Title: Quasinormal modes of a Schwarzschild white hole

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Abstract: We investigate perturbations of the Schwarzschild geometry using a linearization of the Einstein vacuum equations within a Bondi-Sachs, or null cone, formalism. We develop a numerical method to calculate the quasinormal modes, and present results for the case l= 2. The values obtained are different than those of a Schwarzschild black hole, and we interpret them as quasinormal modes of a Schwarzschild white hole.

Zoom link https://pitp.zoom.us/j/93648313308?pwd=akZMZXFKejIwdFphOVU0ZjFpeW13dz09

Quasi-Normal Modes of a Schwarzschild White Hole for the lowest angular momentum

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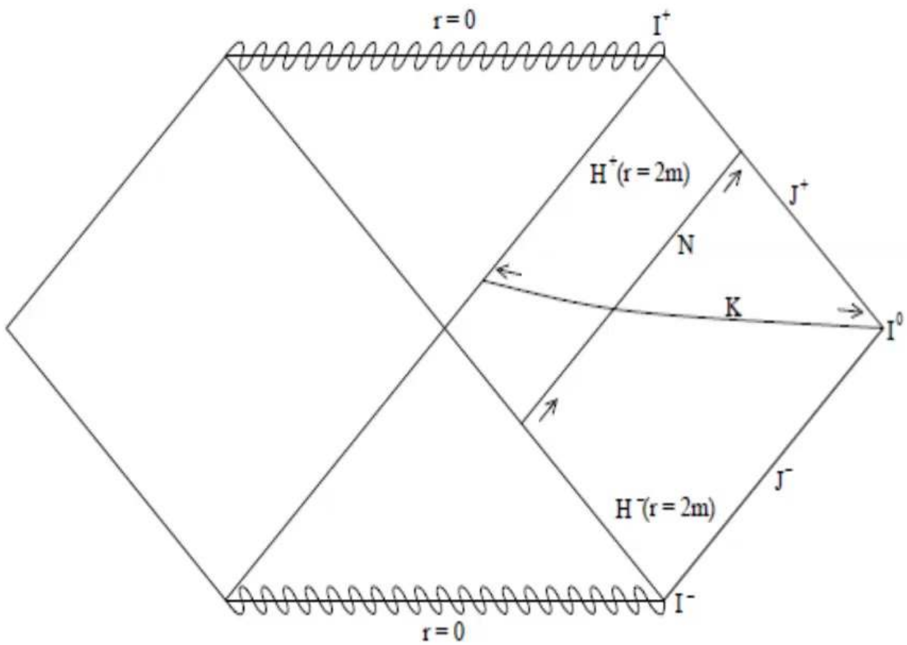
Abstract

We present the first ever few set of results of the quasi-normal modes of a Schwarzschild white hole for lower angular momentum $l = 2$. In determining these normal modes, we use numerical methods to solve the solution of the linearized Einstein vacuum equations in null cone coordinates. Approaching this problem analytically seems to be an impossible task as comprehensively articulated in the literature. However, we experience some difficulty in calculating the quasi-normal modes for higher angular momentum values.

- T. Regge and J. Wheeler, Phys. Rev. 108, 1063 (1957).
- F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).
- C. V. Vishveshwara, Phys. Rev. D 1, 2870 (1970).
- S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London
- A344, 441 (1975).
- S. Chandrasekhar, The Mathematical Theory of Black
- Holes (Oxford University Press, Oxford, England, 1983).

- Bardeen, J.M., and Press, W.H., “Radiation fields in the Schwarzschild background”, J. Math. Phys., 14, 7–19, (1973).

