Title: Quantum metrology in the finite-sample regime - VIRTUAL

Speakers: Johannes Meyer

Series: Machine Learning Initiative

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Abstract: In quantum metrology, one of the major applications of quantum technologies, the ultimate precision of estimating an unknown parameter is often stated in terms of the Cramér-Rao bound. Yet, the latter is no longer guaranteed to carry an operational meaning in the regime where few measurement samples are obtained. We instead propose to quantify the quality of a metrology protocol by the probability of obtaining an estimate with a given accuracy. This approach, which we refer to as probably approximately correct (PAC) metrology, ensures operational significance in the finite-sample regime. The accuracy guarantees hold for any value of the unknown parameter, unlike the Cramér-Rao bound which assumes it is approximately known. We establish a strong connection to multi-hypothesis testing with quantum states, which allows us to derive an analogue of the Cramér-Rao bound which contains explicit corrections relevant to the finite-sample regime. We further study the asymptotic behavior of the success probability of the estimation procedure for many copies of the state and apply our framework to the example task of phase estimation with an ensemble of spin-1/2 particles. Overall, our operational approach allows the study of quantum metrology in the finite-sample regime and opens up a plethora of new avenues for research at the interface of quantum information theory and quantum metrology. TL;DR: In this talk, I will motivate why the Cramér-Rao bound might not always be the tool of choice to quantify the ultimate precision attainable in a quantum metrology task and give a (hopefully) intuitive introduction of how we propose to instead quantify it in a way that is valid in the single- and few-shot settings. We will together unearth a strong connection to quantum multi-hypothesis testing and conclude that there are many exiting and fundamental open questions in single-shot metrology!

Zoom link https://pitp.zoom.us/j/92247273192?pwd=ZkprOFZ0eEdQYjJDY1hneFNLckFDZz09

Quantum metrology in the finite-sample regime

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PERIMETER SEMINAR

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Quantum metrology in the finite-sample regime

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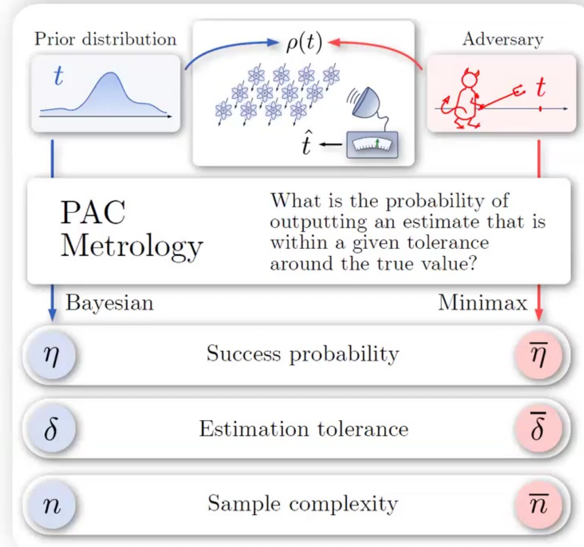
(Dated: July 14, 2023)






Quantum Metrology

metrology

/mɪˈtrɒlədʒi/

noun

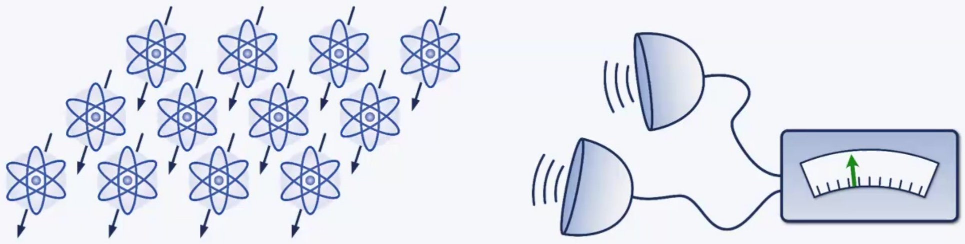
the scientific study of measurement.

