Title: (Weyl-)Fefferman-Graham asymptotic symmetries

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Abstract: To develop a quantum gravity theory, it is fundamental to move away from the gauge-fixing approach and instead employ a gauge-free analysis. There is an increasing body of evidence suggesting that the symmetries employed for gauge-fixing might carry charge. Consequently, setting the associated fields to zero is a physical constraint on the system, which should be avoided. In this talk, we will examine a partial fixing of the Fefferman-Graham (FG) gauge, referred to as the Weyl-Fefferman-Graham (WFG) gauge, which restores boundary Weyl covariance. We will show that the diffeomorphism mapping WFG to FG can be charged and discuss how this relates to holography.

Zoom link https://pitp.zoom.us/j/93359035909?pwd=aHo0TUFXekZaeCtrT2FzNDZKUW15Zz09

(Weyl-)Fefferman-Graham asymptotic symmetries

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Based on [L. Ciambelli, AD, R. Ruzziconi, C. Zwikel (2308.15480)]

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Outline

I. **Plan and Motivations**

II. Fefferman-Graham

III. Weyl-Fefferman-Graham

IV. Summary and Future Possibilities

I. Plan and Motivations 1/2

- Probe the AdS/CFT correspondence

  Study of the classical phase space of asymptotically AdS gravity

  Asymptotic symmetries of AdS $\leftrightarrow$ global symmetries of CFT

- Corner proposal [Donnelly-Freidel '16, Speranza '18, Ciambelli-Leigh '21, ...]

  Gravitational theory $\leftarrow$ charges and their algebra at corners

  Organising principle for quantum observables

- Focus on 3D gravity: no propagation of degrees of freedom

  Presence of black holes [Banados-Teitelboim-Zanelli '92]

  Asympt. symmetry enhancement w.r.t. vacuum isometries [Brown-Henneaux '86]

I. Plan and Motivations 2/2

- Select the allowed metric fluctuations at infinity

  No requirement to fix any particular gauge but it is often convenient

  For example: Fefferman-Graham, Bondi gauge [Starobinsky '83, Fefferman-Graham '85] [Bondi-van der Burg-Metzner '62, Sachs '62]

- Charged diffeomorphisms map inequivalent physical configurations

  Symmetries used to gauge-fix can be charged (see, e.g., [Geiller-Goeller-Zwikel '21])

- In this talk: Weyl-Fefferman-Graham gauge [Ciambelli-Leigh '19, Jia-Karydas '21]

  Interesting implications for holography

Outline

II. Fefferman-Graham gauge in 3D: definition

- Useful gauge fixing for holography: [Fefferman-Graham '85]

Any asymptotically AdS_3 space can be written near the boundary as ($\ell = 1$)

$$ds^2_{AdS} = g_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{\rho^2} + h_{ij}(\rho, x) dx^i dx^j$$

ρ is a spacelike coordinate s.t. $\rho = 0$ locates the bdy
x^i are bdy coords: $x^1 = t$ is timelike and $x^2 = \theta$ is spacelike

- Boundary geometry $\rightarrow$ leading order in the asymptotic expansion

$$h_{ij}(\rho, x) = \frac{1}{\rho^2} h^{(0)}_{ij}(x) + \mathcal{O}(1)$$

- Induced metric on the bdy may be defined if $g \rightarrow \Omega^2 g$ [Penrose '63]

  Ambiguity: $\Omega \rightarrow \Omega' = e^\sigma \Omega \Rightarrow h^{(0)} \rightarrow h^{(0)}_\sigma = e^{2\sigma} h^{(0)}$, σ indep. of ρ

  Bulk metric induces a bdy conformal class $[h^{(0)}]$ [Rooman-Spindel '00]

