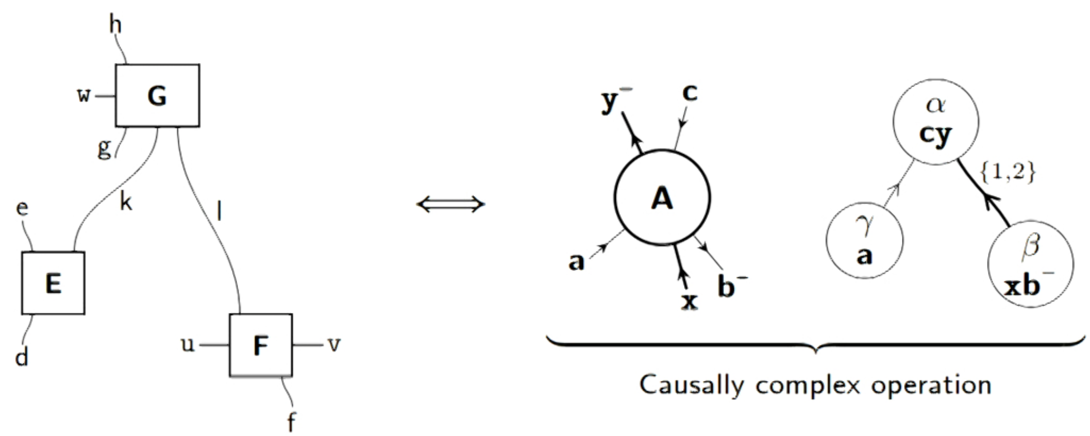


Composition theorem. If we join physical operations together then the resulting network is also physical[a].

[a] for the positivity part of this proof we need to assume that pure preparations and results satisfy tester positivity

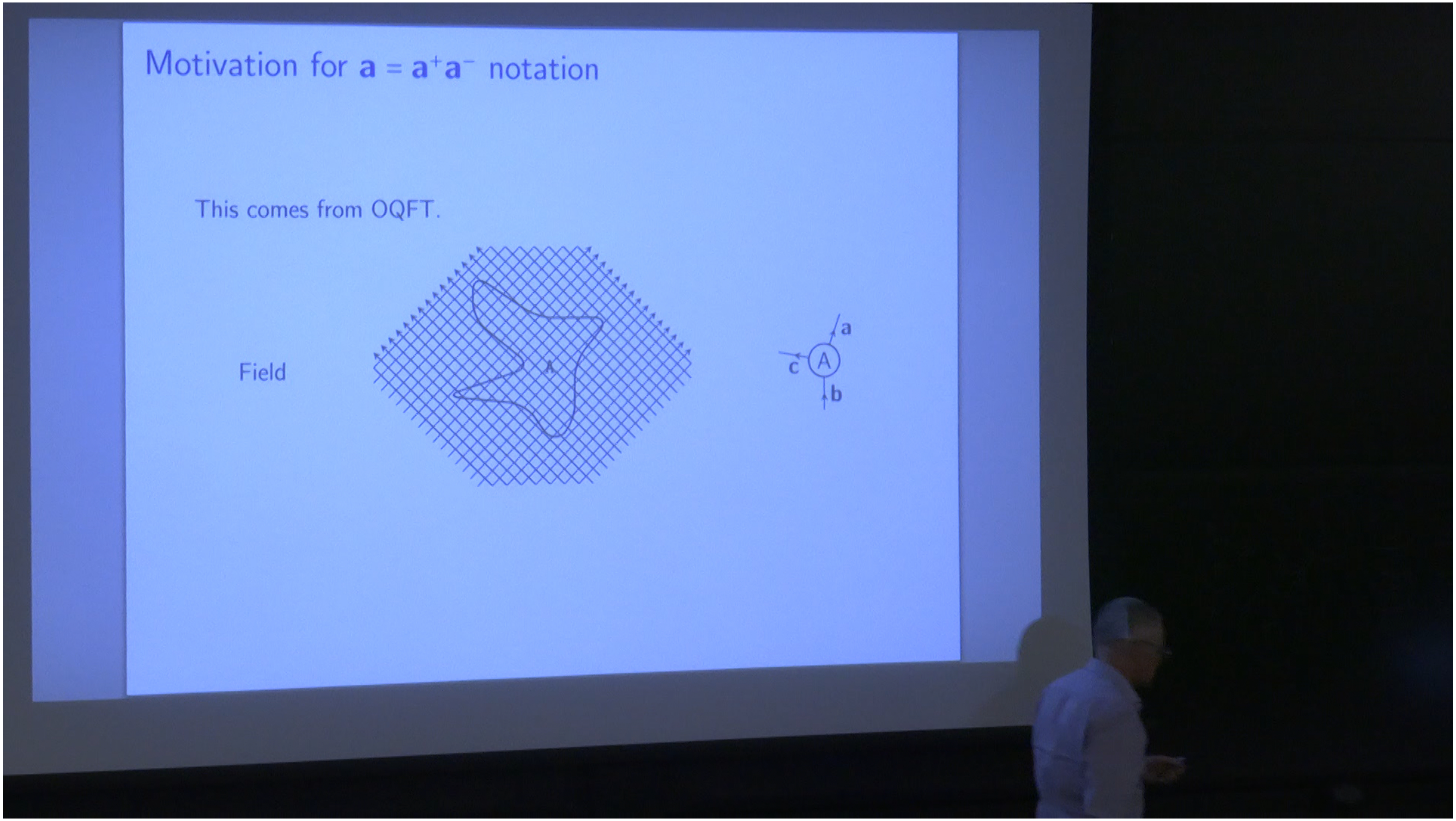
Main idea

Define complex operations to be equal to simple networks

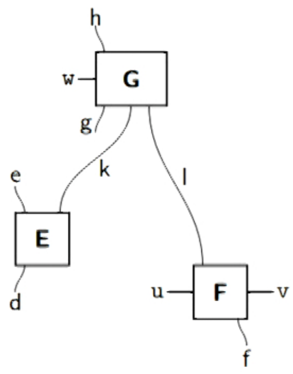


where

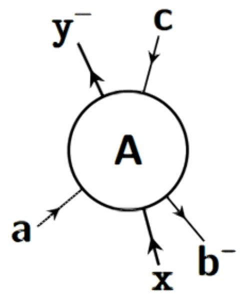
$$\mathbf{c}^+ = g \quad \mathbf{c}^- = h \quad \mathbf{y}^+ = 0 \quad \mathbf{y}^- = w$$
$$\mathbf{a}^+ = d \quad \mathbf{a}^- = e$$
$$\mathbf{b}^+ = 0 \quad \mathbf{b}^- = f \quad \mathbf{x}^+ = u \quad \mathbf{x}^- = v$$


Motivation for $\mathbf{a} = \mathbf{a}^{+}\mathbf{a}^{-}$ notation
This comes from OQFT.
Field
A
a
c
A
b