QUANTUM SENSING
Measure a physical parameter, e.g. a magnetic field

QUANTUM PARAMETER ESTIMATION
Measure a parameter encoded in a quantum system

TOMOGRAPHY
Estimate the quantum state of a system

Traditional Quantum Metrology



$\rho(t)$ $Q(\hat{t})$

Want unbiased estimate

$$\mathbb{E}[\hat{t}] = t$$

with **low variance**

Cramér-Rao Bound

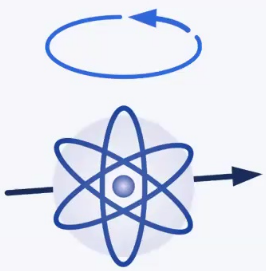
$$\mathrm{Var}(\hat{t}) \geq \frac{1}{\mathcal{F}(t)}$$

- Inherently asymptotic
- Assumes parameter is already approximately known
- Application difficult to justify in the finite-sample regime

An Example

Alice prepares a plus state, lets it evolve under a phase Hamiltonian and sends it to Bob via a channel dephasing it in the plus/minus basis

$$e^{-itZ/2}$$



$$\rho(t) = \cos^2(t/2)|+\rangle\langle+| + \sin^2(t/2)|-\rangle\langle-|$$

$$\mathcal{F} = \begin{cases} 1 & \text{if } t \notin \{0,\pi\} \\ 0 & \text{if } t \in \{0,\pi\} \end{cases}$$

We are guaranteed there exists an observable diagonal in the plus/minus basis with

$$\langle T\rangle = t \quad \langle \Delta T\rangle = 1/\mathcal{F}$$

For small t, this observable has eigenvalues

$$T \simeq \mathrm{diag}(O(t), O(1/t))$$

Occuring with probability

$$P \simeq (O(1), O(t^2))$$

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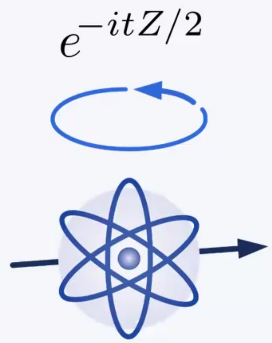

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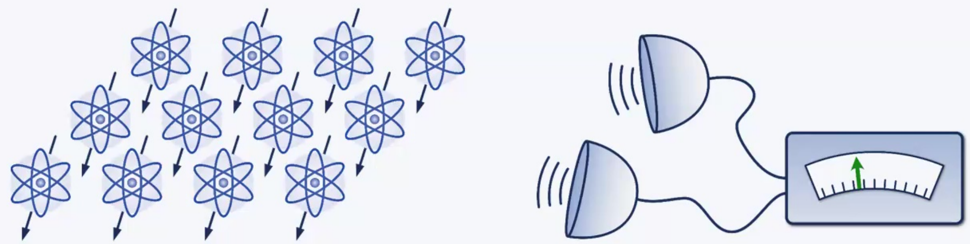
Occuring with probability

$$P \simeq (O(1), O(t^2))$$

Which means resolving t takes $O(t^2)$ samples

Single-shot Quantum Metrology



Is $|t - \hat{t}| \leq \delta$?

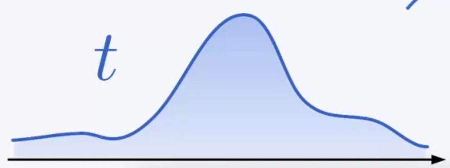
Yes → **Success** ✓

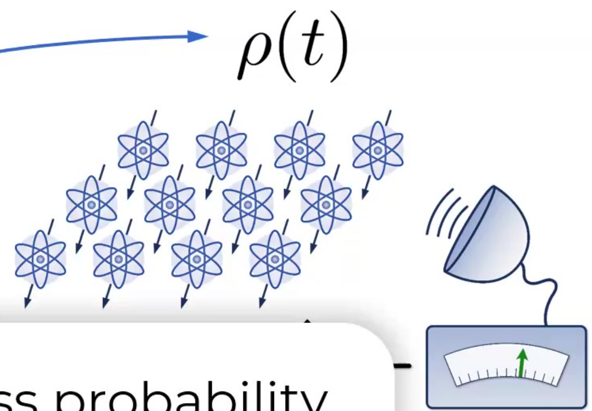
No → **Failure** ✗

What is the probability of successful estimation?

