Title: Equivalence of 1-loop RG flows in 4d Chern-Simons and integrable 2d sigma-models

Speakers: Nat Levine

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Abstract: Costello, Witten and Yamazaki proposed a 4d Chern-Simons theory as a unified way to engineer integrable models. In the presence of 'Disorder' defects (for non-ultralocal 2d theories), this correspondence has been established only classically. As a first quantum check, I will derive the matching of 1-loop divergences between the 4d and 2d theories. My assumptions are general and seem to isolate sigma-models among the 2d theories. (Based on 2309.16753)

Zoom link https://pitp.zoom.us/j/99362983669?pwd=NE1uQ3FmWityQ1R0NUVnZkRPZTRUdz09

Equivalence of 1-loop RG flows in 4d Chern-Simons and Integrable 2d sigma-models

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[NL 2309.16753]
[NL 2209.05502]
[Wallberg, Lacroix, NL 23xx.xxxxx]

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Integrable sigma-models

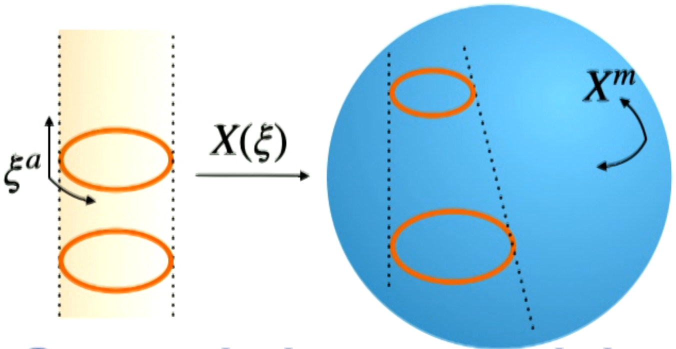
[Pohlmeyer 76] reduction

O(N) sigma-model $\mathscr{L} = (\partial n_a)^2 \quad (n_a^2 = 1)$

$a = 1, \ldots, N$

↓

sine Gordon $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + m^2 \cos g\phi$



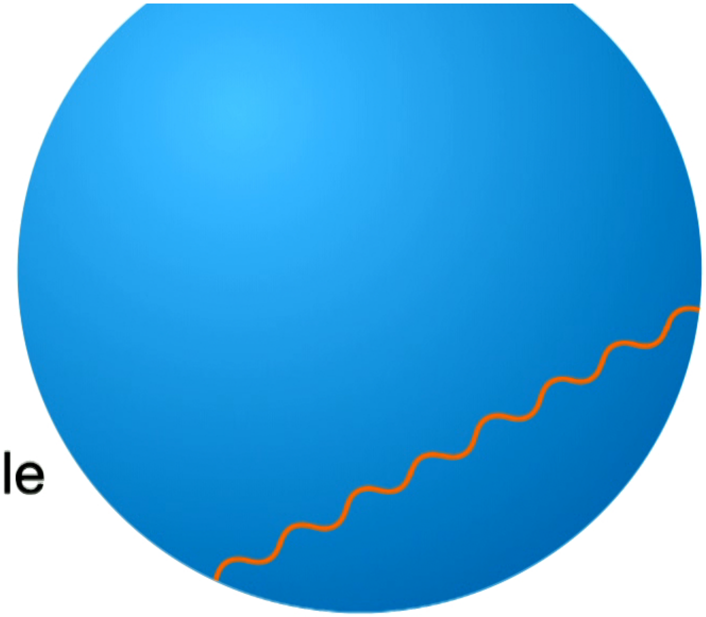
$$S = \int d^2\xi \, \mathscr{L}$$

General sigma-models $\mathscr{L} = (G(X) + B(X))_{MN} \, \partial_+ X^M \, \partial_- X^N$

$(\partial_\pm = \partial_0 \pm \partial_1)$

Classical integrability EOM $\longleftrightarrow$ $F_{+-}\big(L(z, X)\big) \equiv \partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0$

Integrability in string theory



- Strings in curved space: very hard
- Typical case: chaos   Special case: integrable
- Success story: $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM

- Modify while preserving integrability?
- Different dimension… Less supersymmetry… **Deform geometry…**

Integrable landscape

$J_a = g^{-1}\partial_a g\,,\ g \in G$	PCM	PCM + k WZ	η deformed PCM [Klimcik 02,08]	λ deformed PCM [Sfetsos 13]
Lagrangian $\mathscr{L} = (G+B)_{MN}\partial_+ X^M \partial_- X^N$	$h\,\mathrm{Tr}[J_+ J_-]$	$h\,\mathrm{Tr}[J_+ J_-] + k\,\mathrm{WZ}$	$h\,\mathrm{Tr}[J_+ \frac{1}{1-\eta\mathscr{R}_g} J_-]$	$k\Big(\mathcal{L}_{\mathrm{WZW}}(g) + \mathrm{Tr}[J_+ \frac{\mathrm{Ad}_g}{\mathrm{Ad}_g - \lambda^{-1}} J_-]\Big)$
Lax $L_\pm = \frac{1}{1 \pm z}\mathcal{A}_\pm$	$\mathcal{A}_\pm = J_\pm$	$\mathcal{A}_\pm = \left(1 \pm \frac{k}{h}\right) J_\pm$	$\mathcal{A}_\pm = \frac{1+\eta^2}{1 \pm \eta \mathscr{R}_g} J_\pm$	$\mathcal{A}_+ = \frac{2}{1+\lambda}\frac{1}{\lambda^{-1} - \mathrm{Ad}_g^{-1}} J_+$ $\mathcal{A}_- = \frac{2}{1+\lambda}\frac{1}{1 - \lambda^{-1}\mathrm{Ad}_g^{-1}} J_-$

?!?!?

Which sigma-models are integrable?

4d Chern-Simons

[Costello Witten Yamazaki 17,18] [Costello Yamazaki 19]
[Delduc Lacroix Magro Vicedo 20]

$$A = A_+ d\xi^+ + A_- d\xi^- + A_{\bar{z}} d\bar{z} \ (+A_z dz)$$

$$S_{4d}[A] = \int_{\Sigma_2 \times \mathbb{C}P^1} dz\, \phi(z) \wedge \left(A \wedge dA + \frac{2}{3} A^3\right)$$

ξ
2d co-ords

z
spectral parameter

Integrable 2d theory specified by:

Meromorphic 1-form $dz\, \phi(z)$

Zeros	Poles
'Disorder' defects	**2d theory lives here**

Quantum equivalence ?