Causal diagram of complex operation



$\Longleftrightarrow$



$\mathbf{c}^+ = g$	$\mathbf{c}^- = h$
$\mathbf{y}^+ = 0$	$\mathbf{y}^- = w$
$\mathbf{a}^+ = d$	$\mathbf{a}^- = e$
$\mathbf{b}^+ = 0$	$\mathbf{b}^- = f$
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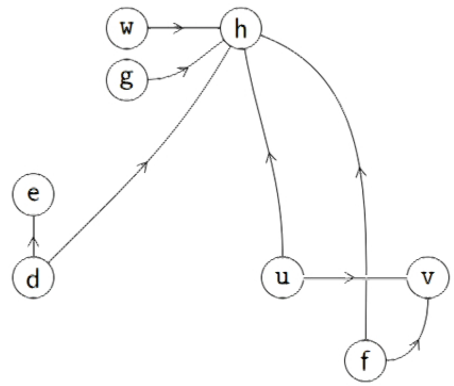
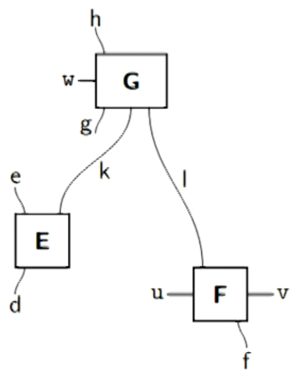
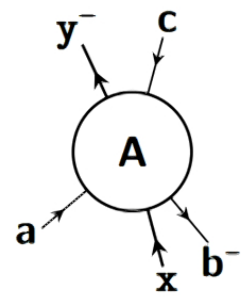


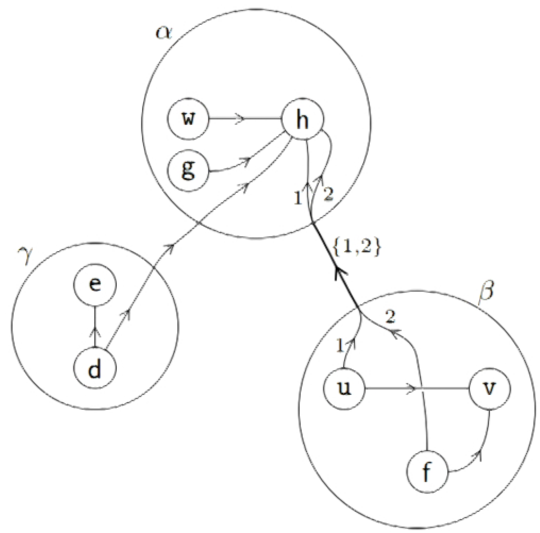
Causal diagram of complex operation



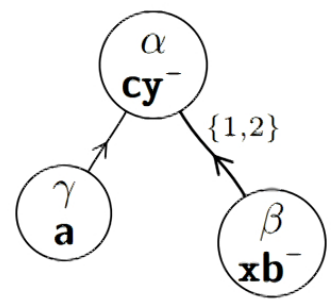
$\Longleftrightarrow$



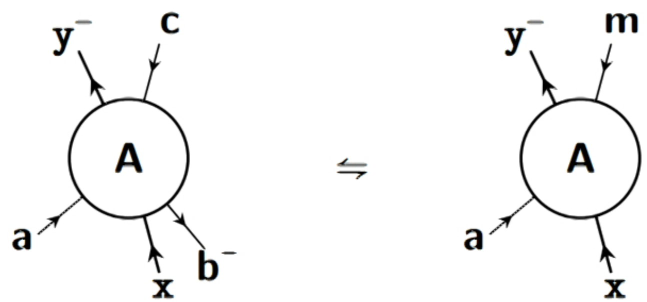
$$
\begin{array}{ll}
\mathbf{c}^+ = \mathrm{g} & \mathbf{c}^- = \mathrm{h} \\
\mathbf{y}^+ = 0 & \mathbf{y}^- = \mathrm{w} \\
\mathbf{a}^+ = \mathrm{d} & \mathbf{a}^- = \mathrm{e} \\
\mathbf{b}^+ = 0 & \mathbf{b}^- = \mathrm{f} \\
\mathbf{x}^+ = \mathrm{u} & \mathbf{x}^- = \mathrm{v}
\end{array}
$$



$\Longleftrightarrow$

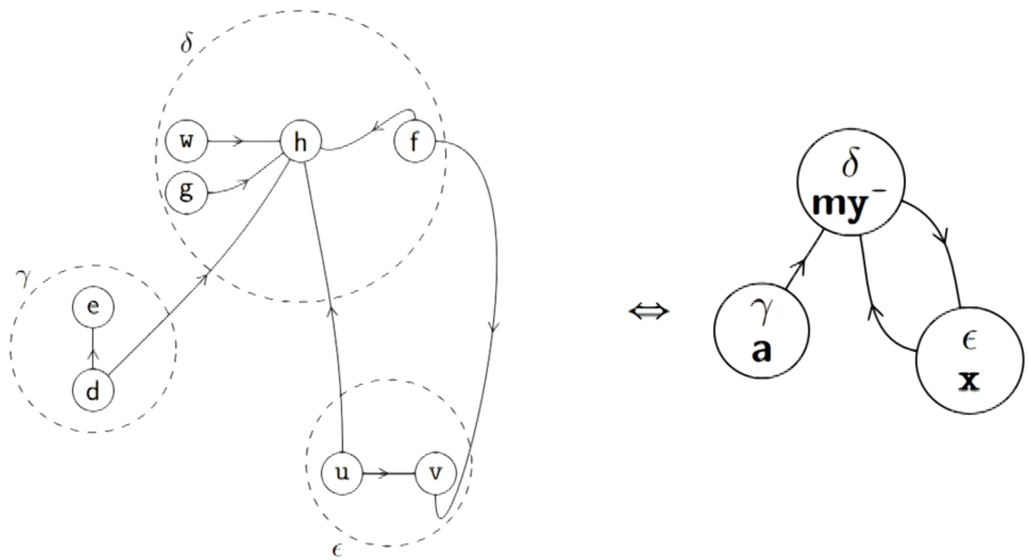


Interconvertible forms



$$\begin{array}{ll} \mathbf{m}^+ = \mathrm{gf} & \mathbf{m}^- = \mathrm{h} \\ \mathbf{y}^+ = 0 & \mathbf{y}^- = \mathtt{w} \\ \mathbf{a}^+ = \mathrm{d} & \mathbf{a}^- = \mathrm{e} \\ \mathbf{x}^+ = \mathrm{u} & \mathbf{x}^- = \mathrm{v} \end{array}$$