II. FG gauge in 3D: solution space

- Finite FG expansion in 3D:

$$h_{ij}(\rho, x) = \rho^{-2} h_{ij}^{(0)}(x) + h_{ij}^{(2)}(x) + \rho^2 h_{ij}^{(4)}(x)$$

with

$$h_{ij}^{(4)} = \frac{1}{4} h_{ik}^{(2)} h_{(0)}^{kl} h_{lj}^{(2)}, \qquad h_{(0)}^{ij} h_{ij}^{(2)} = -\frac{1}{2} R^{(0)}, \qquad D_{(0)}^{i} h_{ij}^{(2)} = -\frac{1}{2} \partial_j R^{(0)}$$

- Holographic stress-energy tensor: [Henningson-Skenderis '98, Balasubramanian-Kraus '99]

$$T_{ij} = \frac{1}{8\pi G} \left(h_{ij}^{(2)} + \frac{1}{2} h_{ij}^{(0)} R^{(0)} \right)$$

such that ($c = \frac{3\ell}{2G}$ is the BH central charge)

$$T_i{}^i = \frac{c}{24\pi} R^{(0)}, \qquad D_i^{(0)} T^{ij} = 0$$

- Bdy conformally-flat parametrization: [Alessio-Barnich-Ciambelli-Mao-Ruzziconi '20]

$$h_{ij}^{(0)}(x) = e^{2\phi(x)} \eta_{ij}$$

II. FG gauge in 3D: symmetries

- Residual symmetries: $(\mathcal{L}_\xi g_{\rho\rho} = 0,\ \mathcal{L}_\xi g_{\rho i} = 0,\ \mathcal{L}_\xi h_{ij} = \mathcal{O}(\rho^{-2}))$

$$\xi^\rho = \rho\,\omega + \mathcal{O}(\rho^3), \qquad \xi^\pm = Y^\pm + \rho^2 \zeta^\pm + \mathcal{O}(\rho^4)$$

where $(x^\pm = \theta \pm t)$

$$\omega(x^+, x^-) = -\sigma + \tfrac{1}{2}\partial_i Y^i + Y^i \partial_i \phi\,,$$
$$\zeta^\pm(x^+, x^-) = \mathrm{e}^{-2\phi}\left[\partial_\mp \sigma - \left(Y^i \partial_i + \partial_\mp Y^\mp\right)\partial_\mp \phi - \tfrac{1}{2}\partial_\mp \partial_i Y^i\right]$$

- Modified Lie bracket: [Barnich-Troessaert '10]

$$\left[\xi_1, \xi_2\right]_M := \left[\xi_1, \xi_2\right] - \delta_{\xi_1}\xi_2 + \delta_{\xi_2}\xi_1$$

Residual algebra: Witt $\oplus$ Witt $\oplus$ u(1)

$$\hat{Y}^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm, \qquad \hat{\sigma} = 0$$

- Gauge transformations:

$$\delta_\xi \ell_\pm = Y^\pm \partial_\pm \ell_\pm + 2\ell_\pm \partial_\pm Y^\pm - \frac{1}{2}\partial_\pm^3 Y^\pm\,, \qquad \delta_\xi \phi = \sigma$$

II. FG gauge in 3D: holographic renormalization

- Renormalized action: [Henningson-Skenderis '98, Papadimitriou-Skenderis '05]

$$S_{ren} = \frac{1}{16\pi G}\int d^3x\,(R+2) + \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,(K-1) + \frac{\rho^2\log\rho}{16\pi G}\int d^2x\sqrt{-\gamma}\,R^{(0)}$$

where $n_\mu = -\sqrt{-\gamma}\delta^\rho_\mu$, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K = g^{\mu\nu}\nabla_\mu n_\nu$

- On-shell variation of the renormalized action:

$$\delta S_{ren} \approx \frac{1}{2}\int d^2x\sqrt{-h^{(0)}}\,T^{ij}\delta h^{(0)}_{ij} = \frac{c}{24\pi}\int d^2x\sqrt{-h^{(0)}}\,R^{(0)}\delta\phi$$

Under Weyl diffeos: presence of Weyl anomaly $(\delta_\sigma h^{(0)}_{ij} = 2\sigma h^{(0)}_{ij})$

$$\delta_\sigma S_{ren} = \int d^2x\sqrt{-h^{(0)}}\mathcal{A}\,\sigma, \quad \mathcal{A} = \frac{c}{24\pi}R^{(0)}$$