- Equivalence established: **classical**

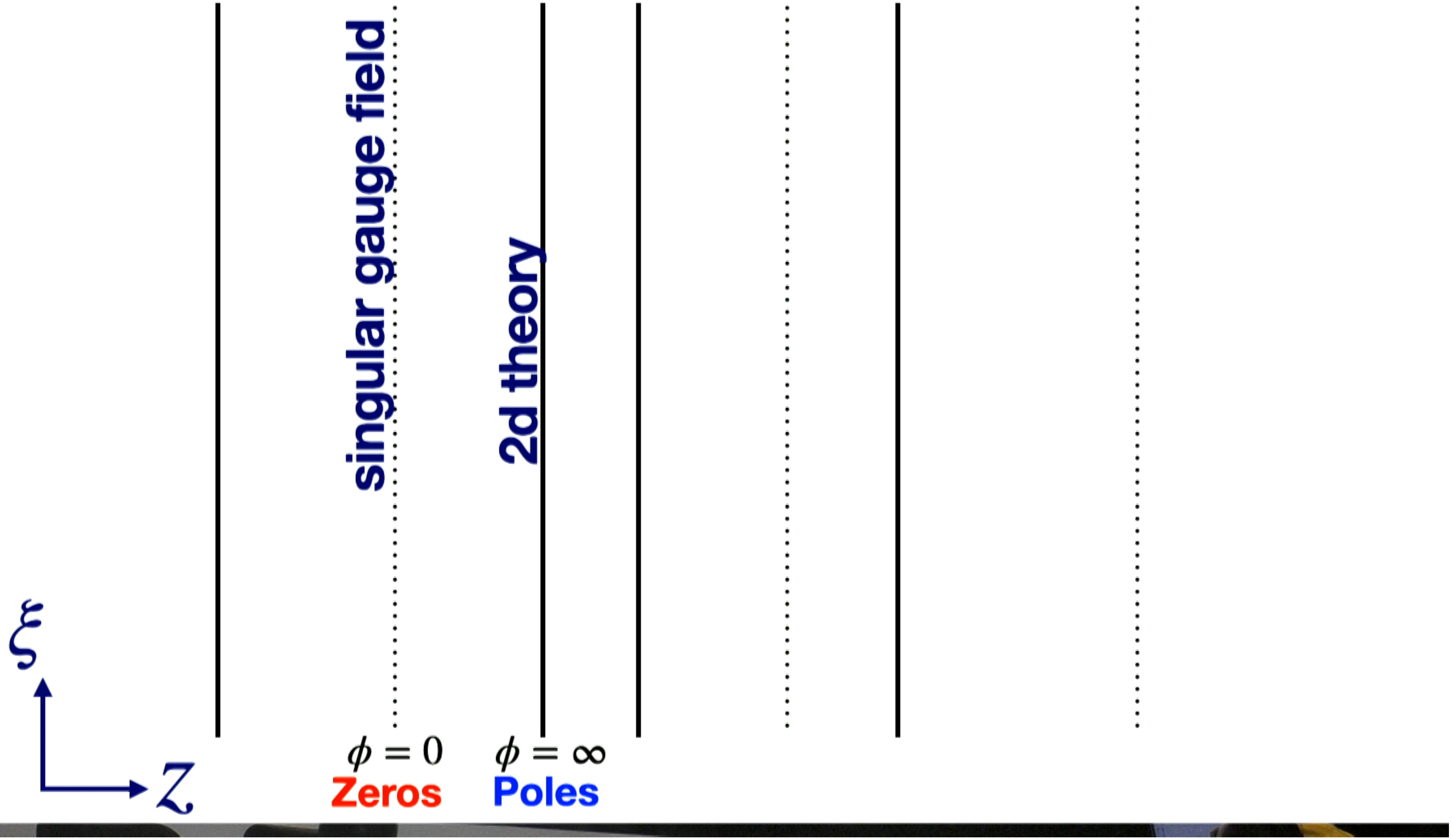
  [Costello Yamazaki 19] [Benini Schenkel Vicedo 20]
  [Delduc Lacroix Magro Vicedo 20] [Lacroix Vicedo 21]

Cf. 'Order' defects [Costello Yamazaki 19]
And lattice models [Ashwinkumar Sakamoto Yamazaki 23]

- Quantum equivalence?

- Watch out: — Cf. [Fadeev Reshetikhin 86]
  May have classical equivalence between different quantum theories

  — Disorder defects ~ places where $\hbar \to \infty$


singular gauge field
2d theory
ξ
Z
$\phi = 0$
Zeros
$\phi = \infty$
Poles

Classical equivalence

Review

$$S_{4d}[A] = \int_{\Sigma_2 \times \mathbb{C}P^1} dz\, \phi(z) \wedge \left(A \wedge dA \, + \, \frac{2}{3} A^3\right)$$

EOMs
$$\begin{cases} \phi(z)\, F_{\bar{z}+} = 0 \\ \phi(z)\, F_{\bar{z}-} = 0 \\ \phi(z)\, F_{+-}(L) = 0 \end{cases}$$

Classical equivalence

Review

$$S_{4d}[A] = \int_{\Sigma_2 \times \mathbb{C}P^1} dz\, \phi(z) \wedge \left(A \wedge dA + \frac{2}{3} A^3\right)$$

EOMs
$$\begin{cases} \phi(z)\, F_{\bar{z}+} = 0 \\ \phi(z)\, F_{\bar{z}-} = 0 \\ \phi(z)\, F_{+-}(L) = 0 \end{cases}$$

Change variable $(A_{\bar{z}}, A_{\pm}) \to (g, L_{\pm})$

$$A_{\bar{z}} = -\,\partial_{\bar{z}} g\, g^{-1}$$

$$A_{\pm} = g\, L_{\pm}\, g^{-1} - \partial_{\bar{z}} g\, g^{-1}$$

EOMs
$$\begin{cases} \phi(z)\, \partial_{\bar{z}} L_{\pm} = 0 \\ F_{+-}(L) = 0 \end{cases}$$

$\phi(z)\, \partial_{\bar{z}} L_{\pm} = 0 \to$ $L_{\pm}(z, C)$ meromorphic. Poles specified by ϕ