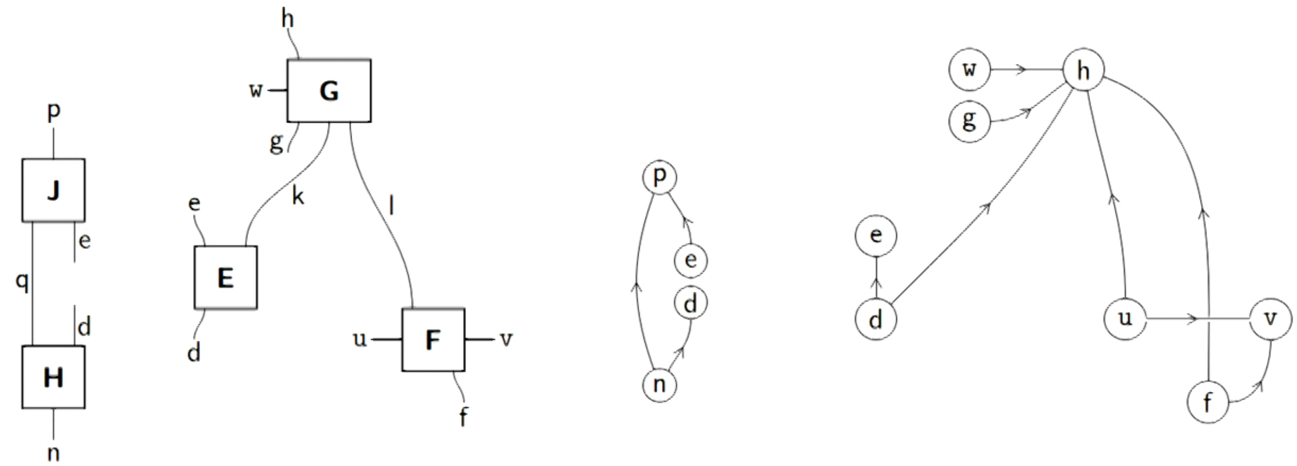
Have causal diagram



Still a DAG

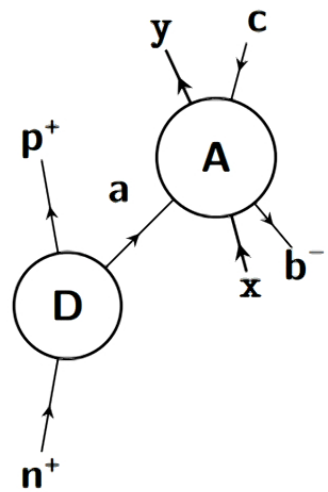
Fusing causal diagrams

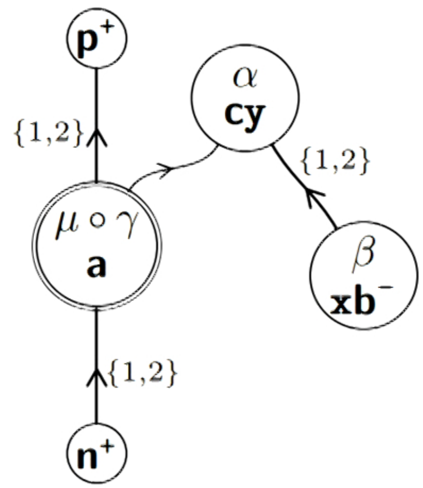
Consider joining



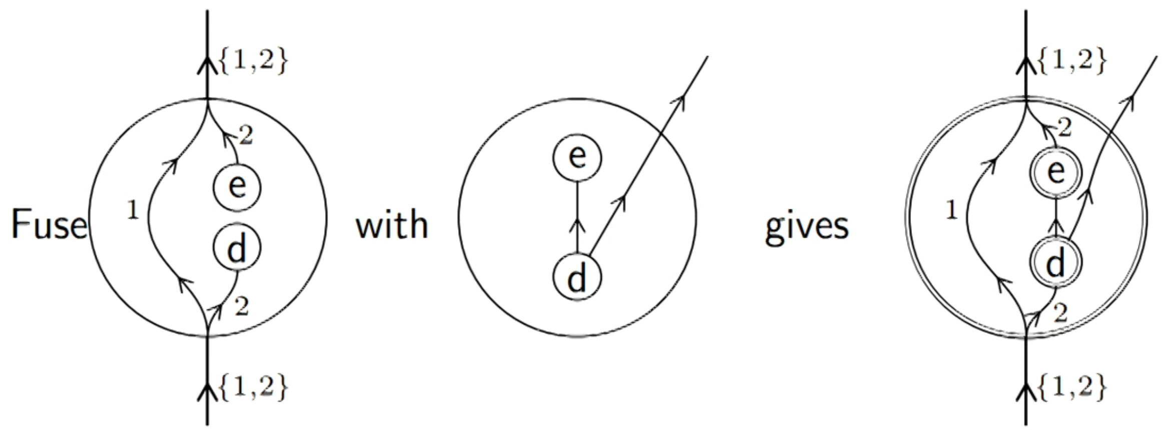
Fusing causal diagrams

Consider joining

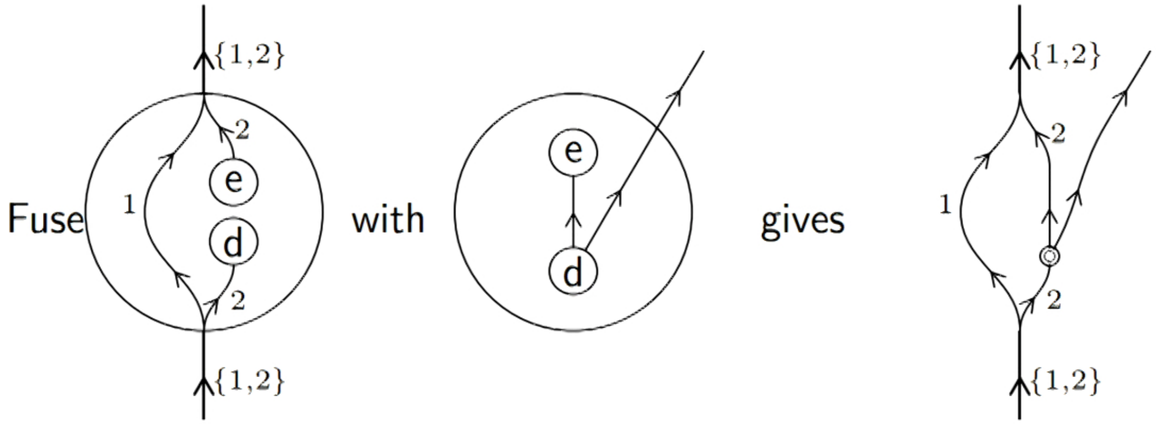




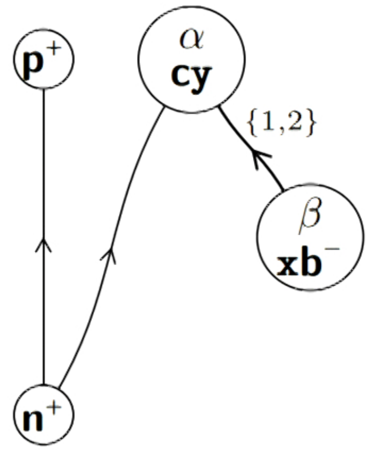
Fusing nodes



Fusing nodes

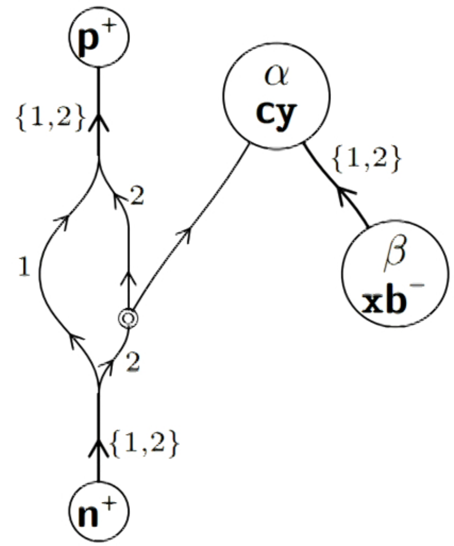


. . . so the resulting causal diagram is



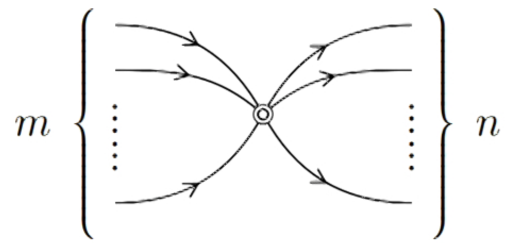
This is a DAG

... so the resulting causal diagram is



Causal spiders

The causal spider

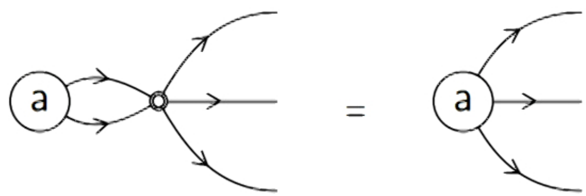


means that have a causal link connecting every incoming to every outgoing wire.