II. FG gauge in 3D: charges

- Renormalized symplectic current:

$$\delta S_{ren} \approx \int d^2x\, \Theta_{ren}, \qquad \omega_{ren} = \delta\Theta_{ren} = \frac{1}{2}\delta\left(\sqrt{-h^{(0)}}\, T^{ij}\right) \wedge \delta h^{(0)}_{ij}$$

- Surface charges: finite, integrable, non conserved

$$Q_\xi = -\frac{1}{8\pi G}\int_0^{2\pi} d\theta \Big(\ell_+ Y^+ - \ell_- Y^- + \phi\, \partial_t \sigma - \sigma\, \partial_t \phi\Big)$$

- Charge algebra: $\{Q_{\xi_1}, Q_{\xi_2}\} = \delta_{\xi_2} Q_{\xi_1} = Q_{[\xi_1,\xi_2]_M} + \mathcal{K}[\xi_1, \xi_2]$

$\hookrightarrow$ Virasoro $\oplus$ Virasoro $\oplus$ Weyl ($Y^\pm \sim e^{inx^\pm}$, $\sigma \sim e^{ipx^+} e^{iqx^-}$)

$$\{Q_{\xi^{Y^\pm}_n}, Q_{\xi^{Y^\pm}_m}\} = i(n-m) Q_{\xi^{Y^\pm}_{n+m}} - i m^3 \frac{c}{12}\delta_{n+m,0},$$

$$\{Q_{\xi^\sigma_{pq}}, Q_{\xi^\sigma_{rs}}\} = -i(r-q)\frac{c}{3} e^{2i(q+s)t}\delta_{p+r,q+s}$$

II. FG gauge in 3D: Penrose-Brown-Henneaux transfos

- Issue: Under a Weyl transformation on the boundary [Henningson-Skenderis '98]

$$\rho \to \rho' = \frac{\rho}{\mathfrak{B}(x)}, \qquad x^i \to x'^i = x^i + \xi^i(\rho, x)$$

$\hookrightarrow$ FG expansion does not transform Weyl-covariantly

- Solution: relax the FG ansatz to the WFG gauge [Ciambelli-Leigh '19]

$$\mathrm{d}s^2_{\mathrm{AdS}} = \left(\frac{\mathrm{d}\rho}{\rho} - k_i(\rho, x)\mathrm{d}x^i\right)^2 + h_{ij}(\rho, x)\mathrm{d}x^i\mathrm{d}x^j\,, \quad k_i(\rho, x) = k_i^{(0)}(x) + \mathcal{O}(\rho^2)$$

$\hookrightarrow$ $h_{ij}^{(0)}(x)$ is the bdy metric and $k_i^{(0)}(x)$ is a bdy Weyl connection

(1) Weyl rescalings at the bdy $\leftarrow$ purely radial bulk diffeos

(2) Full Weyl geometry at the bdy

- Question: Is the Weyl structure associated with an asymptotic symmetry?

III. Weyl-Fefferman-Graham gauge in 3D: PBH transfos

- Reminder: WFG ansatz

$$ds^2_{\text{AdS}} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\frac{\mathrm{d}\rho}{\rho} - k_i(\rho, x)\mathrm{d}x^i\right)^2 + h_{ij}(\rho, x)\mathrm{d}x^i \mathrm{d}x^j$$

- Under radial diffeos inducing bdy Weyl transfos: WFG is preserved

$$\rho \to \rho' = \frac{\rho}{\mathfrak{B}(x)}, \qquad x^i \to x'^i = x^i$$

$\hookrightarrow$ radial expansion transforms Weyl-covariantly

$$k_i^{(2n)}(x) \to k_i^{(2n)}(x)\,\mathfrak{B}(x)^{2n} - \delta_{n,0}\,\partial_i \ln \mathfrak{B}(x)\,,$$
$$h_{ij}^{(2n)}(x) \to h_{ij}^{(2n)}(x)\,\mathfrak{B}(x)^{2n-2}$$