$F_{+-}(L) = 0 \to$ zero-curvature eqs

Boundary conditions

Boundary term $\delta S \propto \int d^2z \, d^2\xi \; \partial_{\bar{z}}\phi(z) \, \mathrm{Tr}[A_+ \delta A_- - A_- \delta A_+]$

localised to poles of $\phi(z)\, dz$

→ BCs at poles: Admissible BCs studied in

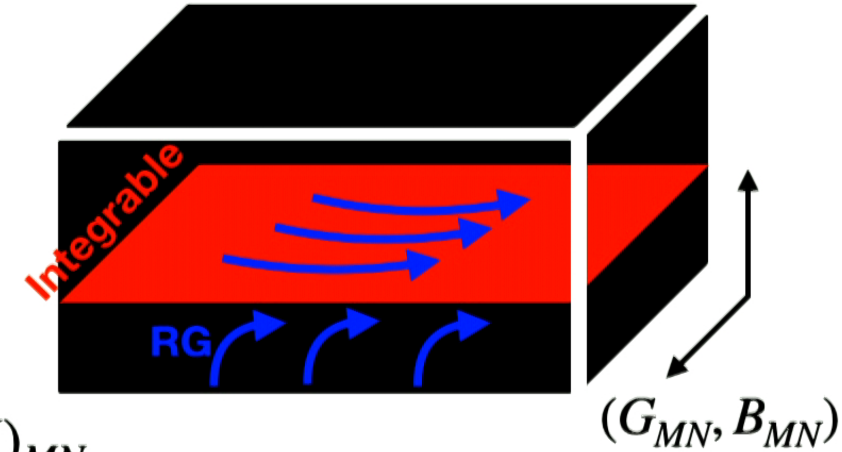
[Delduc Lacroix Magro Vicedo 20]
[Benini Schenkel Vicedo 20]
[Lacroix Vicedo 21]

$$\mathcal{A} = \mathcal{A}(g|_P)$$

g_{bulk} pure gauge

$$S_{4d} = S_{2d}[g\,|_P]$$ (affine Gaudin models)

Sigma-model RG flow



1-loop ~ Ricci flow   (bosonic)

$$\frac{d}{dt}(G_{MN} + B_{MN}) = R_{MN} - \frac{1}{4}(H^2)_{MN} - \frac{1}{2}(\nabla H)_{MN}$$

[Friedan 80] ...

$(t = \log \mu)$ $H = dB$

Conjecture: Integrable models will be renormalizable

[Fateev Onofri Zamolodchikov 93] [Lukyanov 12] [Litvinov et al]

Example: PCM + cousins

$$L_\pm = \frac{1}{1 \pm z} \mathcal{A}_\pm$$

$$F_{+-}(L) = \frac{1}{1 - z^2}(\partial \cdot \mathcal{A}) + \frac{z}{1 - z^2} F_{+-}(\mathcal{A})$$

EOM $\longleftrightarrow$ $\partial \cdot \mathcal{A} = F_{+-}(\mathcal{A}) = 0$

Flat, conserved current $\mathcal{A}_\pm$

$J_a = g^{-1}\partial_a g \, , \, g \in G$	**PCM**
Lagrangian $\mathscr{L} = (G+B)_{MN}\partial_+ X^M \partial_- X^N$	$h\,\mathrm{Tr}[J_+ J_-]$
Lax $L_\pm = \frac{1}{1 \pm z}\mathcal{A}_\pm$	$\mathcal{A}_\pm = J_\pm$

$J_a = g^{-1}\partial_a g\,,\ g \in G$	**PCM**	**PCM + k WZ**	η **deformed PCM** [Klimcik 02,08]	λ **deformed PCM** [Sfetsos 13]
Lagrangian $\mathscr{L} = (G+B)_{MN}\partial_+ X^M \partial_- X^N$	$h\,\mathrm{Tr}[J_+ J_-]$	$h\,\mathrm{Tr}[J_+ J_-] + k\,\mathrm{WZ}$	$h\,\mathrm{Tr}[J_+ \frac{1}{1-\eta \mathscr{R}_g} J_-]$	$k\Big(\mathcal{L}_{\mathrm{WZW}}(g) + \mathrm{Tr}[J_+ \frac{\mathrm{Ad}_g}{\mathrm{Ad}_g - \lambda^{-1}} J_-]\Big)$
Lax $L_\pm = \frac{1}{1 \pm z}\mathcal{A}_\pm$	$\mathcal{A}_\pm = J_\pm$	$\mathcal{A}_\pm = \left(1 \pm \frac{k}{h}\right) J_\pm$	$\mathcal{A}_\pm = \frac{1+\eta^2}{1 \pm \eta \mathscr{R}_g} J_\pm$	$\mathcal{A}_+ = \frac{2}{1+\lambda}\frac{1}{\lambda^{-1} - \mathrm{Ad}_g^{-1}} J_+$ $\mathcal{A}_- = \frac{2}{1+\lambda}\frac{1}{1 - \lambda^{-1}\mathrm{Ad}_g^{-1}} J_-$

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1-loop RG flow	$\frac{d}{dt} h = c_G$	$\frac{d}{dt} h = c_G\left(1 - \frac{k^2}{h^2}\right)$ $\frac{d}{dt} k = 0$	$\frac{d}{dt} h = c_G (1+\eta^2)^2$ $\frac{d}{dt} \eta = c_G \frac{\eta}{h} (1+\eta^2)^2$	$\frac{d}{dt}\lambda = -2\frac{c_G}{k}\left(\frac{\lambda}{1+\lambda}\right)^2$ $\frac{d}{dt} k = 0$
$\vert\vert\vert$ $\frac{d}{dt}\widehat{\mathscr{L}}^{(1)}$	$c_G\,\mathrm{Tr}[J_+ J_-]$	$c_G\left(1 - \frac{k^2}{h^2}\right) \mathrm{Tr}[J_+ J_-]$		
$\mathrm{Tr}[\mathcal{A}_+ \mathcal{A}_-]$	$\mathrm{Tr}[J_+ J_-]$	$\left(1 - \frac{k^2}{h^2}\right) \mathrm{Tr}[J_+ J_-]$		

Universal 1-loop divergences [NL 2209.05502]

cf. [Appadu Hollowood 15]

Claim: For integrable sigma-models* with $L_\pm = \dfrac{1}{1 \pm z} \mathcal{A}_\pm$,

$$\frac{d}{dt} \widehat{\mathscr{L}}^{(1)} = c_G \, \mathrm{Tr}[\mathcal{A}_+ \mathcal{A}_-]$$

*under some technical assumptions

General claim: For any pole structure $L_\pm = \displaystyle\sum_{i,p} \frac{1}{(z - z_i)^p} \mathcal{A}_\pm^{i,p}$,

$\dfrac{d}{dt} \widehat{\mathscr{L}}^{(1)}(\mathcal{A})$ is universal (depends only on poles + multiplicities)