Simplifying causal diagrams

We have the following causal spider identities

and



If causal diagrams are wired together in a non-DAG fashion we will get "causal loops" and the diagram will not simplify further.

Simplifying causal diagrams

We have the following causal spider identities

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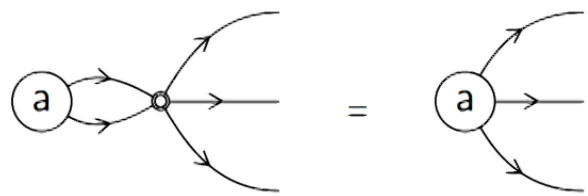
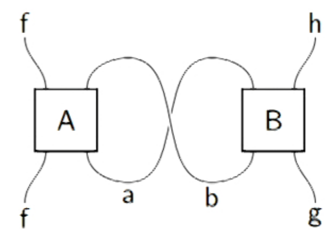


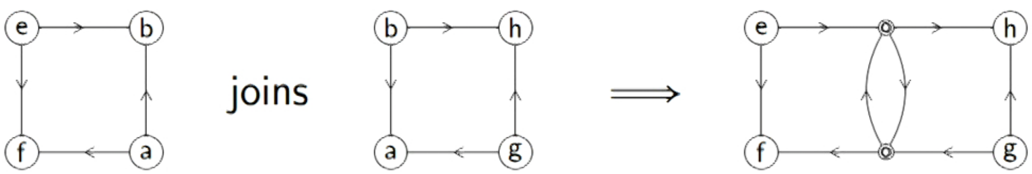
If causal diagrams are wired together in a non-DAG fashion we will get "causal loops" and the diagram will not simplify further.

Example of a non-DAG causal diagram

Consider the simple network

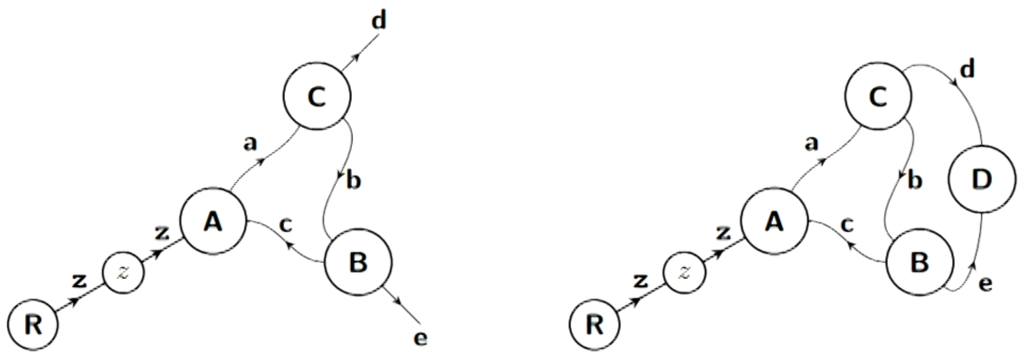


We obtain the causal diagram as follows



The loop cannot be removed by simplification.

Networks and circuits



- Networks and circuits do not appear to be DAG.
- However causal diagrams must be DAGs.
- A circuit has no wires left open so it must simplify to the empty set diagram. If there are causal loops they block this simplification.

Physicality

We will have two physicality requirements

- Tester positivity.
- Double causality.

Special complex operations

We have the following special complex operations to consider

- **readout boxes**
- **R boxes**
- **I boxes**

Physicality

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- Tester positivity.
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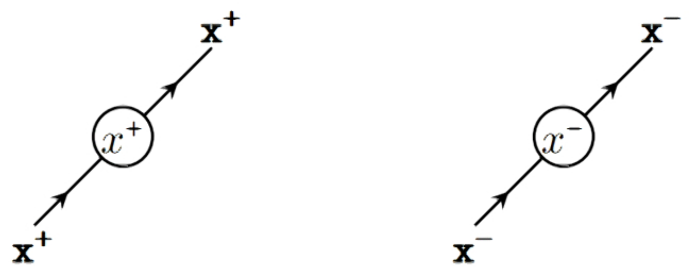
Special complex operations

We have the following special complex operations to consider

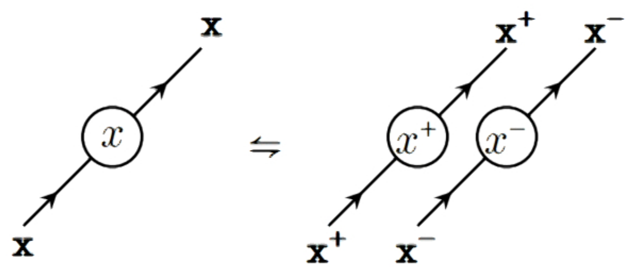
- **readout boxes**
- **R boxes**
- **I boxes**

Readout boxes

We have x^+ and x^- readout boxes



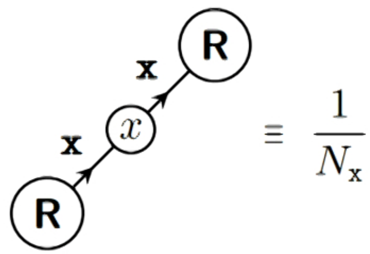
We can combine these in the compact notation



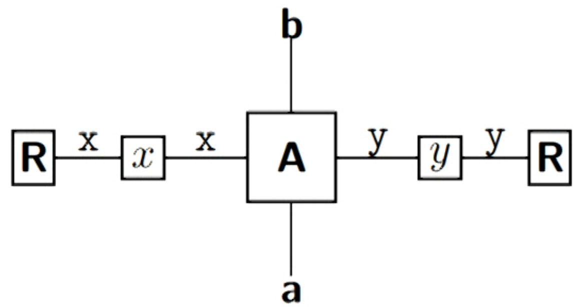
where $x = (x^+, x^-)$.

R boxes

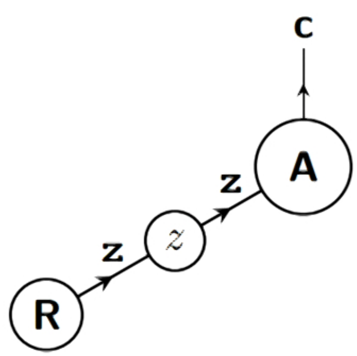
where



Illustrating compact notation



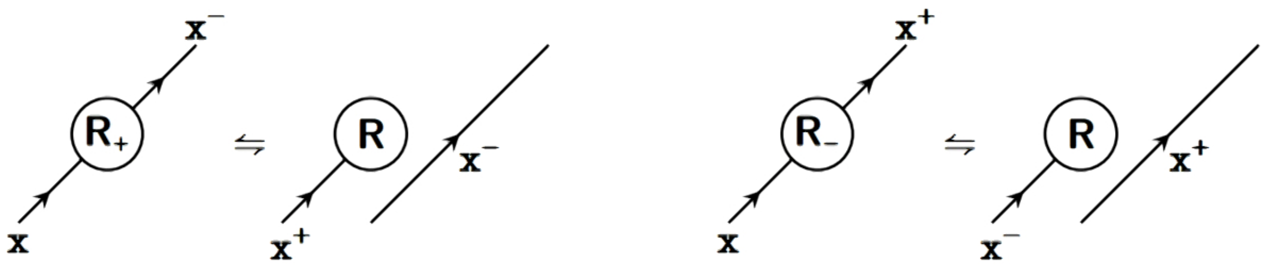
can be written as



where $\mathbf{z}^+ = \mathrm{x}$, $\mathbf{z}^- = \mathrm{y}$, $\mathbf{c}^+ = \mathrm{b}$, and $\mathbf{c}^- = \mathrm{a}$.