Penrose diagram

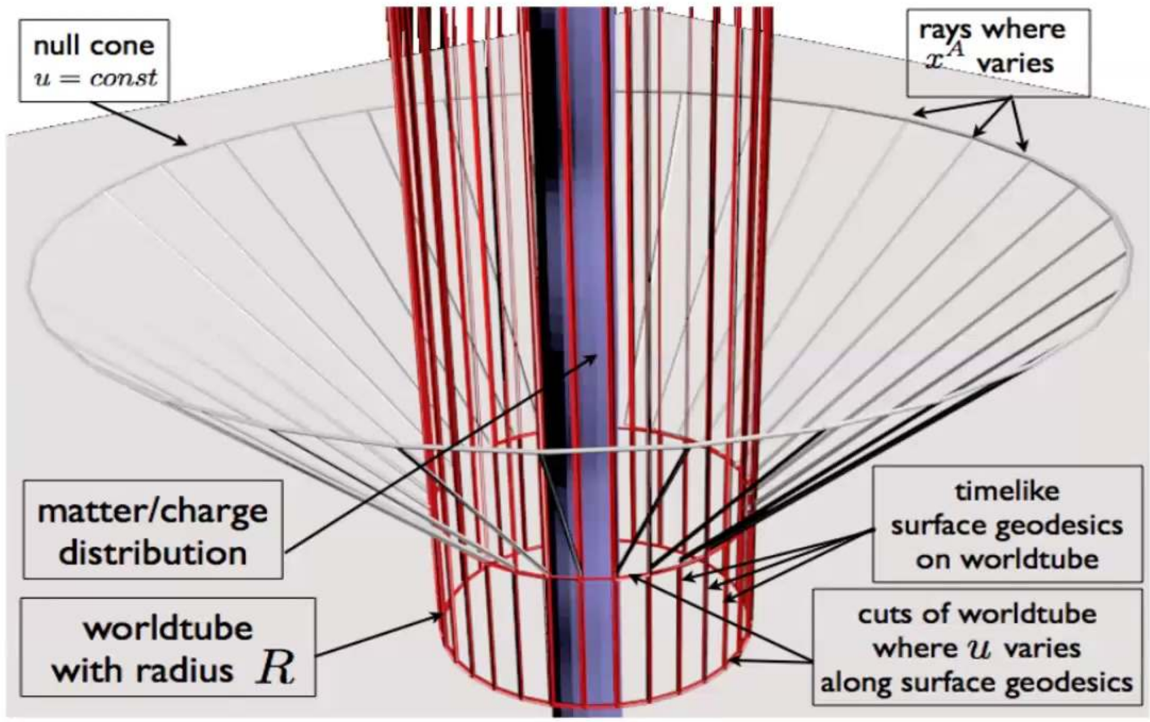


Bondi-Sachs formalism OR Null cone Formalism

Bondi-Sachs coordinates

- The Bondi-Sachs formalism uses coordinates $x_i = (u, r, x_A)$ based upon a family of outgoing null hypersurfaces.

- We label these hypersurfaces by $u = const.$, null rays by $x^A (A = 2,3)$, and the surface area coordinate by *r*.



null cone
$u = const$
rays where
x^A varies
matter/charge
distribution
timelike
surface geodesics
on worldtube
worldtube
with radius R
cuts of worldtube
where u varies
along surface geodesics

Bondi-Sachs metric

- In this coordinates system the Bondi-Sachs metric takes the form

$$ds^2 = -\left[e^{2\beta}\left(1+\frac{W}{r}\right) - r^2 h_{AB}U^A U^B\right]du^2 - 2e^{2\beta}dudr$$
$$-2r^2 h_{AB}U^B dudx^A + r^2 h_{AB}dx^A dx^B,$$

- Where $h^{AB}h_{BC} = \delta^A_B$ and $\det(h_{AB}) = \det(q_{AB})$, with q_{AB} being a unit sphere metric,
- h_{AB} conformal 2-metric, and has only two degrees of freedom ($+$ and $\times$ polarization of gravitational waves).

- U is the complex spin-weighted field given by $U = U^A q_A$.

Problem: Thin Matter Shell

We linearize the Einstein equations when:

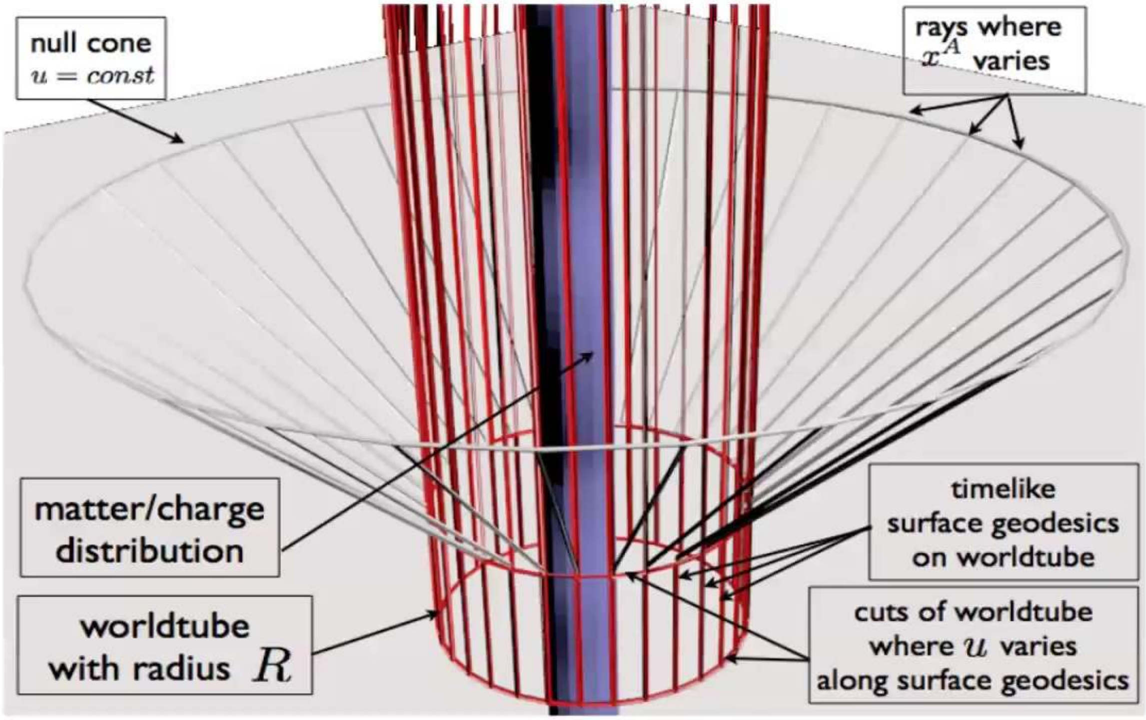
- the metric is Bondi-Sachs,
- the background is Schwarzschild, and
- when there is a matter source in the form of a thin shell whose density varies with time and angular position.

The linearized Einstein equations

Metric quantities and spherical harmonics

We regard the density and metric quantities as being small, i.e.
ρ, J, β, U, w with $W = -2M + w$
And we assume the following ansatz

- $J = Re(J_0(r)e^{i\sigma u})\,{}_2Z_{lm}$, $U = Re(U_0(r)e^{i\sigma u})\,{}_2Z_{lm}$, $\beta = Re(\beta_0(r)e^{i\sigma u})\,{}_2Z_{lm}$
- $\omega = Re(\omega_0(r)e^{i\sigma u})\,{}_2Z_{lm}$,
- (4)
- where l, r_0, and σ are fixed.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- $\omega = Re(\omega_0(r)e^{i\sigma u})_2 Z_{lm}$,
- (4)
- where l, r_0, and σ are fixed.

Equations decompose into

- Hypersurface equations for β, U, W:

$$R_{11}: \frac{4}{r}\beta_{,r} = 8\pi T_{11}$$

$$q^A R_{1A}: \frac{1}{2r}\left(4\eth\beta - 2\mathrm{r}\eth\beta_{,r} + r\bar{\eth} J_{,r} + r^3 U_{,rr} + 4r^2 U_{,r}\right) = 8\pi q^A T_{1A}$$

$$h^{AB} R_{AB}: \left(4 - 2\eth\bar{\eth}\right)\beta + \frac{1}{2}\left(\bar{\eth}^{\,2} J + \eth^{\,2}\bar{J}\right) + \frac{1}{2r^2}\left(r^4 \eth\,\bar{U} + r^4 \bar{\eth}\,U\right)_{,r}$$
$$-2\omega_{,r} = 8\pi\left(h^{AB} T_{AB} - r^2 T\right)$$

- Evolution equations for J
- $q^A q^B R_{AB}: -2\eth^{\,2}\beta + (r^2 \eth\, U)_{,r} - 2(r - M)J_{,r} - \left(1 - \frac{2M}{r}\right) r^2 J_{,rr}$
$$+2r(rJ)_{,ur} = 8\pi q^A q^B T_{AB}$$

Constraints R_{Oj} $(R_{00}, R_{01}, q^A R_{0A})$

System of ordinary differential

- From the Hypersurface and evolution equations, we get the following system of ordinary differential

$$x^3(1-2xM)\frac{d^2J_2}{dx^2}+2\frac{dJ_2}{dx}(2x^2+i\sigma x-7x^3M)$$
$$-2\left(\frac{x(l^2+l-2)}{2}+8Mx^2-i\sigma\right)J_2=0$$

Where: $J_2(x)\equiv\frac{d^2J_{0+}}{dx^2}$ and $x=\frac{1}{r}$, and the formula for general of the angular momentum of the system l.