$J_a = g^{-1}\partial_a g\,,\ g \in G$	**PCM**	**PCM + k WZ**	η **deformed PCM** [Klimcik 02,08]	λ **deformed PCM** [Sfetsos 13]
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Lax $L_\pm = \frac{1}{1\pm z}\mathcal{A}_\pm$	$\mathcal{A}_\pm = J_\pm$	$\mathcal{A}_\pm = \left(1 \pm \frac{k}{h}\right) J_\pm$	$\mathcal{A}_\pm = \frac{1+\eta^2}{1\pm\eta\mathscr{R}_g} J_\pm$	$\mathcal{A}_+ = \frac{2}{1+\lambda}\frac{1}{\lambda^{-1}-\mathrm{Ad}_g^{-1}} J_+$ $\mathcal{A}_- = \frac{2}{1+\lambda}\frac{1}{1-\lambda^{-1}\mathrm{Ad}_g^{-1}} J_-$

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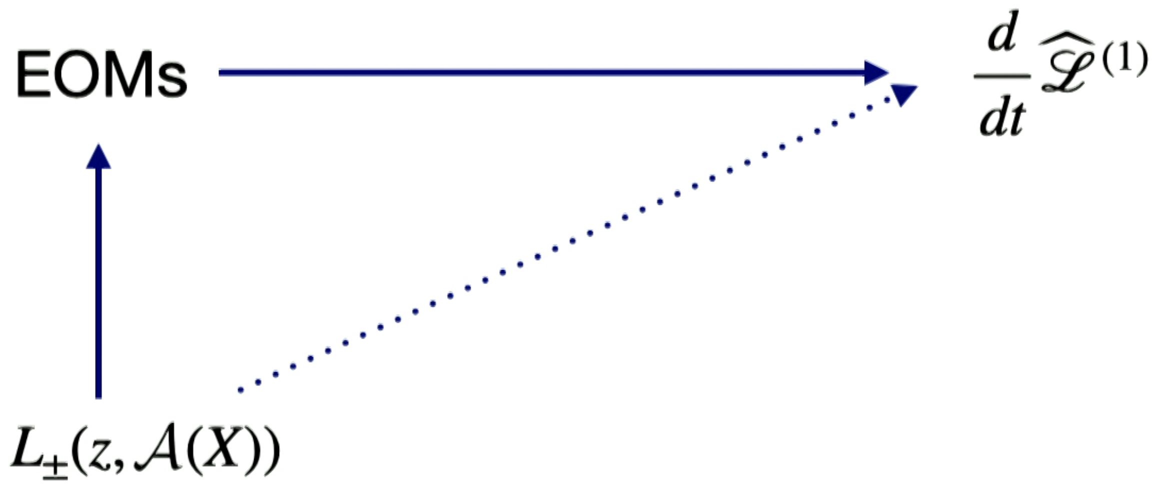
The idea

$$\frac{d}{dt}\widehat{\mathscr{L}}^{(1)}$$

$$L_{\pm}(z, \mathcal{A}(X))$$

The idea



The subtlety

$$F_{+-}\big(L(z,\mathcal{A})\big) = 0 \quad \forall z \qquad \Longleftrightarrow \qquad \{\mathrm{zc}_S(\mathcal{A}) = 0\}$$

$$\Longleftrightarrow \qquad \{\mathrm{EOM}_s(\mathcal{A}) = 0\} + \{\mathrm{Bianchi}_\sigma(\mathcal{A}) \equiv 0\}$$

$$S = (s, \sigma)$$

The split is non-universal

E.g. $L_\pm = \frac{1}{1 \pm z}\mathcal{A}_\pm \qquad \{\mathrm{zc}_S(\mathcal{A}) = 0\} = \{\partial \cdot \mathcal{A} = 0,\ F_{+-}(\mathcal{A}) = 0\}$

PCM	$\mathcal{A}_\pm = J_\pm$
EOM	$\partial \cdot \mathcal{A} = 0$
Bianchi	$F_{+-}(\mathcal{A}) \equiv 0$

PCM + k WZ	$\mathcal{A}_\pm = (1 \pm \frac{k}{h})J_\pm$
EOM	$\partial \cdot \mathcal{A} = 0$
Bianchi	$F_{+-}(\mathcal{A}) + \frac{k}{h}(\partial \cdot \mathcal{A}) \equiv 0$

The resolution (informal version)

1-loop (on-shell) divergences aren't sensitive to the difference between EOMs and Bianchis

$$\langle \text{Bianchi} \cdots \rangle = 0 \ , \qquad \langle \text{EOM} \cdots \rangle = \hbar\,(\text{contact})$$

Path integral argument

"Bianchi Completeness" Assumption

$$B_\sigma(\bar{X}) \cdot \alpha = 0 \quad \Longleftrightarrow \quad \bar{A} + \alpha = \mathcal{A}(\bar{X} + Y)$$

for some Y

- i.e. all Bianchi identities follow from zero-curvature
- seems to pick out sigma-models
- sufficient condition: $\mathcal{A}_{\pm} = O_{\pm}(g) \cdot g^{-1}\partial_{\pm} g$

Path integral argument

$$\hat{S}^{(1)}[\bar{X}] = -\,i \log \int \mathcal{D}Y^s \exp i \int \underbrace{Y^s O_{st}(\bar{X})\, Y^t}_{\mathscr{L}(\bar{X}+Y) - \mathscr{L}(\bar{X})} = \frac{i}{2} \log \det O(\bar{X})$$

alternate
path integral

$$= -\frac{i}{2} \log \int \mathcal{D}u^s\ \underline{\mathcal{D}Y^s}\ \exp i \int u^s\, O_{st}(\bar{X})\, Y^t$$

$$= \int \mathcal{D}\alpha_\pm\, \delta\big(B_\sigma(\bar{X}) \cdot \alpha\big) \qquad E_s(\bar{X}) \cdot \alpha$$

change variables

$X \to \mathcal{A}_\pm$

$Y \to \alpha_\pm$

$$= -\frac{i}{2} \log \int \mathcal{D}u^s\ \mathcal{D}\alpha_\pm\ \delta(B_\sigma(\bar{X}) \cdot \alpha)\ \exp i \int u^s E_s(\bar{X}) \cdot \alpha$$

Path integral argument

$$\widehat{S}^{(1)}[\bar{X}] = -\,i \log \int \mathcal{D}Y^s \exp i \int \underbrace{Y^s\, O_{st}(\bar{X})\, Y^t}_{\mathscr{L}(\bar{X}+Y)-\mathscr{L}(\bar{X})} = \frac{i}{2} \log \det O(\bar{X})$$