$\mathbf{R}_+$ and $\mathbf{R}_-$ boxes

These special boxes

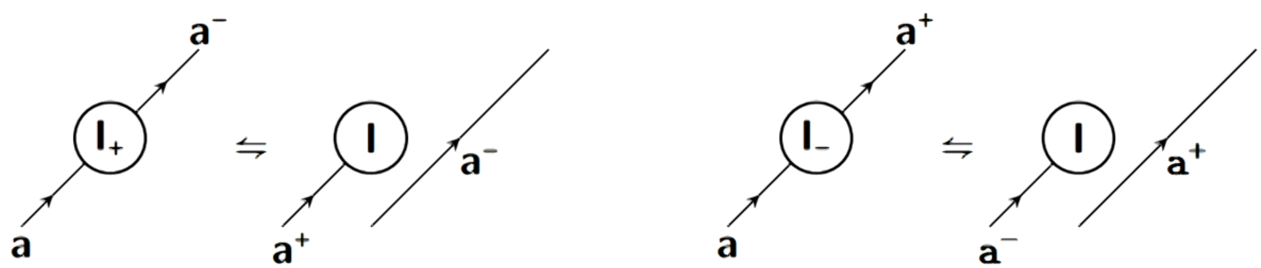


are used in the double causality condition.

I boxes

$\mathbf{I}_+$ and $\mathbf{I}_-$ boxes

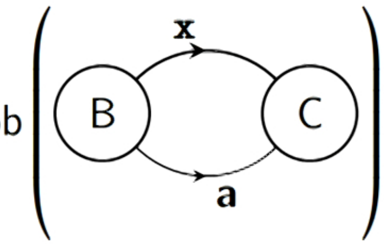
These special boxes



are used in the double causality condition.

Strong tester positivity

Strong tester positivity of B is the condition that

prob


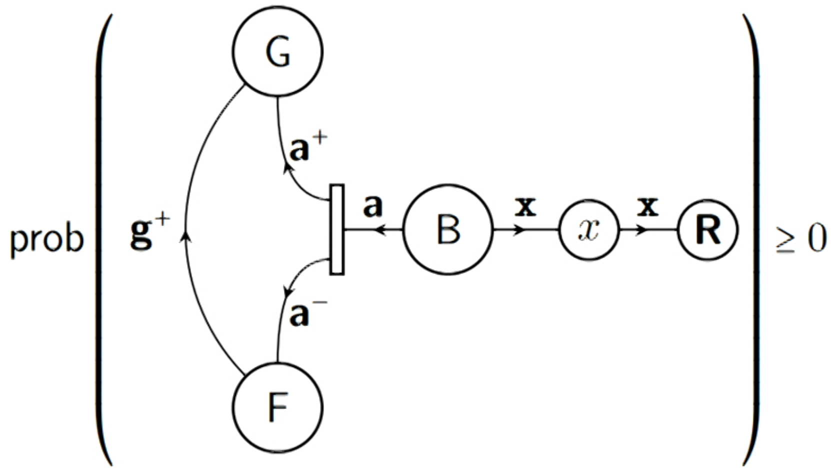
$\geq 0 \qquad \forall\ \mathsf{C}$

But this condition is too strong as it implicitly entails assuming that all pure C are tester positive.

A weaker and more interesting condition is . . .

Tester positivity

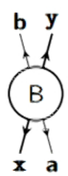
The tester positivity condition on B is that



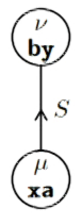
for all pure preparations G and pure results F.

Synchronous partition assumption with tester positivity.

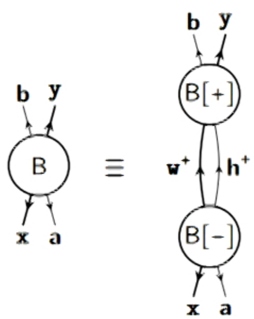
For any tester positive causally complex operation



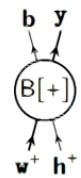
having c.d.



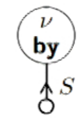
we can write



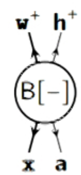
for some $\mathbf{h}^+$ *and* $\mathbf{w}^+$*, where*



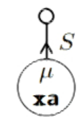
has c.d.



and

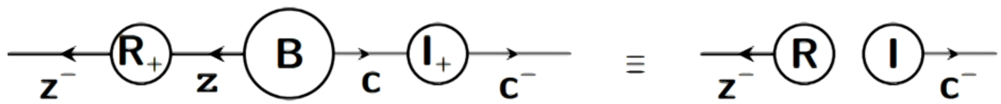


has c.d.

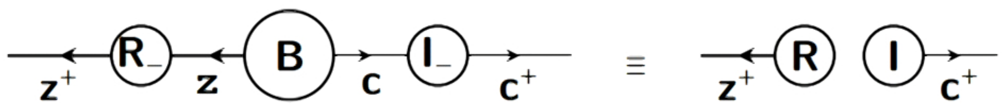


where the causally complex operations B[+] *and* B[−] *are tester positive.*

Simple double causality conditions on a deterministic complex operations B *consists of the simple forward causality condition is*



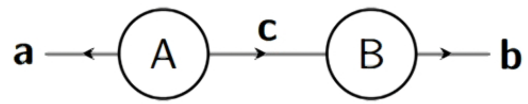
and the simple backward causality condition is



This is the same as the simple double causality condition given earlier. These simple causality conditions are not the full set of causality conditions in the causally complex case.

Positivity composition theorem

Positivity composition theorem. *Assume all pure preparations and pure results satisfy tester positivity and that the synchronous partition assumption with T-positivity holds. If we wire together two or more causally complex operations, each satisfying tester positivity, then the resulting network will satisfy tester positivity.*



The proof of this is not trivial since the **c** wire joining A and B can consist of many + and − parts. The proof is iterative and involves taking "bites" out of the causal diagram from one side, then the other.

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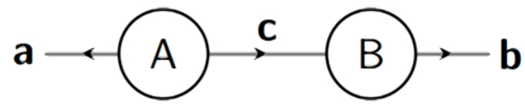

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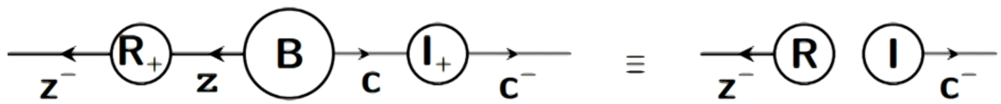
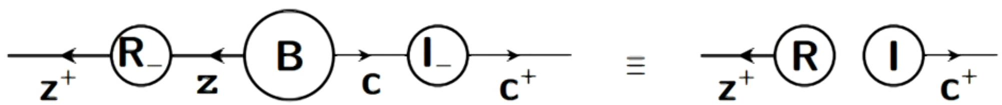


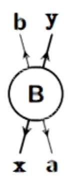
and the simple backward causality condition is



This is the same as the simple double causality condition given earlier. These simple causality conditions are not the full set of causality conditions in the causally complex case.