Single-shot Quantum Metrology

Prior distribution





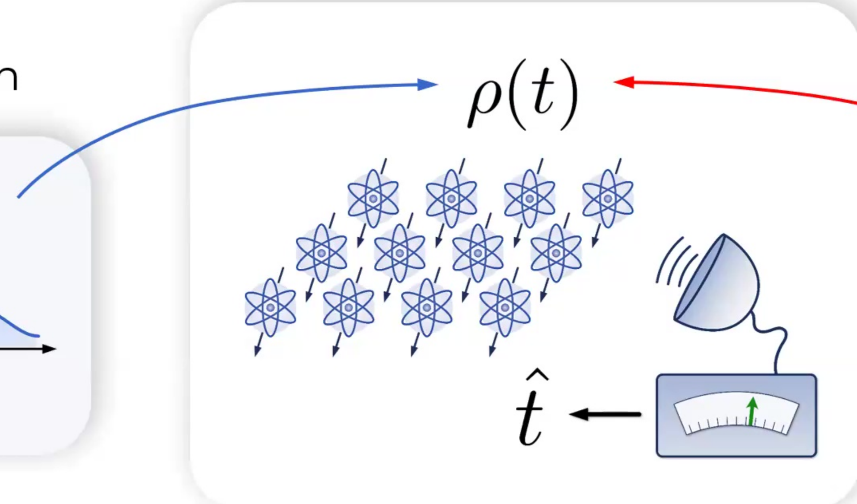
„Expected" success probability

Bayesian success probability

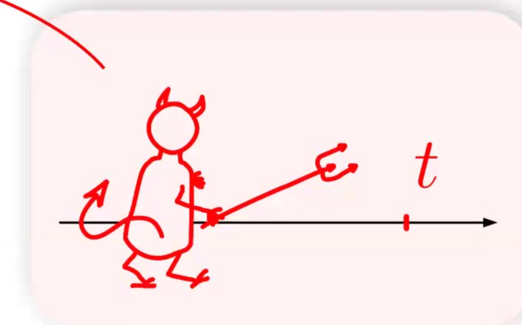
$$\eta = \int \mathrm{d}\mu(t) \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Single-shot Quantum Metrology

Prior distribution



Adversary

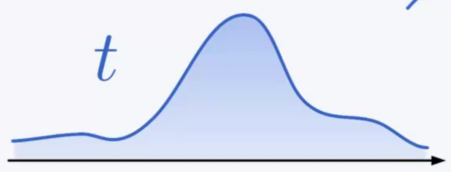


Bayesian success probability

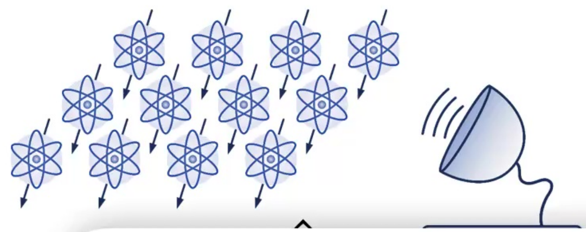
$$\eta = \int \mathrm{d}\mu(t) \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Single-shot Quantum Metrology

Prior distribution



$\rho(t)$



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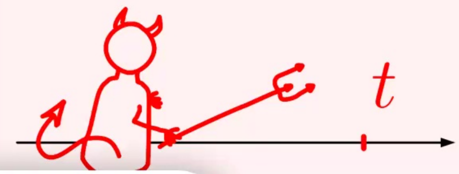


„Worst-case“ success probability

Bayesian success probability

$$\eta = \int \mathrm{d}\mu(t) \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Minimax success probability

$$\overline{\eta} = \min_{t} \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

The PAC Metrology Framework

$\overline{\eta}$

SUCCESS PROBABILITY

What is the probability of obtaining an estimate within a fixed tolerance?

$\overline{\delta}$

ESTIMATION TOLERANCE

What is the smallest tolerance that still guarantees a fixed success probability?

$\overline{n}$

SAMPLE COMPLEXITY

How many copies of a state do I need to guarantee a fixed success probability and tolerance?

Optimal Measurements

Optimal minimax success probability

$$\overline{\eta}^* = \max_{Q(\hat{t})} \left\{ \min_t \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Constitutes a **semi-infinite program**,
think a continuous semi-definite program

Optimal Measurements

Optimal minimax success probability

$$\overline{\eta}^* = \max_{Q(\hat{t})} \left\{ \min_t \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

- We give a dual formulation without duality gap
- We generalize it to the parametrized channels where we optimize over combs or strategies with indefinite causal order
- We also give post-processing strategies for fixed measurements