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Path integral argument

$$\widehat{S}^{(1)}[\bar{X}] = -\frac{i}{2}\log\int \mathcal{D}u^s\ \mathcal{D}v^\sigma\ \mathcal{D}\alpha_\pm\ \exp i\int \left[u^s E_s(\bar{X}) + v^\sigma B_\sigma(\bar{X})\right]\cdot\alpha$$

$$= -\frac{i}{2}\log\int \mathcal{D}U^S\ \mathcal{D}\alpha_\pm\ \exp i\int U^S\, \mathrm{zc}_{\mathrm{S}}\left(\bar{\mathcal{A}} + \alpha\right)$$

= Vol { flat connections with given pole structure }

Universal ✓

Much simpler to compute than direct approach

Examples

1) PCM + cousins

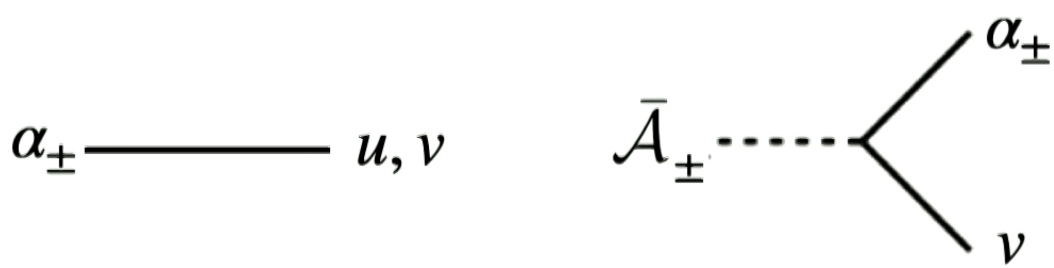
$$L_\pm = \frac{1}{1 \pm z} \mathcal{A}_\pm$$

$$\left\{ \mathrm{zc}_S(\mathcal{A}) = 0 \right\} = \left\{ \partial \cdot \mathcal{A} = 0,\ F_{+-}(\mathcal{A}) = 0 \right\}$$

$u, v \in \mathfrak{g}$

$$\widehat{S}^{(1)}[\bar{C}] = -\frac{i}{2} \log \int \mathcal{D}u\ \mathcal{D}v\ \mathcal{D}\alpha_\pm$$

$$\exp i \int \mathrm{Tr}\big[u\ \partial \cdot (\bar{\mathcal{A}} + \alpha) + v\ F_{+-}(\bar{\mathcal{A}} + \alpha)\big]$$



2) General simple pole theory

$$L_+ = \sum_{z_i} \frac{1}{z - z_i} \mathcal{A}^i_+ \,, \qquad L_- = \sum_{w_j} \frac{1}{z - w_j} \mathcal{A}^j_- \qquad \text{all } z_i \neq w_j$$

Universal result

$$\frac{d}{dt} \widehat{\mathscr{L}}^{(1)} = -4c_G \sum_{z_i, w_j} \frac{1}{(z_i - w_j)^2} \mathrm{Tr}[\mathcal{A}^i_+ \mathcal{A}^j_-]$$

E.g. affine Gaudin models (with simple zeros in twist)

- G^N models [Delduc Lacroix Magro Vicedo 18,19]
- η/λ deformations of them [Bassi Lacroix 20] ✔

Other examples more closely related to string theory: $\mathbb{Z}_T$ cosets

3) General case (curiosity)

$$\frac{d}{dt}\widehat{\mathscr{L}}^{(1)} = 4c_G \oint_{+\gamma_-} \frac{dz}{2\pi i}\, \mathrm{Tr}[L_+\, \partial_z L_-]$$

A 4d origin?

$$\frac{d}{dt}\widehat{S} = -i\log\int \mathcal{D}\alpha_{\pm}^{i,n}\,\mathcal{D}u^{s}\;\; \exp i\int d^2\xi\;\, u^{s}\;\, \mathrm{zc}_{s}\,(\bar{\mathcal{A}}+\alpha)$$

= Vol { flat connections with given pole structure }

$$\sim \text{“}{-i}\log\int [\mathcal{D}\ell_{\pm}]_{\text{fix poles}}\,\mathcal{D}u \;\; \exp i\int \mathrm{d}^2\xi\,\mathrm{d}^2 z\;\, \mathrm{Tr}\Big[u\,F_{+-}\,(\bar{L}+\ell)\Big]\text{”}$$

1-loop divergences in 4d

[NL 2309.16753]

$$S_{4d}[\bar{A} + a] = S_{4d}[\bar{A}] + \int \mathrm{Tr}[a^m \, \mathcal{O}_{mn}(\bar{A}) \, a^n] + \ldots$$

$(m, n = + \, , \, - \, , \bar{z})$

$$\widehat{S}^{(1)}_{4d}[\bar{A}] = -i \log \int_{\mathrm{B.C.s}} \frac{\mathcal{D}a_m}{\mathrm{gauge}} \, e^{i \int \mathrm{Tr}[a^m \, \mathcal{O}_{mn}(\bar{A}) \, a^n]}$$

$$= \tfrac{i}{2} \log \det{}' \mathcal{O}(\bar{A})$$

Same trick: alternate representation of $\det$

$$\widehat{S}^{(1)}_{4d}[\bar{A}] \;=\; \tfrac{i}{2}\log\det{}'\mathcal{O}(\bar{A})$$

$$= -\tfrac{i}{2}\log\int_{\text{B.C.s}} \frac{\mathcal{D}a_m\,\mathcal{D}u^m}{\text{gauge}}\;\exp i\int\;\text{Tr}[u^m\,\mathcal{O}_{mn}(\bar{A})\,a^n]$$

$$= -\tfrac{i}{2}\log\int_{\text{B.C.s}} \frac{\mathcal{D}a_m\,\mathcal{D}u^m}{\text{gauge}}\;\exp i\int\;\phi(z)\,\text{Tr}[u^{\bar{z}}\,F_{+-}(A)$$

$$+\,u^-\,F_{\bar{z}+}(A) + u^+\,F_{\bar{z}-}(A)]$$

Changing variables

$$(A_{\bar{z}}, A_{\pm}) \to (g, L_{\pm})$$

$$\begin{cases} A_{\bar{z}} = -\,\partial_{\bar{z}} g\, g^{-1} \\ A_{\pm} = g\, L_{\pm}\, g^{-1} - \partial_{\bar{z}} g\, g^{-1} \end{cases}$$