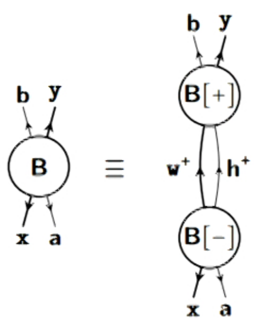
Synchronous partition assumption with simple double causality.

For any deterministic complex operator

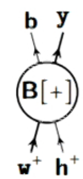


having c.d.

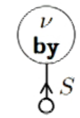
satisfying simple double causality we can write



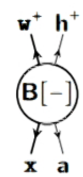
for some $\mathbf{h}^+$ *and* $\mathbf{w}^+$*, where*



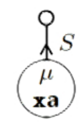
has c.d.



and



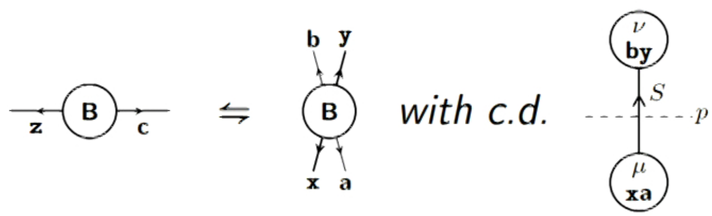
has c.d.



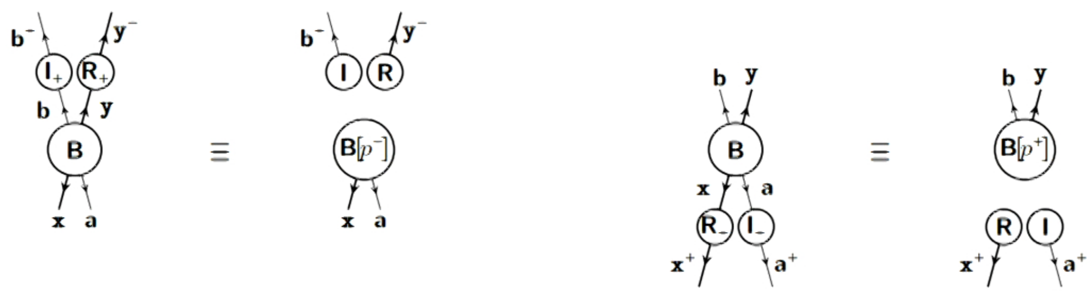
where the causally complex operations B[+] *and* B[−] *each satisfy simple double causality.*

Double causality theorem part A

Double causality theorem: Part A. *Assume the synchronous partition assumption with simple double causality holds. Consider a deterministic causally complex operation* **B** *satisfying simple double causality such that*

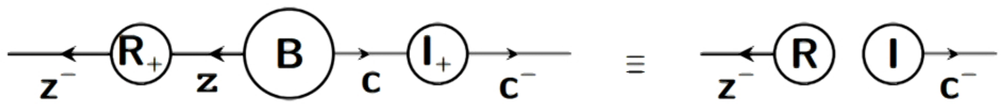


then we have the double causality conditions

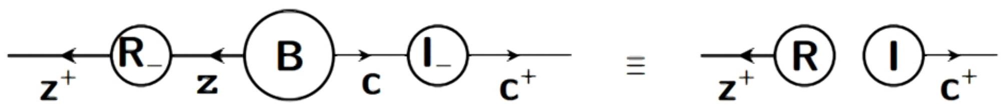


where $\mathbf{B}[p^{\pm}]$ *are deterministic complex operations satisfying simple double causality.*

Simple double causality conditions on a deterministic complex operations B *consists of the simple forward causality condition is*



and the simple backward causality condition is



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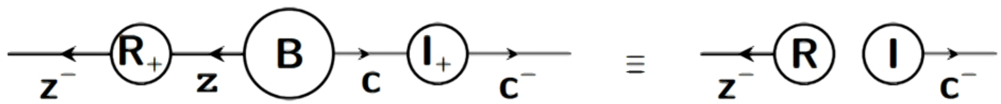
Simple double causality conditions on a deterministic complex operations B *consists of the simple forward causality condition is*



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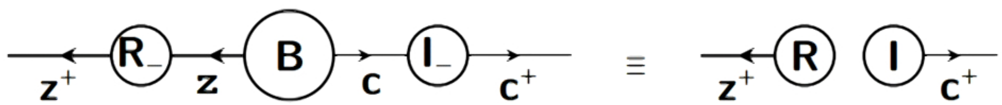

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Double causality theorem part A

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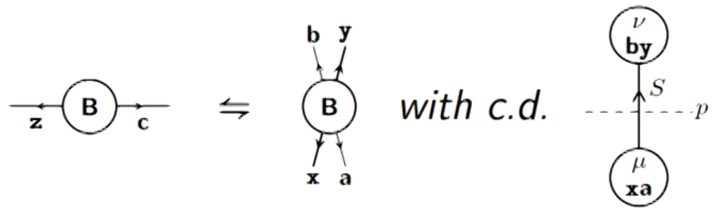
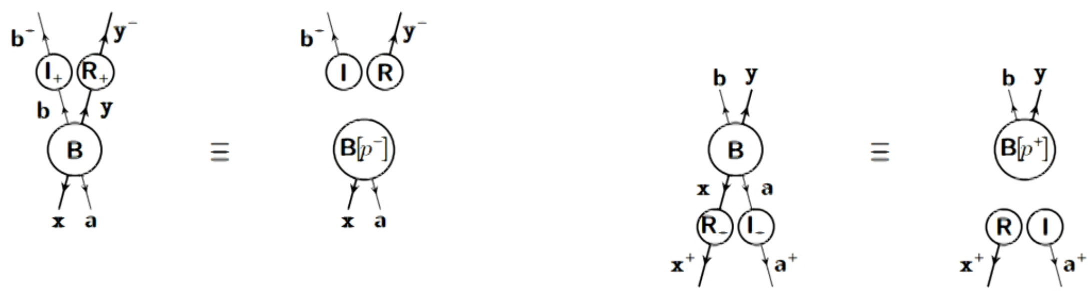


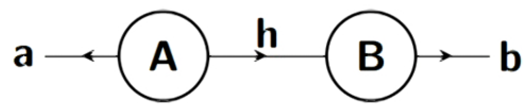
then we have the double causality conditions



where $\mathbf{B}[p^{\pm}]$ *are deterministic complex operations satisfying simple double causality.*