- The leading term in k_i transforms as a Weyl connection

$$k_i^{(0)} \to k_i^{(0)} - \partial_i \ln \mathfrak{B}$$

III. WFG gauge in 3D: geometry 1/2

- Choice of dual form and vector basis for the WFG ansatz:

$$E^\rho = \frac{d\rho}{\rho} - k_i(\rho, x)dx^i\,, \qquad E^i = dx^i\,,$$

$$E_\rho = \rho\partial_\rho \equiv D_\rho\,, \qquad E_i = \partial_i + \rho k_i(\rho, x)\partial_\rho \equiv D_i$$

- Coefficients of bulk Levi-Civita connection ∇ in $\{D_\rho, D_i\}$:

$$\nabla_{D_i} D_j = \Gamma^k_{ij} D_k + \Gamma^\rho_{ij} D_\rho$$

$\hookrightarrow$ at leading order:

$$(\Gamma^{(0)})^k_{ij} = \frac{1}{2} h^{kl}_{(0)} \left[(\partial_i - 2k^{(0)}_i) h^{(0)}_{jl} + (\partial_j - 2k^{(0)}_j) h^{(0)}_{il} - (\partial_l - 2k^{(0)}_l) h^{(0)}_{ij} \right]$$

$\hookrightarrow$ torsion-free connection with Weyl metricity [G. B. Folland '70]

III. WFG gauge in 3D: geometry 2/2

- Induced connection $\nabla^{(0)}$: on the boundary metric

$$\nabla_i^{(0)} h_{jk}^{(0)} = 2k_i^{(0)} h_{jk}^{(0)}$$

- Weyl covariant connection $\hat{\nabla}^{(0)}$: on a generic Weyl-weight ω_T tensor

$$\hat{\nabla}_i^{(0)} T \equiv \nabla_i^{(0)} T + \omega_T k_i^{(0)} T$$

$\hookrightarrow$ $\hat{\nabla}^{(0)}$ is metric and $\hat{\nabla}_i^{(0)} T$ is Weyl covariant

- All geometric quantities built with $\hat{\nabla}^{(0)}$ are Weyl covariant

- Conclusion: WFG is equipped with a full Weyl geometry, a metric $h_{ij}^{(0)}$ and a Weyl connection $k_i^{(0)}$ at the bdy

- Motivated in the ambient construction [Jia-Karydas-Leigh '23]

III. WFG gauge in 3D: solution space and symmetries

- Infinite WFG expansion in 3D:

$$h_{ij} = \rho^{-2} h_{ij}^{(0)} + h_{ij}^{(2)} + \rho^2 h_{ij}^{(4)} + \mathcal{O}(\rho^6)\,, \qquad k_i = k_i^{(0)} + \rho^2 k_i^{(2)} + \mathcal{O}(\rho^4)$$

where $(x^\pm = \theta \pm t,\ K_i^{(0)} = k_i^{(0)} - \partial_i \phi,\ \partial_\pm \ell_\mp = 0)$

$$h_{\pm\pm}^{(0)} = 0\,, \qquad h_{\pm\pm}^{(2)} = \ell_\pm - (K_\pm^{(0)})^2 - \partial_\pm K_\pm^{(0)}, \qquad \text{etc.}$$

$$h_{+-}^{(0)} = \tfrac{1}{2} e^{2\phi}, \qquad h_{+-}^{(2)} = -\tfrac{1}{2}(\partial_- K_+^{(0)} + \partial_+ K_-^{(0)})$$

- Residual symmetries: Witt $\oplus$ Witt $\oplus$ u(1) $\oplus$ u(1)

$$\xi^\rho = \rho\,\omega + \mathcal{O}(\rho^3)\,, \qquad \xi^\pm = Y^\pm + \rho^2 \zeta^\pm + \mathcal{O}(\rho^4)$$

where $(H_i^{(0)} = h_i^{(0)} - \partial_i \sigma,\ \omega = -\sigma + \frac{1}{2}\partial_i Y^i + Y^i \partial_i \phi)$

$$\zeta^\pm = e^{-2\phi}\left(K_\mp^{(0)} \partial_\mp Y^\mp - H_\mp^{(0)} - \tfrac{1}{2}\partial_\mp \partial_i Y^i + Y^i \partial_i K_\mp^{(0)}\right)$$