Fluctuations

$$\begin{cases} g = \bar{g}\,\gamma \\ L_{\pm} = \bar{L}_{\pm} + \ell_{\pm} \end{cases}$$

$$\widehat{S}^{(1)}_{4d} = -\tfrac{i}{2} \log \int_{\mathrm{B.C.s}} \frac{\mathcal{D}\gamma\ \mathcal{D}\ell_{\pm}\ \mathcal{D}u^m}{\text{gauge}} \exp i \int \phi(z)\, \mathrm{Tr}[u^{\bar{z}} F_{+-}(L) + u^{-}\, \partial_{\bar{z}} L_{+} + u^{+}\, \partial_{\bar{z}} L_{-}]$$

Note extra “2d gauge” sym

$$g \to g\, q \qquad \partial_{\bar{z}} q = 0$$

$$L_{\pm} \to q^{-1} L_{\pm}\, q + q^{-1}\, \partial_{\pm} q$$

Solving meromorphicity equations

Integrate u^+, u^- $\quad\longrightarrow\quad$ $L_{\pm}(z, \tilde{\mathcal{A}})$ meromorphic with fixed poles

$$\int \mathcal{D}\ell_{\pm}\, \mathcal{D}u^{\pm}\; \exp i \int \phi(z)\, \mathrm{Tr}[u^-\, \partial_{\bar{z}} L_+ + u^+\, \partial_{\bar{z}} L_-]$$

$$= \int \mathcal{D}\ell_{\pm}\, \delta^{(4)}(\phi(z)\partial_{\bar{z}} L_+)\, \delta^{(4)}(\phi(z)\partial_{\bar{z}} L_-) = \int \mathcal{D}\tilde{\alpha}^{i,p}_{\pm}$$

$$(\ \tilde{\mathcal{A}}^{i,p} = \bar{\mathcal{A}}^{i,p} + \tilde{\alpha}^{i,p}\ )$$

Neglecting **constant** Jacobian for $\quad L_{\pm} \to \left(\phi(z)\, \partial_{\bar{z}} L_{\pm}\,,\ \mathcal{A}^{i,p}_{\pm}(L)\right)$

$$\mathcal{A}^{i,p}_{\pm}(L) := L_{\pm}(z - a_i)^{p_i}\Big|_{z=a_i}$$

Gauge fixing

4d gauge sym : $A_m \to h^{-1} A_m h + h^{-1} \partial_m h$

$$\widehat{S}^{(1)}_{4d} \sim -\tfrac{i}{2} \log \int_{\mathrm{B.C.s}} \frac{\mathcal{D}\gamma \ \mathcal{D}\tilde{\alpha}^{i,p}_{\pm} \ \mathcal{D}u^{\bar{z}}}{\mathrm{gauge}} \ \exp i \int \phi(z) \, \mathrm{Tr}[u^{\bar{z}} F_{+-}(L(z,\tilde{\mathcal{A}}))]$$

$$\boxed{\mathcal{D}\gamma = \mathcal{D}\tilde{\gamma}|_P \ \mathcal{D}\gamma|_{\mathrm{bulk}}} \qquad \mathcal{D}\gamma|_{\mathrm{bulk}} = \mathcal{D}\gamma \ \delta(\gamma|_P = \tilde{\gamma}|_P)$$

$\gamma|_{\mathrm{bulk}}$ is pure gauge :

fix any smooth field config interpolating between the poles

Boundary conditions

$$\int_{\mathrm{B.C.s}} \mathcal{D}\alpha_{\pm}^{i,p}\, \mathcal{D}\tilde{\gamma}|_P = \int \mathcal{D}\alpha^{i,p}\, \mathcal{D}\tilde{\gamma}|_P\, \delta\big(\tilde{\mathcal{A}} - \tilde{\mathcal{A}}(\tilde{g}|_P)\big)$$

$$= \int \mathcal{D}\alpha_{\pm}^{i,p} \prod_s \delta(B_s \cdot \alpha)$$

$$= \int \mathcal{D}\alpha_{\pm}^{i,p}\, \mathcal{D}\tilde{v}^s\, \exp i \int d^2\xi\, \tilde{v}^s\, (B_s \cdot \tilde{\alpha})$$

B.C.s ⟶ **Bianchi identities**

Bianchi Completeness Assumption: Let us assume that the linearised Bianchi identities $B_s \cdot \alpha = 0$ following from the boundary conditions are a subset of the linearised zero-curvature equations $F_{+-}\big(L(z, \bar{A} + \tilde{\alpha})\big) = 0$.

Overcounting due to B.C.s

$$\widehat{S}^{(1)}_{4d} \sim -\tfrac{i}{2}\log \int \frac{\mathcal{D}\tilde{v}^{s}\, \mathcal{D}\tilde{\alpha}^{i,p}_{\pm}\, \mathcal{D}u^{\bar{z}}}{\text{gauge}} \exp i \int \phi(z)\, \mathrm{Tr}[u^{\bar{z}} F_{+-}(L(z,\tilde{\mathcal{A}}))]$$

No B.C.s:

'Universality'

$$\int \frac{\mathcal{D}\tilde{v}^{s}}{\text{gauge}} = 1$$

overcounted equations that are trivialised by B.C.s

$$\sim -\tfrac{i}{2}\log \int \frac{\mathcal{D}\tilde{\alpha}^{i,p}_{\pm}\, \mathcal{D}u^{\bar{z}}}{\text{gauge}} \exp i \int \phi(z)\, \mathrm{Tr}[u^{\bar{z}} F_{+-}(L(z,\tilde{\mathcal{A}}))]$$

Matching 4d and 2d

$$\widehat{S}^{(1)}_{4d} \sim -\frac{i}{2}\log \int \frac{\mathcal{D}\tilde{\alpha}^{i,n}_{\pm}\ \mathcal{D}u^{\bar{z}}}{\text{gauge}}\ \exp i \int\ \phi(z)\,\mathrm{Tr}[u^{\bar{z}}F_{+-}(L(z,\tilde{\mathcal{A}}))]$$

$$\overset{?}{=} -i\log \int \mathcal{D}\tilde{\alpha}^{i,n}_{\pm}\mathcal{D}\tilde{u}^{S}\ \exp i \int d^2\xi\ \tilde{u}^{S}\,\mathrm{zc}_{S}(\bar{\mathcal{A}}+\tilde{\alpha})$$