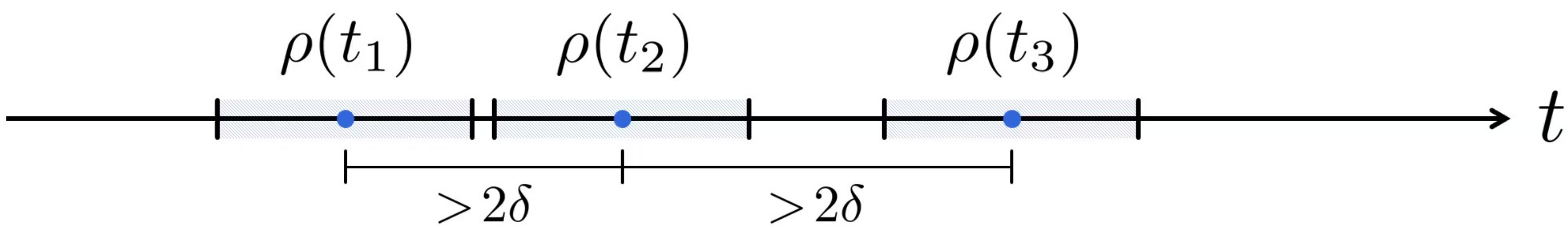
Connection to Hypothesis Testing

Metrology problem

$$Q(\hat{t}) \to \hat{t} \qquad \mathbb{P}[|t_1 - \hat{t}| \leq \delta] \geq \overline{\eta}$$

Connection to Hypothesis Testing

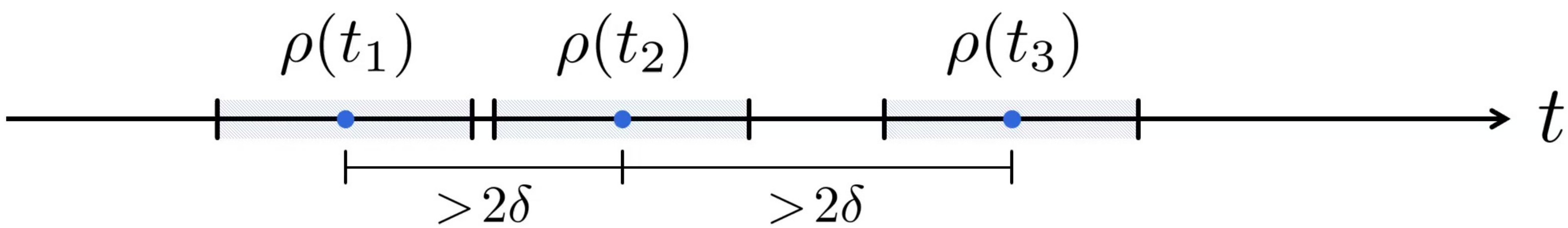
Multi-hypothesis testing problem



$Q(\hat{t}) \rightarrow \hat{t} \rightarrow \text{closest } t_i$ $\quad\quad$ $\mathbb{P}[\hat{i} = i] \geq \overline{\eta}$

Connection to Hypothesis Testing

Multi-hypothesis testing problem



$Q(\hat{t}) \to \hat{t} \to$ closest t_i $\qquad \mathbb{P}[\hat{i} = i] \geq \overline{\eta}$

We conclude that

$$\overline{\eta} \leq \overline{P}_s(\{\rho(t_i)\}) \text{ as long as } |t_i - t_j| > 2\delta$$

Estimation Tolerance

So far, we analyzed the success probability at fixed tolerance. But in applications, we often care about the achievable precision at fixed success probability.

Minimax estimation tolerance

$$\overline{\delta}(\overline{\eta}) = \inf \left\{ \delta' \geq 0 \;\middle|\; \overline{\eta} \leq \min_t \int_{-\delta'}^{\delta'} \mathrm{d}\tau \; \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Finite-sample Cramér-Rao bound

Cramér-Rao bound

$$\sigma(\hat{t}) \geq \frac{1}{\sqrt{\mathcal{F}(t)}}$$

Our bound

$$\bar{\delta} \geq \frac{O\left(\sqrt{\log \frac{1}{1-\bar{\eta}}} - q \log \frac{1}{1-\bar{\eta}}\right)}{\sqrt{\min_t \mathcal{F}(t)}}$$

In the i.i.d. case

$$q = O\left(\frac{1}{\sqrt{n}}\right)$$

Sample Complexity

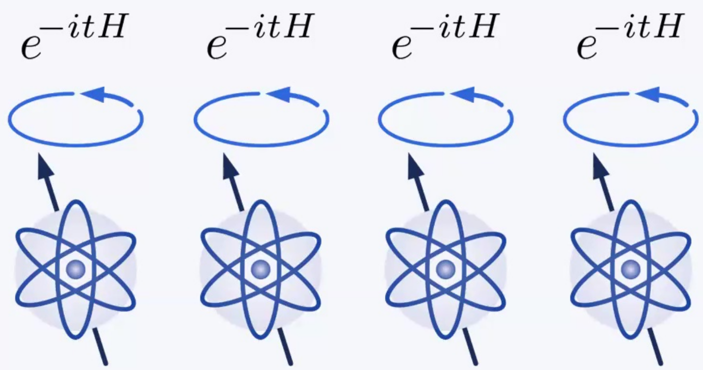
What if we care about both the achievable precision and the success probability? Then we have to ask how many copies of a state we need to achieve it.