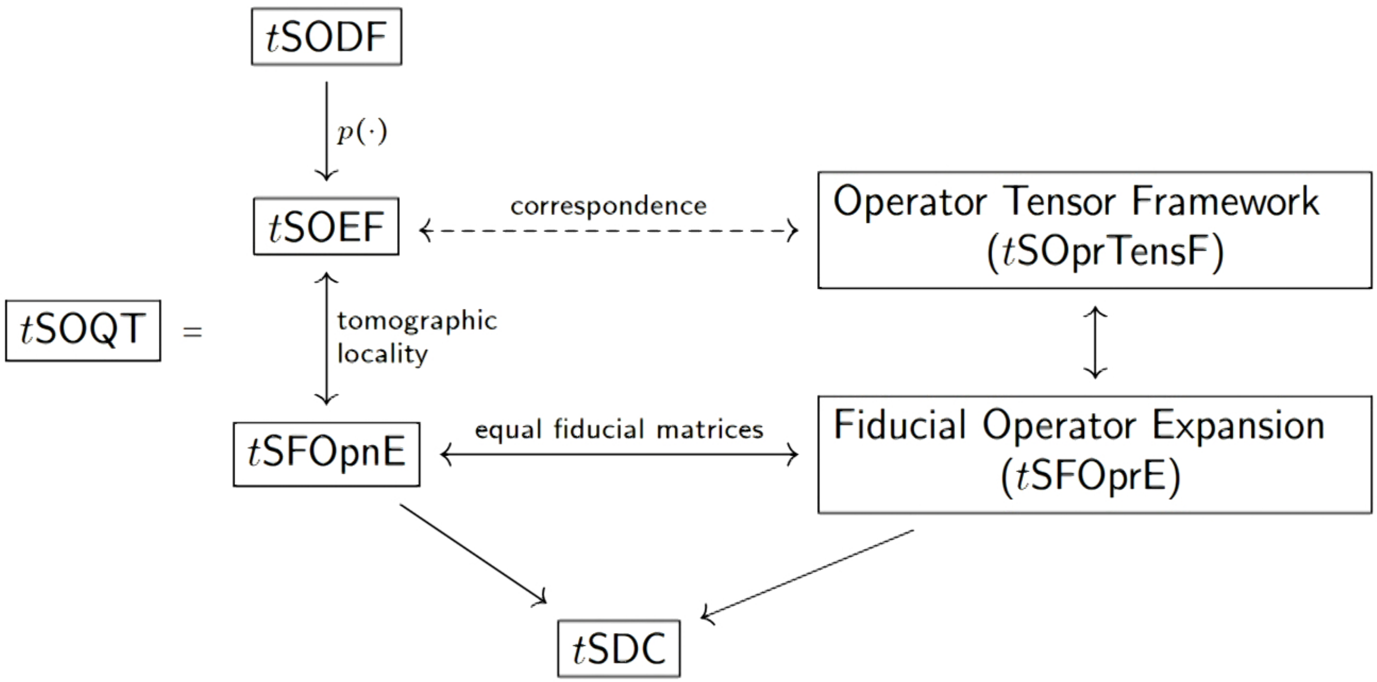
Double causality composition theorem

Double causality composition theorem. *If we join together two or more deterministic causally complex operations each of which satisfies the double causality conditions then the resulting network will also satisfy double causality.*



The proof of this is nontrivial. It involves taking iterative "bites" out of the causal diagram.

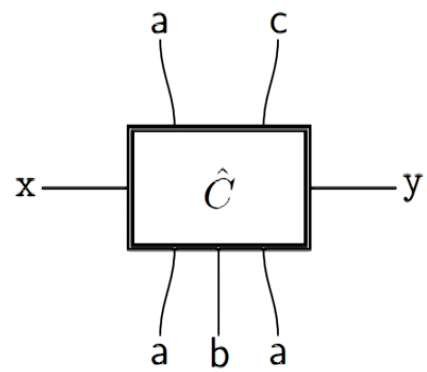
Flowchart for tSOQT



We have t =TS.

Operator Tensors

We represent operator tensors as



This is an element of the space

$$\mathcal{P}_{x_1} \otimes \mathcal{V}_{a_3} \otimes \mathcal{V}_{b_4} \otimes \mathcal{V}_{a_5} \otimes \mathcal{P}^{y_2} \otimes \mathcal{V}^{a_6} \otimes \mathcal{V}^{a_6} \otimes \mathcal{V}^{c_7}$$

where

$$\mathcal{V}_{a_1} \subset \mathcal{L}_{a_1} := \mathcal{H}_{a_1} \otimes \overline{\mathcal{H}}_{a_1}$$

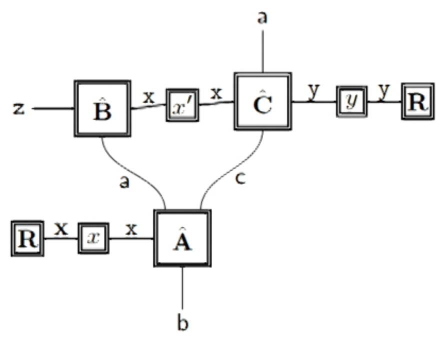
is the set of Hermitian operators in $\mathcal{L}_{a_1}$ and

the income space $\mathcal{P}_{x_1}$ is a real vector space of dimension N_x

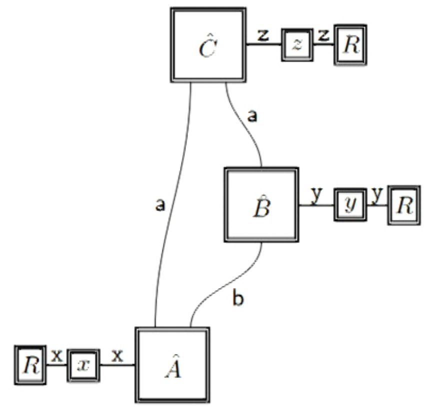
the outcome space $\mathcal{P}^{y_2}$ is a real vector space of dimension N_y

Networks and circuits

Network



Circuit.



Operator Tensors

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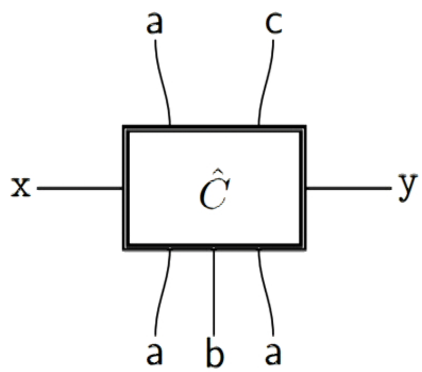

This is an element of the space

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Networks and circuits

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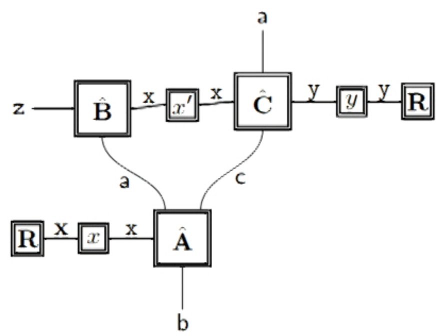
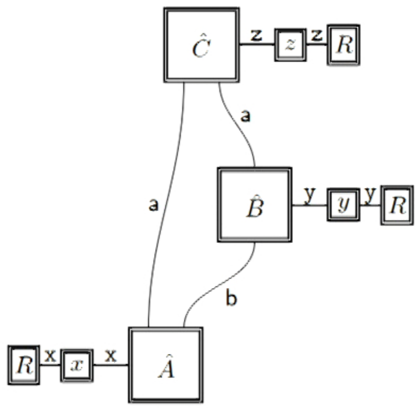
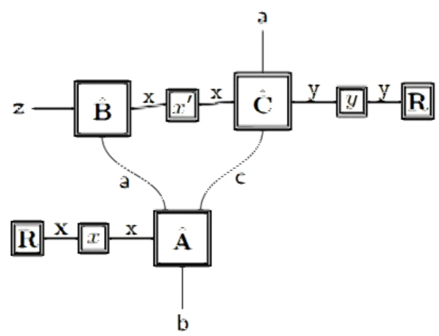


Circuit.