Matching 4d and 2d

$$\widehat{S}^{(1)}_{4d} \sim -\frac{i}{2}\log \int \frac{\mathcal{D}\tilde{\alpha}^{i,n}_{\pm}\ \mathcal{D}u^{\bar{z}}}{\text{gauge}} \exp i \int \phi(z)\,\mathrm{Tr}[u^{\bar{z}} F_{+-}(L(z,\tilde{\mathcal{A}}))]$$

Overcounted 2d equations as 4d ones

Only depends on $\tilde{u}^I := \int d^2z\,\phi(z)\,u^{\bar{z}}\,f^I$ $\left(F_{+-}(L(z,\tilde{\mathcal{A}})) = \sum_I f^I(z)\,\mathrm{zc}_I(\tilde{\mathcal{A}})\right)$

→ Gauge sym $u^{\bar{z}} \to u^{\bar{z}} + \lambda^{\bar{z}}$, $\int \mathrm{d}^2z\ \phi(z)\ \lambda^{\bar{z}}\,f^I = 0$

Fix gauge $u^{\bar{z}} = \sum_I \left(\int d^2z\ \phi(z)\ u^{\bar{z}}\,f^I\right) g_I = \sum_I \tilde{u}^I\,g_I$

$\left(\int d^2z\,\phi(z)\ f^I\,g_J = \delta^I{}_J\right)$

Matching 4d and 2d

$$\widehat{S}^{(1)}_{4d} \sim -\tfrac{i}{2}\log \int \frac{\mathcal{D}\tilde{\alpha}^{i,p}_{\pm}\,\mathcal{D}u^{\bar{z}}}{\text{gauge}}\,\exp i \int \phi(z)\,\mathrm{Tr}[u^{\bar{z}}F_{+-}(L(z,\tilde{A}))]$$

$$= -i\log \int \mathcal{D}\tilde{\alpha}^{i,p}_{\pm}\mathcal{D}\tilde{u}^{S}\,\exp i \int d^2\xi\;\tilde{u}^{S}\,\mathrm{zc}_{S}(\bar{A}+\tilde{\alpha})$$

= Vol { flat connections with given pole structure }

$$\widehat{S}^{(1)}_{4d} \sim \widehat{S}^{(1)}_{2d}$$

Measure factors

Have been careful about **background field-dependent** measure factors

NOT about **constant** ones

$$\mathcal{D}\gamma = \mathcal{D}\tilde{\gamma}|_P \; \mathcal{D}\gamma|_{\text{bulk}}$$

Boundary conditions

$$\int_{\text{B.C.s}} \mathcal{D}\alpha_{\pm}^{i,p} \; \mathcal{D}\tilde{\gamma}|_{P} = \int \mathcal{D}\alpha^{i,p} \; \mathcal{D}\tilde{\gamma}|_{P} \, \delta\big(\tilde{\mathcal{A}} - \tilde{\mathcal{A}}(\tilde{g}|_{P})\big)$$

$$= \int \mathcal{D}\alpha_{\pm}^{i,p} \prod_s \delta(B_s \cdot \alpha)$$

$$= \int \mathcal{D}\alpha_{\pm}^{i,p} \, \mathcal{D}\tilde{v}^s \; \exp i \int d^2\xi \; \tilde{v}^s \, (B_s \cdot \tilde{\alpha})$$

Solving meromorphicity equations

Integrate u^+, u^- $\quad\longrightarrow\quad$ $L_{\pm}(z, \tilde{\mathcal{A}})$ meromorphic with fixed poles

$$\int \mathcal{D}\ell_{\pm} \, \mathcal{D}u^{\pm} \; \exp i \int \phi(z) \, \mathrm{Tr}[u^- \, \partial_{\bar{z}} L_+ + u^+ \, \partial_{\bar{z}} L_-]$$

$$= \int \mathcal{D}\ell_{\pm} \, \delta^{(4)}(\phi(z)\partial_{\bar{z}} L_+) \, \delta^{(4)}(\phi(z)\partial_{\bar{z}} L_-) = \int \mathcal{D}\tilde{\alpha}_{\pm}^{i,p}$$

Measure factors

Have been careful about **background field-dependent** measure factors

NOT about **constant** ones

- Dropping constant term in effective action (const factor in Z)

- Extra divergence? Or prescription for 4d measure?

- In any case: renormalisability + RG flow basically unaffected

Measure factors

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On gauge symmetries

Treated all local syms of linearised theory as gauge

Why?

- 4d gauge: natural. $A_m \to h^{-1} A_m h + h^{-1} \partial_m h$
  Gauge transf vanishes "at defects"

- "2d gauge": $g \to g\, q\,, \quad L_\pm \to q^{-1} L_\pm q + q^{-1} \partial_\pm q \quad (\partial_{\bar{z}} q = 0)$
  Artefact of change of variables. Must gauge

- "Overcounting" of EOMs:
  Artefact of our formalism. Should gauge

Faddeev-Popov determinants

All obvious except $$G := u^{\bar{z}} - \sum_I \left(\int d^2z \; \phi(z) \; u^{\bar{z}} \, f^I \right) g_I = 0$$

$$\delta u^{\bar{z}} \;=\; \lambda^{\bar{z}} \qquad\qquad \int \mathrm{d}^2 z \; \phi(z) \; \lambda^{\bar{z}} \, f^I = 0$$

$$\boxed{\frac{\delta G}{\delta \lambda^{\bar{z}}} \;=\; \frac{\delta u^{\bar{z}}}{\delta \lambda^{\bar{z}}} \;=\; \mathbb{1}}$$

On Disorder defects and semi-classics

$$S_{4d}[A] = \int_{\Sigma_2 \times \mathbb{C}P^1} dz \, \frac{\phi(z)}{\hbar} \wedge \left(A \wedge dA \, + \, \tfrac{2}{3} A^3 \right)$$

Disorder defects ($\phi = 0$) are like $\hbar = \infty$ … Danger?

1-loop divergences seem ok

Higher-loops?

Unifying unifying approaches

[Wallberg, Lacroix, NL 23xx.xxxxx]

4d

1-loop RG

2d

Flow of periods of $\omega = \phi(z)\, dz$

[Derryberry 21] [Costello]

Conjecture 1 (Costello). Let $\mathcal{N}$ denote the moduli space of l
holomorphic 1-form ω that has only simple zeroes and double pc
of the zeroes of ω into two equally sized groups, D_1 and D_2.

Then the modified Ricci flow on the space of metrics on the targ
by a certain flow on $\mathcal{N}$, where:

- The closed periods $\oint \omega$ are held fixed.
- The periods $\int_p^q \omega$ are held fixed when p, q are both in either
- When $p \in D_1$ and $q \in D_2$ the periods $\int_p^q \omega$ satisfy $\frac{d}{d\epsilon} \int_p^q \omega \Big|_{\epsilon=}$
  the flow on $\mathcal{N}$.