This equations cannot be solved analytically but numerically.

Numerical problem specification

- The ODE above has singularities;

  at x = 0 (essential, in the case of a Schwarzschild black hole), and

  $x = 0.5M$ (regular, in the case of Schwarzschild white hole).

- Then our problem translates to searching for values of σ for which a solution to the ODE that is regular everywhere in the interval [0,0.5M] exist; these values of σ are the QNMs.

- NB! This is the same situation that is faced when finding the quasi-normal modes of a black hole

A. Asymptotic series solution about the essential singularity at x = 0

- The series solution $J_2(x) = \sum_{n=1}^{\infty} a_n x^n$ to the ODE can be generated by the recurrence relation

$$a_n = -a_{n-1}\frac{n^2+n-6}{2i\sigma(n-1)} + a_{n-2}M\frac{2n(n+2)}{2i\sigma(n-1)}$$ with

$$a_1 = 1, \quad a_2 = 0.$$

- This series solution has radius of convergence zero, although it is asymptotic

Asymptotic theory

- We use the asymptotic theory by F. Olver " Asymptotics and special functions (Academic Press, New York, 1974) " to investigate the above series solution about the singularity at infinity,
- by transforming the ODE to its asymptotic form
- $$z^2(z-2)\frac{d^2J_2(z)}{dz^2} - z(2z^2 i\sigma + 2z - 10)\frac{dJ_2(z)}{dz} - (2z^2 i\sigma + 4z + 16)J_2(z) = 0$$

using the transformation $x \to z = \frac{1}{x}$, and have normalized the scaling of z by setting $M = 1$.

- Then the solutions can be written as

$$J_{2j}(z) \asymp \exp(\lambda_j z) z^{\mu_j} \sum_{s=0}^{\infty} \frac{a_{s,j}}{z^s}$$

Where $\lambda_1 = 0, \mu_1 = 1, \lambda_2 = 2i\sigma, \mu_2 = 3 + 4i\sigma.$

By the asymptotic theory, we let the solution to the ODE to be written as

$$J_2(z) = L_n(z) + \epsilon_n(z)$$

Where $L_n(z) = \exp(\lambda_1 z) z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s}$, and we define the residual $R_n(z)$ by

$$\frac{d^2 L_n(z)}{dz^2} + f(z)\frac{dL_n(z)}{dz} + g(z)L_n(z) = \frac{R_n(z)}{z}$$

- With

$$|R_n(z)| \leq \frac{B_n}{z^{n+1}}$$

NB! In some region, and where B_n is calculable, $|z| > b$.

Olver (1974) obtained the bound on $\epsilon_n(z)$ provided the quantity $C(n, b, \sigma)$ defined by

$$C(n, b, \sigma) = \frac{\beta\sqrt{\pi}\Gamma(\frac{1}{2}(n+1)+1))}{|2i\sigma|\Gamma(\frac{1}{2}(n+1)+\frac{1}{2}))(n+1)}$$

Where β is bounded by

$$\beta \leq |4i\sigma| + \left|8\frac{1+i\sigma}{b-2}\right| + \left|32\frac{1}{b(b-2)}\right| + |2i\sigma|(|2+4i\sigma| + \left|2\frac{3-4i\sigma}{b-2}\right|)$$

- Given σ and b, we use numerics to determine conditions on n such that $C < 0.99$ and then we bound $\epsilon_n(z)$ by

$$|\epsilon_n(z)| \leq \frac{2B_n}{\beta(1 - C(n,b,\sigma))|z|^{n+1}}$$

We also bound the error $\epsilon_n'(x)$ in using a finite series to estimate $\frac{dJ_2(x)}{dx}$, by noting that

$$\frac{dJ_2(x)}{dx} = -z^2 \frac{dJ_2(x)}{dz}$$

The bound on the error is

$$|\epsilon_n'(x)| \leq \frac{2|i\sigma|\beta_n}{\beta(1 - C(n,b,\sigma))z^{n-1}}$$