III. WFG gauge in 3D: holographic renormalization

- Renormalized action: ($n_\mu = -\sqrt{-\gamma}\delta^\rho_\mu$, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K = g^{\mu\nu}\nabla_\mu n_\nu$)

$$S_{ren} = \frac{1}{16\pi G}\int \mathrm{d}^3x\,(R+2) + \frac{1}{8\pi G}\int \mathrm{d}^2x\sqrt{-\gamma}\,(K-1)$$
$$+ \frac{\rho^2\log\rho}{16\pi G}\int \mathrm{d}^2x\sqrt{-\gamma}\,\hat{R}^{(0)} + \frac{1}{16\pi G}\int \mathrm{d}^2x\sqrt{-\gamma}\,k_i\,\gamma^{ij}\,k_j$$

- Renormalized presymplectic potential:

$$\Theta_{ren} = -\sqrt{-h^{(0)}}\left(\frac{1}{2}T^{ij}\delta h^{(0)}_{ij} - J^i\delta k^{(0)}_i\right)$$

where

$$T^{ij} = -\frac{2}{\sqrt{-h^{(0)}}}\frac{\delta S_{ren}}{\delta h^{(0)}_{ij}} \approx \frac{1}{8\pi G}\left(h^{ij}_{(2)} + \frac{1}{2}h^{ij}_{(0)}R^{(0)} + \hat{\nabla}^{(i}_{(0)}k^{j)}_{(0)}\right),$$

$$J^i = \frac{1}{\sqrt{-h^{(0)}}}\frac{\delta S_{ren}}{\delta k^{(0)}_i} \approx \frac{1}{8\pi G}k^i_{(0)}$$

III. WFG gauge in 3D: Ward identities and charges

- Under bdy diffeos: $(\delta_\xi h_{ij}^{(0)} = \mathcal{L}_Y h_{ij}^{(0)} - 2\,\omega\, h_{ij}^{(0)},\ \delta_\xi k_i^{(0)} = \mathcal{L}_Y k_i^{(0)} - \partial_i\omega - 2\,\zeta_i)$

$$\nabla_i^{(0)} T^i{}_j = J^i f_{ij}^{(0)} + \nabla_i^{(0)} J^i\, k_j^{(0)}$$

where

$$f_{ij}^{(0)} = \nabla_i^{(0)} k_j^{(0)} - \nabla_j^{(0)} k_i^{(0)}$$

- Under bdy Weyl transfos: Weyl-covariant Weyl anomaly

$$T^i{}_i + \hat{\nabla}_i^{(0)} J^i = \frac{c}{24\pi}\hat{R}^{(0)}$$

- Surface charges: finite, integrable, non conserved

$$Q_\xi = -\frac{1}{8\pi G}\int_0^{2\pi} \mathrm{d}\theta \Big(\ell_+ Y^+ - \ell_- Y^- + \phi\,\partial_t\sigma - \sigma\partial_t\phi\Big)$$

$\hookrightarrow$ The Weyl connection is not associated with an asymptotic symmetry!

III. WFG gauge in 3D: Chern-Simons

- Isometry algebra of AdS$_3$ = $\mathfrak{so}(2,2)$

$$[M_a, M_b] = \epsilon_{abc} M^c, \quad [M_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \left(\frac{G}{\ell}\right)^2 \epsilon_{abc} M^c$$

- Dreibein or first order formalism ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$)

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b, \qquad de^a + \epsilon^{abc} \omega_b \wedge e_c = 0$$

- Einstein gravitation → Chern-Simons gauge theory ($\mathscr{A} = \frac{1}{G} e^a P_a + \omega^a M_a$)
  [Achucarro-Townsend '86, Witten '88]

$$S_{EH} = \frac{1}{16\pi} \int \text{Tr} \left(\mathscr{A} \wedge d\mathscr{A} + \frac{2}{3} \mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A} \right)$$