'Universal' formulas for 1-loop RG

$$\frac{d}{dt}\widehat{\mathcal{L}}^{(1)} = 4c_G \sum_{z_i w_j} \frac{1}{(z_i - w_j)^2} \mathrm{Tr}[C_+^i C_-^j]$$

2d twist function flow

[Delduc Lacroix Sfetsos Siampos 20]

$$\frac{d}{dt}\phi(z) = c_G\, \partial_z h(z)$$

$h(z_i) = 1$

$h(w_j) = -1$

same poles as ϕ

Unifying unifying approaches

[Wallberg, Lacroix, NL 23xx.xxxxx]

4d

$$\frac{d}{dt}\widehat{S}^{(1)} = c_G \int_{\Sigma_2\times\mathbb{C}P^1} dz\, \partial_z h(z) \wedge CS_{(3)}(A)$$

✔ 1-loop RG

$$\frac{d}{dt}\widehat{\mathscr{L}}^{(1)} = 4c_G \oint_\gamma \frac{dz}{2\pi i}\mathrm{Tr}[L_+\,\partial_z L_-]$$

2d

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4d

$$\frac{d}{dt}\widehat{S}^{(1)} = c_G \int_{\Sigma_2 \times \mathbb{C}P^1} dz\, \partial_z h(z) \wedge CS_{(3)}(A)$$

Derive directly in 4d?

✓ 1-loop RG

2d

'Universal' formulas for 1-loop RG

$$\frac{d}{dt}\widehat{\mathscr{L}}^{(1)} = 4c_G \sum_{z_i w_j} \frac{1}{(z_i - w_j)^2} \mathrm{Tr}[C_+^i C_-^j]$$

Flow of periods of $\omega = \phi(z)\, dz$

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Messages

- 1-loop divergences match:

  4d Chern-Simons $\leftrightarrow$ 2d integrable sigma-models

- Take correspondence seriously at quantum level

- Integrability constrains RG: ‘universal’ structure

- Can derive from 4d

Future directions

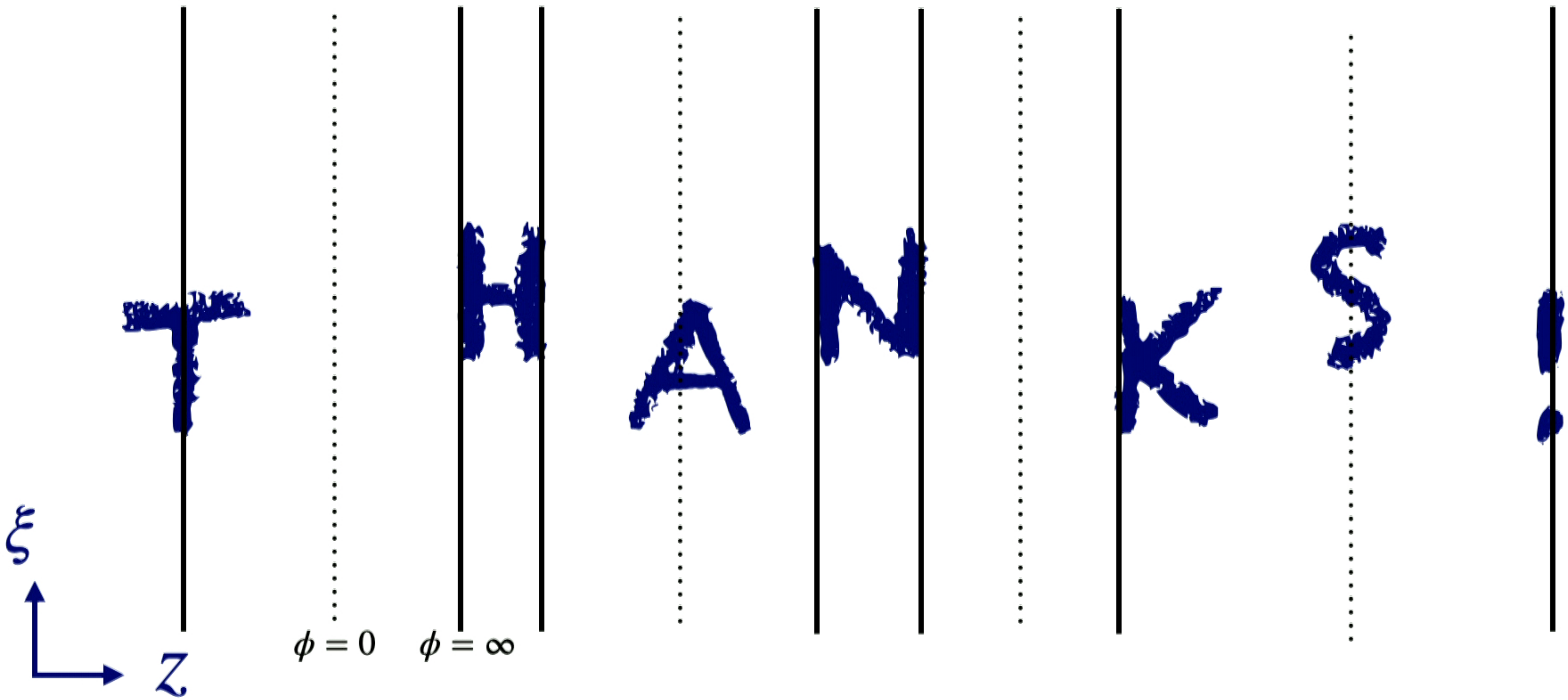
- Other integrable theories, e.g. sine-Gordon?

- Curiosity: $L_\pm = \mathrm{P}_0 C_\pm + z^{\pm 1}\,\mathrm{P}_1 C_\pm$ $\quad C_\pm \sim \partial_\pm \phi\, T_1 + \cos\phi\, T_2 + \sin\phi\, T_3$

  ~ $\mathbb{Z}_2$ **coset** $\quad \in \mathrm{SU}(2)$ $\quad$ **extra Bianchi identities!**

  Universal result $\frac{d}{dt}\widehat{\mathscr{L}}^{(1)} = 2c_G\,\mathrm{Tr}[C_+\,\mathrm{P}_1\,C_-]$ ✔

- Explanation **(?)** S^2 sigma-model $\xrightarrow{\text{Pohlmeyer}}$ sine Gordon


THANKS!
ξ
Z
$\phi = 0$
$\phi = \infty$