Numerical implementation

By the transformation

$$J_2(x) \to u(x) = \frac{1}{J_2(x)}\frac{dJ_2(x)}{dx} \quad (*)$$

And defining the new independent variable $v(x)$, $u(x) \to v(x) = \frac{1}{u(x)}$

and noting that at small $x, v \sim x$, the ODE transform into a first-order Ricatti equation

$$\frac{dv}{dx} = 1 + \frac{2v}{x^2(1-2x)}\left((x-v)\left(2+\frac{i\sigma}{x}\right) - x(7x+8v)\right) \quad (**)$$

- Then the solutions can be written as

$$J_{2j}(z) \asymp \exp(\lambda_j z) z^{\mu_j} \sum_{s=0}^{\infty} \frac{a_{s,j}}{z^s}$$

Where $\lambda_1 = 0, \mu_1 = 1, \lambda_2 = 2i\sigma, \mu_2 = 3 + 4i\sigma.$

By the asymptotic theory, we let the solution to the ODE to be written as

$$J_2(z) = L_n(z) + \epsilon_n(z)$$

Where $L_n(z) = \exp(\lambda_1 z) z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s}$, and we define the residual $R_n(z)$ by

$$\frac{d^2 L_n(z)}{dz^2} + f(z) \frac{dL_n(z)}{dz} + g(z) L_n(z) = \frac{R_n(z)}{z}$$

B. Series solution about the regular singularity at x = 0.5M

- We make the transformation

$$x \to s = 1 - 2x$$

And the ODE transform to

$$s(1-s)^3\frac{d^2J_2(s)}{ds^2} - (1-s)(4i\sigma - 3 + 10s - 7s^2)\frac{dJ_2(s)}{ds} - 4(i\sigma + 3 - 5s + 2s^2)J_2(s) = 0 \quad (***)$$

This equation has a series solution $\sum_0^\infty a_n s^n$ that satisfies the recurrence relation

$$a_0 = 1,\ a_1 = 4\frac{3+i\sigma}{3-4i\sigma},\ a_2 = \frac{15(4+3i\sigma)}{2(1-i\sigma)(3-4i\sigma)},$$

$$a_n = a_{n-1}\frac{4ni\sigma - 8i\sigma - 5 - 3n^2 - 4n}{n(4i\sigma - n - 2)} + a_{n-2}\frac{4+3n^2+2n}{n(4i\sigma - n - 2)} + a_{n-3}\frac{(1-n)(1+n)}{n(4i\sigma - n - 2)}$$

Numerical implementation

The radius of convergence of the above series is $s < 1$, and, given σ, the numerical evaluation of the coefficients, and then of the series, is straightforward.

Using x_c = 0.25 means that we need to evaluate the series at $s = 0.5$.

C. Quasi-normal modes

- We have written a Matlab program that, given a value of v,

(a)First, for the essential singularity, use the asymptotic series about $x = 0$ to find the value v_0 of $v(x)$ as defined in (*) at $x = x_0 = 1/b$, and, then integrate numerically the Ricatti Eq. (**) between x_0 and x_c = 0.25, to obtain a complex number $v_+ = v(x_c)$;

(b) secondly, use the regular series about $x = 0.5$ and integrate numerically Eq. (***) between $x = 0.5$ and $x_c = 0.25$ to find the complex number $v_- = v(x_c)$

- By defining $g_\sigma = v_+(\sigma) - v_-(\sigma)$, then the quasi-normal modes are those values of v such that g_σ is indistinguishable from zero, and we found to be