Minimax sample complexity

$$\overline{n}(\overline{\eta}, \overline{\delta}) = \min \left\{ n' \in \mathbb{N} \,\middle|\, \overline{\eta} \leq \min_t \int_{-\overline{\delta}}^{\overline{\delta}} \mathrm{d}\tau \ \mathrm{Tr}[\rho^{\otimes n'}(t) Q_{n'}(t+\tau)] \right\}$$

Phase estimation

Local evolution of
an ensemble of spins under
the same phase Hamiltonian



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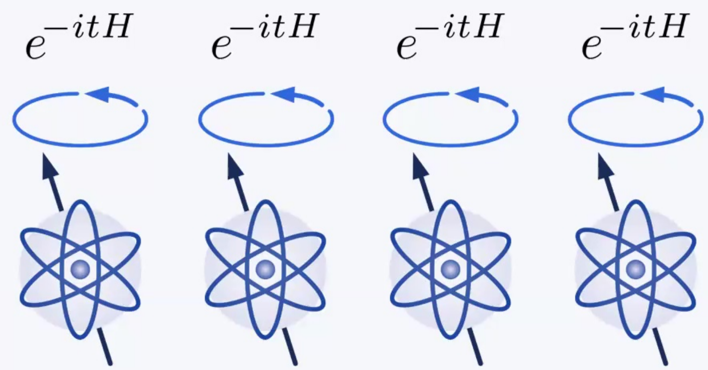


For the regular phase Hamiltonian and $t \in [0, 2\pi)$
this yields a **covariant** set of states

Optimal Measurement

We show that the **pretty good measurement** is optimal for covariant state sets

We use this result to obtain a closed-form solution for the minimax success probability

$$\overline{\eta}^*(\delta,\psi) = \sum_{\lambda,\lambda'} |\psi_\lambda||\psi_{\lambda'}| \frac{\sin(\delta(\lambda-\lambda'))}{\pi(\lambda-\lambda')}$$

Optimal Measurement

We show that the **pretty good measurement**
is optimal for covariant state sets

$$H = \sum_\lambda \lambda \Pi_\lambda$$

$$|\psi\rangle = \sum_\lambda \Pi_\lambda |\psi\rangle = \sum_\lambda \psi_\lambda |\psi_\lambda\rangle$$

o obtain a closed-form solution
max success probability

$$\overline{\eta}^*(\delta, \psi) = \sum_{\lambda, \lambda'} |\psi_\lambda||\psi_{\lambda'}| \frac{\sin(\delta(\lambda - \lambda'))}{\pi(\lambda - \lambda')}$$

Comparison of Probe States

The closed-form solution factilitates a numerical comparison of different probe states