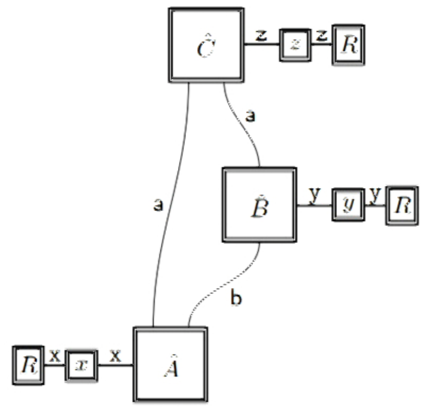


Networks and circuits

Network



Circuit.



The physical wires between boxes (e.g. a) correspond to taking the partial trace. The pointer wires between boxes (e.g. x) correspond to the usual tensor summation.

The circuit evaluates to a real number (since the operator tensors are Hermitian).

Special operator tensors

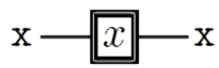
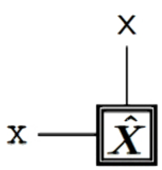
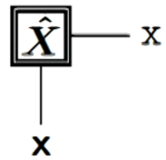
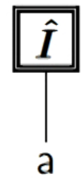
 and 	$\boldsymbol{R}^{\mathrm{x}_1}$ and $\boldsymbol{R}_{\mathrm{x}_1}$	Flat distribution operators
	$O_{\mathrm{x}_1}^{\mathrm{x}_2}[x]$	Readout box operator
 and 	$\hat{\boldsymbol{X}}_{\mathrm{x}_1}^{\mathrm{x}_2}$ and $\hat{\boldsymbol{X}}_{\mathrm{x}_2}^{\mathrm{x}_1}$	Maximal output and maximal input operators
 and 	$\hat{\boldsymbol{I}}^{\mathrm{a}_1}$ and $\hat{\boldsymbol{I}}_{\mathrm{a}_1}$	Ignore output and input operators

Table: Operators that play a special role. We provide diagrammatic and symbolic notation for these operators.

Normalisation gauge parameters α_{a} and α^{a}

$= \alpha^{\mathsf{a}} \hat{\mathbb{1}}^{\mathsf{a}_1}$



$= \alpha_{\mathsf{a}} \hat{\mathbb{1}}_{\mathsf{a}_1}$

where

$$\mathbb{1}^{\mathsf{a}_1} = \sum_{a=1}^{N_{\mathsf{a}}} |a\rangle^{\mathsf{a}_1} \langle a| \qquad \mathbb{1}_{\mathsf{a}_1} = \sum_{a=1}^{N_{\mathsf{a}}} |a\rangle_{\mathsf{a}_1} \langle a|$$

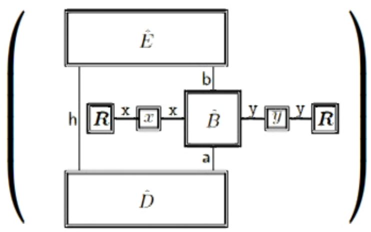
where

$$\alpha^{\mathsf{a}} \alpha_{\mathsf{a}} = 1$$

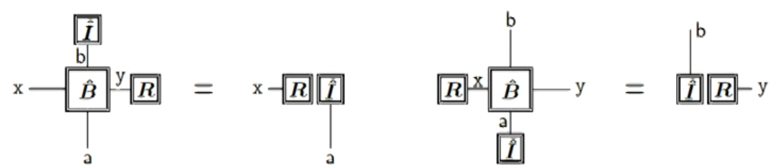
Physicality conditions

We get physicality conditions from TSSOPT by correspondence

Tester positivity

$0 \leq \text{prob}$ 

Simple double causality



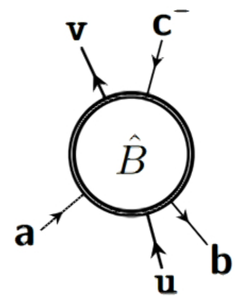
Axioms for Simple Quantum Theory

For DAG circuits we have the following axioms:

Axioms for Simple Operational Quantum Theory.

1. *All operations are physical.*
2. *All physical operators have an operation corresponding to them.*

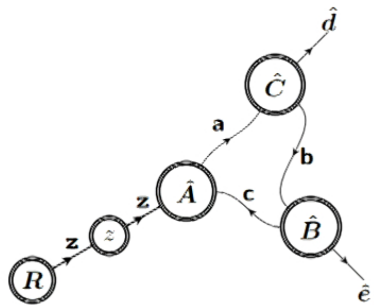
Basic object - complex operator tensor

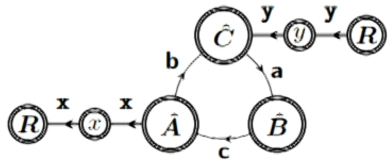


This is an element of the space

$$\mathcal{V}_{\mathbf{a}_1} \otimes \mathcal{V}_{\mathbf{c}_2^-} \otimes \mathcal{P}_{\mathbf{u}_3} \otimes \mathcal{V}^{\mathbf{b}_4} \otimes \mathcal{P}^{\mathbf{v}_5}$$

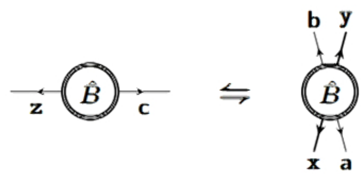
Networks and Circuits



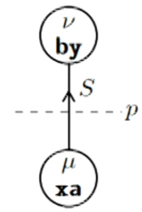


Physicality conditions - double causality

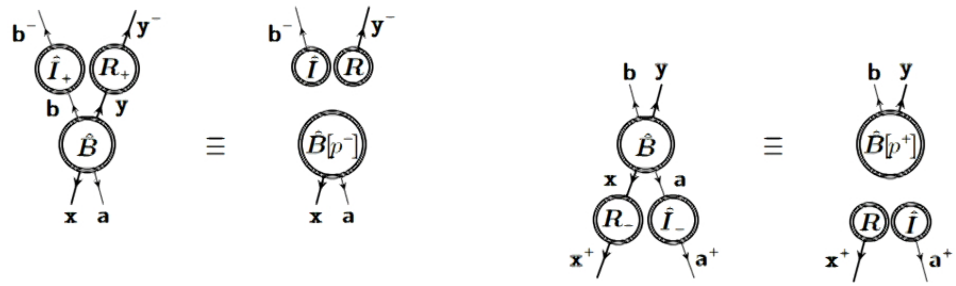
If we have



with c.d.



then



where $\hat{B}[p^{\pm}]$ also satisfy these double causality conditions in place of $\mathbf{B}$.

Axioms for Complex Quantum Theory

Axioms for Complex Operational Quantum Theory.

0 *The causal diagram for any circuit is empty.*
1 *All operations are physical.*
2 *All physical operators have an operation corresponding to them.*

The causal circuit for a circuit can only be empty when we join DAG causal structures appropriately.

Axioms for Complex Quantum Theory

Axioms for Complex Operational Quantum Theory.

0. *The causal diagram for any circuit is empty.*
1. *All operations are physical.*
2. *All physical operators have an operation corresponding to them.*

The causal circuit for a circuit can only be empty when we join DAG causal structures appropriately.

Conclusions

- We have seen how to formulate the causally simple and complex case for OPT and OQT.
- There is much I haven't been able to show you. The treatment of Hilbert space is particularly interesting in the complex case.
- This work is meant to be a stepping stone to a study of indefinite causal structure in Quantum Gravity.
- The Quantum Equivalence Principle (if true) would be a place where understanding causality would be important.