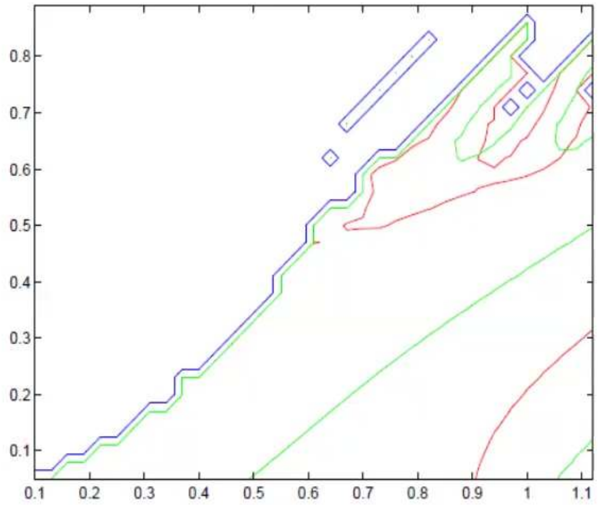
l	$M\sigma$	$\lvert\frac{\partial\sigma}{\partial g_\sigma}\times g_\sigma\rvert = error$
2	$0.883 + 0.614i$	$\lvert(3.95+0.69i)\times(6.02+5.87i)\times10^{-4}\rvert = 0.003$
	$0.916 + 0.630i$	$\lvert(0.578+2.085i)\times(4.776+2.212i)\times10^{-4}\rvert = 0.001$
	$1.063 + 0.631i$	$\lvert(-3.614+0.639i)\times(-0.5861-7.792i)\times10^{-5}\rvert = 0.0003$
	$1.199 + 0.624i$	$\lvert(-3.681-7.989i)\times(0.234-3.876i)\times10^{-5}\rvert = 0.0003$
	$1.318 + 0.611i$	$\lvert(-1.044-8.920i)\times(6.402-3.489i)\times10^{-6}\rvert = 0.000007$

For Schwarchild while hole mass $M = 1$ and $l = 2$.

Numerical illustrations with the fundamental quasi-normal mode

Contour plot

- We calculated g_σ for values of σ in the range $\sigma = a + ib$, $0.1 \le a \le 1.07$, $0.05 \le b \le 0.89$, in increments of 0.03.
- The results are shown in the contour plot below



Contour plot in the complex plane of σ showing the zero contours where $\Re(g) = 0$ (red) and $\Im(g) = 0$ (green).

blue line is the boundary of a region where the computation is probably unreliable (because the computed curve oscillates, indicating that a smaller step-length is required

- from the plot we can read off an estimate for the lowest mode, $\sigma = 0.9 + 0.63i$.
- We then applied a secant method, obtaining a final estimate for the lowest quasi-normal mode at

$$\sigma = 0.883 + 0.614i. \quad (****)$$

- In this case, $x_0 = 0.036493228795438$,

$$v_0 = 0.036838521818950 + 0.000637428772012i,$$

$$\beta = 0.988517790240599, \text{ and}$$

62 terms were used in the asymptotic series. The contour plot indicates another

- By defining $g_\sigma = v_+(\sigma) - v_-(\sigma)$, then the quasi-normal modes are those values of v such that g_σ is indistinguishable from zero, and we found to be

l	$M\sigma$	$\lvert\frac{\partial\sigma}{\partial g_\sigma}\times g_\sigma\rvert = error$
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For Schwarchild while hole mass $M = 1$ and $l = 2$.