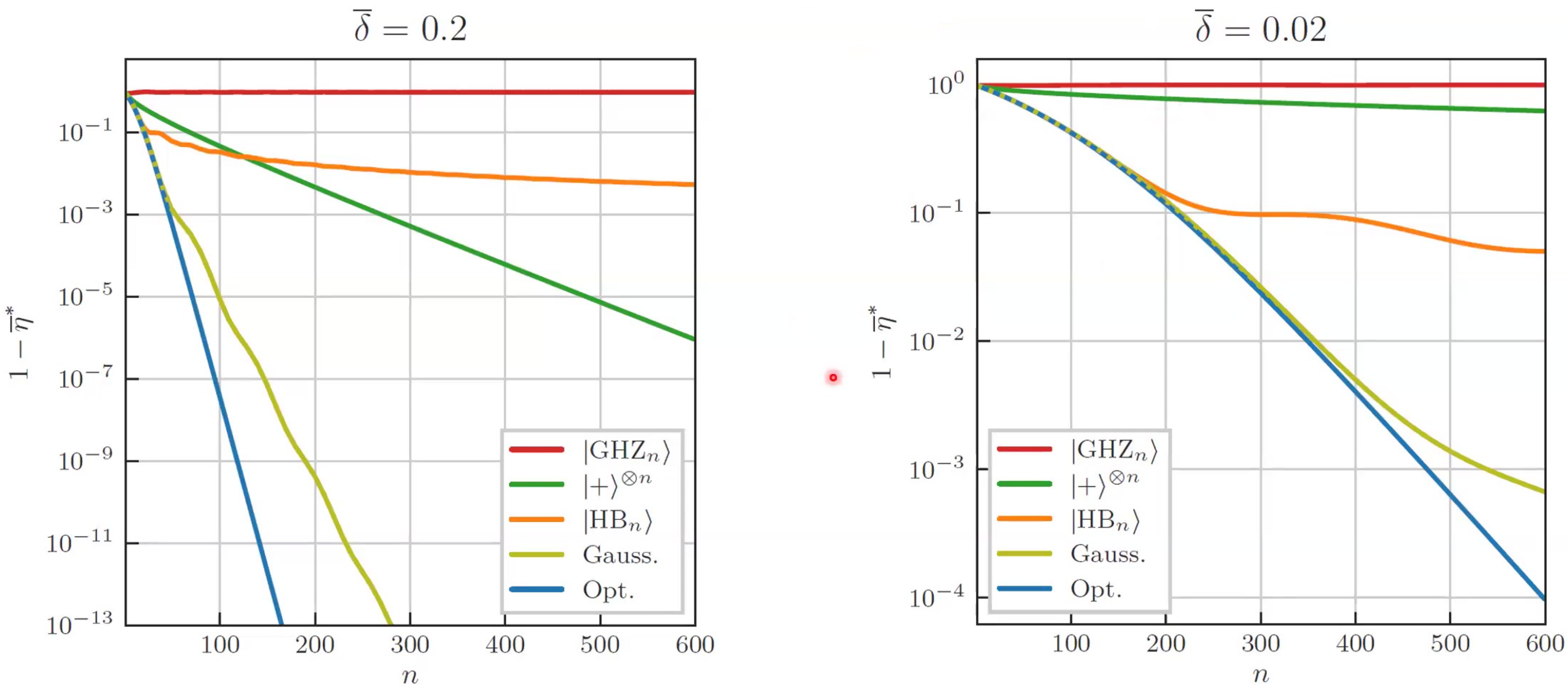
GHZ

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$$

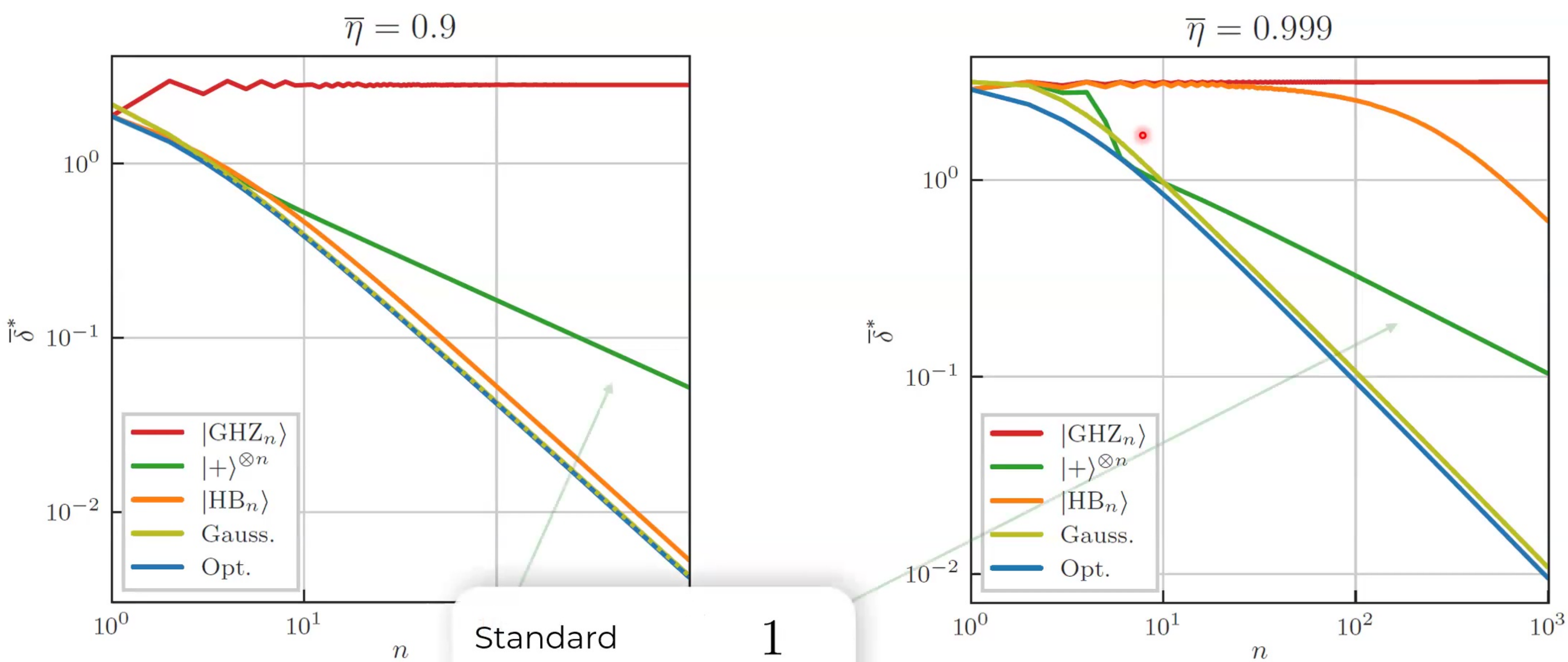
Holland-Burnett

$$|\mathrm{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |n\rangle)$$

Success Probability

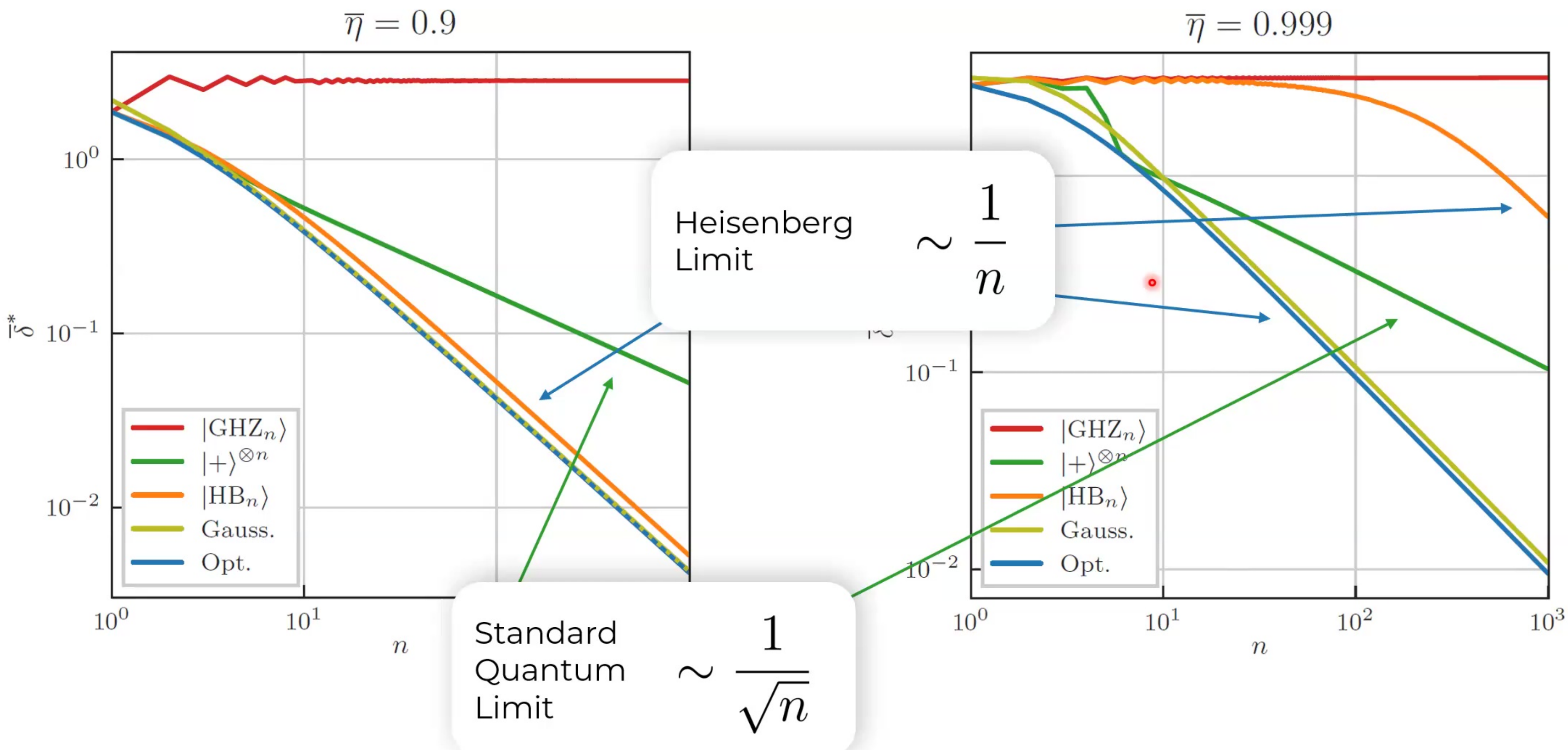


Estimation Tolerance



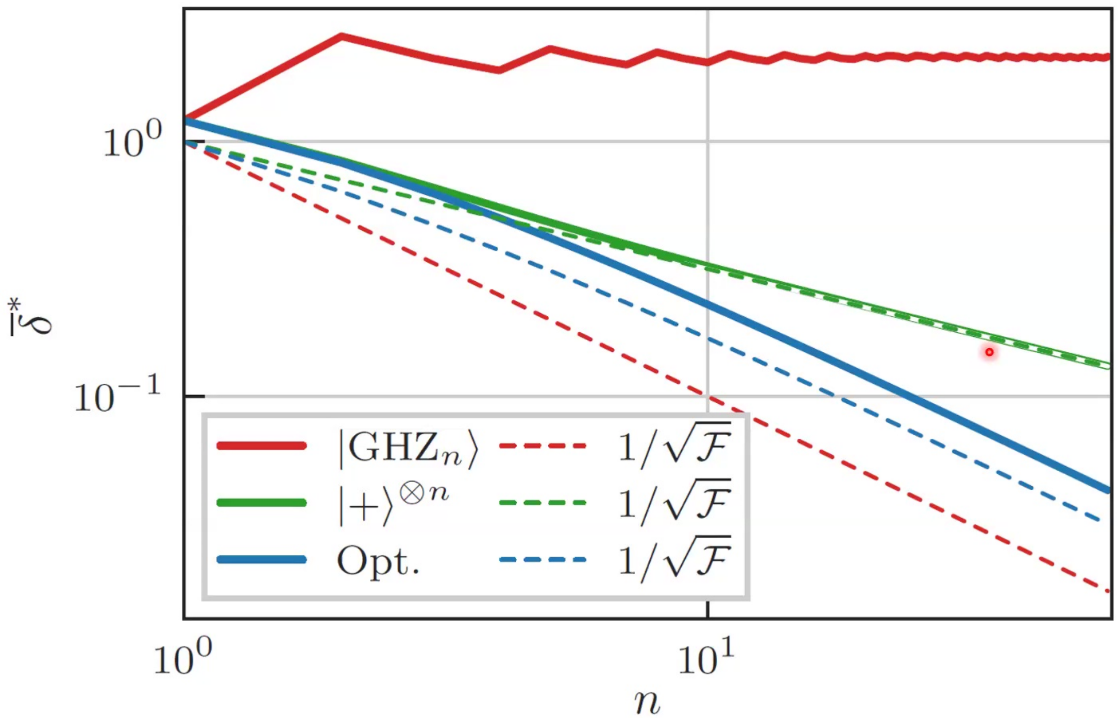
Standard Quantum Limit $\sim \dfrac{1}{\sqrt{n}}$

Estimation Tolerance



Comparison with QCRB

$$\overline{\eta} = \text{erf}(1/\sqrt{2})$$



Further Results in the Paper

- We connect our quantities to single-shot entropy measures
- We lift the hypothesis testing connection to quantum channels with different access models
- We discuss many possible extensions of our results and definitions, e.g. the multi-parameter case
- We give an overview of open questions

Open Questions

- What measurements (i.e. POVMs) give good out-of-the-box performance guarantees? Pretty good measurement?
- Improved finite-sample analogues of the Cramér-Rao bound
- Understanding the advantages of adaptive processing and entanglement
- What are the admissible scalings with mixed asymptotics?
- How do noise and error correction fit into this picture?

Summary

- We give new tools to understand quantum metrology in the single-shot regime
- Our framework is very close to quantum information theory both in tools as in results
- A plethora of open questions ranging from practically oriented to completely information-theoretic
- An exciting opportunity to explore new directions in quantum metrology!

Thank you for your attention!

Slides

arXiv:2307.06370

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