- Surface charges for WFG: Weyl connection → asymptotic symmetry!

$$Q_\xi = -\frac{1}{8\pi G} \int_0^{2\pi} d\theta \Big(\ell_+ Y^+ - \ell_- Y^- - \phi(h_t^{(0)} - \partial_t \sigma) + \sigma(k_t^{(0)} - \partial_t \phi) \Big)$$

III. WFG gauge in 3D: finite boundary term

- Option 1: Add a finite bdy counterterm to the action

$$\bar{S}_{ren} = S_{ren} + S_\circ, \qquad S_\circ = \int d^2x\, L_\circ[h^{(0)}_{ij}, k^{(0)}_i]$$

with the following choice

$$L_\circ = \lim_{\rho\to 0}\left[\frac{1}{16\pi G} k_i \gamma^{ij}\partial_j\sqrt{-\gamma}\right] = \frac{1}{16\pi G} k^{(0)}_i h^{ij}_{(0)} \partial_j \sqrt{-h^{(0)}}$$

- Presymplectic potential:

$$\bar{\Theta}_{ren} = -\sqrt{-h^{(0)}}\left(\frac{1}{2}\bar{T}^{ij}\delta h^{(0)}_{ij} - \bar{J}^i \delta k^{(0)}_i\right)$$

where ($J^i = \frac{1}{8\pi G} k^i_{(0)}$)

$$\bar{T}^{ij} = T^{ij} + J^{(i}\partial^{j)}\sqrt{-h^{(0)}} + \frac{1}{2} h^{ij}_{(0)} \nabla^{(0)}_k J^k,$$

$$\bar{J}^i = J^i + \frac{1}{16\pi G}\partial^i \log\sqrt{-h^{(0)}}$$

III. WFG gauge in 3D: finite corner term

- Option 1: Add a finite corner counterterm to the action [McNees-Zwikel '23]

$$\tilde{S}_{ren} = S_{ren} + S_C, \qquad S_C = \int d^2x\, \partial_i L^i_C[h^{(0)}_{ij}, k^{(0)}_i]$$

with the following choice

$$L^i_C = \lim_{\rho\to 0}\left[-\frac{1}{16\pi G}\sqrt{-\gamma}\gamma^{ij}k_j\right] = -\frac{1}{16\pi G}\sqrt{-h^{(0)}}\,h^{ij}_{(0)}k^{(0)}_j$$

- Presymplectic potential:

$$\tilde{\Theta}_{ren} = -\sqrt{-h^{(0)}}\left(\frac{1}{2}\tilde{T}^{ij}\delta h^{(0)}_{ij} - J^i\delta K^{(0)}_i\right)$$

where

$$\tilde{T}^{ij} = T^{ij} + \frac{1}{2}h^{ij}_{(0)}\hat{\nabla}^{(0)}_k J^k, \qquad K^{(0)}_i = k^{(0)}_i - \frac{1}{2}\partial_i \ln\sqrt{-h^{(0)}}$$

IV. Summary

Main goal:

- Explore the charges of 3D gravity in Weyl-Fefferman-Graham gauge

Results:

- Restored Weyl covariance of the boundary
- Residual symmetries, surface charges and anomalies
- FG fixing can constrain the physical solution space

Future possibilities:

- Explicit diffeomorphisms, synthesis of other gauge relaxation proposals [Grumiller-Riegler '16, Geiller-Goeller-Zwikel '21, Campoleoni-AD-Ciambelli-Marteau-Petropoulos-Ruzziconi '22]
- Extension to higher dimensions [Ciambelli-Leigh '19, Petkou-Petropoulos-Rivera-Siampos '22, Campoleoni-AD-Pekar-Petropoulos-Rivera-Vilatte '23]
- Coupling to higher spins [Campoleoni-Fredenhagen-Pfenninger-Theisen '10, Li-Theisen '15, WIP with A. Campoleoni]