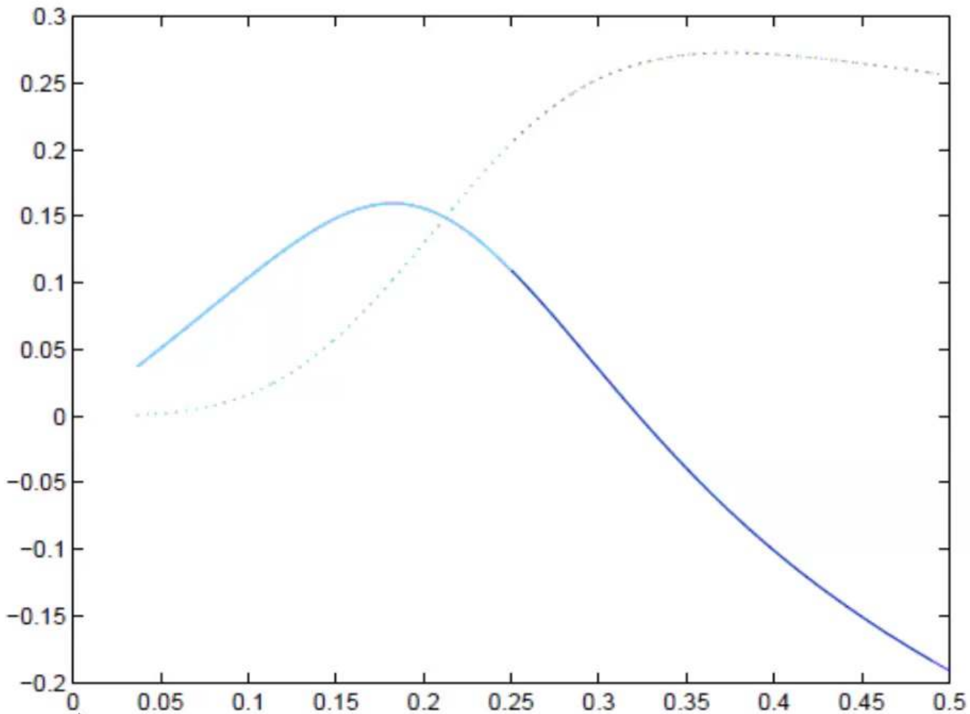
Error analysis

- We use the obtained value (****) and vary the numerical methods so as to determine the accuracy with g_σ is determined.

- The integration between x_0 and x_c is carried out with different values of MaxStep, by 2×10^{-6}, 10^{-6}, and 5×10^{-7}, and also an error of an amount $(1 + i) \times 10^{-15}$ is introduced into the value of v_0 at x_0 in the case MaxStep = 2×10^{6}.

- Also, numerical integration of Eq. (22) as well as a series solution is used in the range (x_c, 0.5).

- So in total, there were four numerical methods comparing to each other.



The real (solid line) and imaginary (dotted line) parts of $\sigma(x)$ in the quasi-normal mode case $\sigma = 0.883 + 0.614i$.

- Also, numerical integration of Eq. (***) as well as a series solution is used in the range $(x_c, 0.5)$.
- The various curves lie on top of each other and are visually indistinguishable.
- Taking all these options into account, the maximum value noted for

g_σ was $(6.02 + 5.87i) \times 10^{-4}$.

- So, using intermediate results from the secant root-finding process to estimate

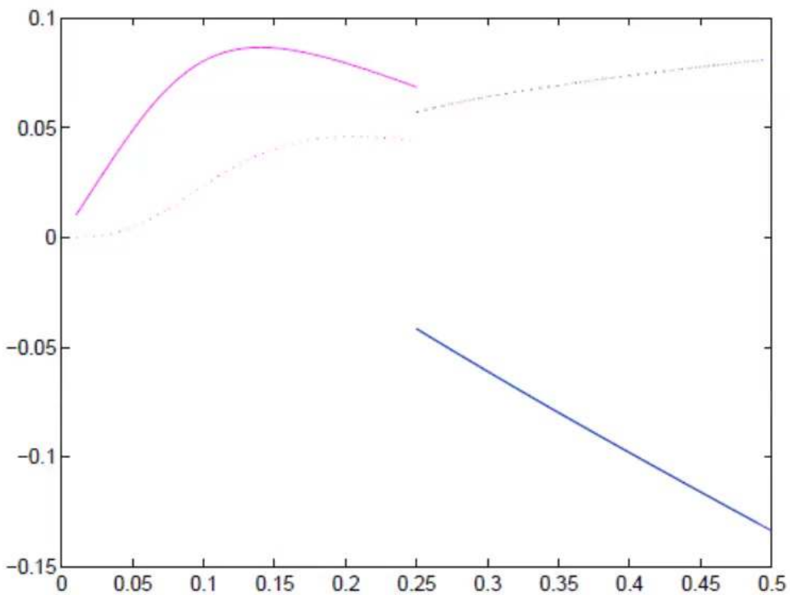
$$\frac{\partial \sigma}{\partial g_\sigma} = 3.95 + 0.69i$$

Analysis of the ODE with the quasi-normal mode of a Schwarzschild black hole

- The lowest quasi-normal mode of a Schwarzschild black hole is at

$$\sigma = 0.37367 + 0.08896i$$

We have used this value in our code, and obtained figure below



The real (solid line) and imaginary (dotted line) parts of $\sigma(x)$ in the case $\sigma = 0.37367 + 0.08896i$, indicating that the lowest quasi-normal mode of a Schwarzschild black hole is not a quasi-normal mode of ODE.

- T. Regge and J. Wheeler, Phys. Rev. 108, 1063 (1957).
- F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).
- C. V. Vishveshwara, Phys. Rev. D 1, 2870 (1970).
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