Title: What becomes of vortices when they grow giant?

Speakers: Alexander Penin

Series: Particle Physics

Date: November 06, 2023 - 1:00 PM

URL: https://pirsa.org/23110048

Abstract: Quantum vortices are two-dimensional solitons which carry a topological charge - the first Chern number n. They play a crucial role in many physical concepts from cosmic strings to mirror symmetry and dualities of supersymmetric models. When n grows the vortices become giant. The giant vortices are observed experimentally in a variety of quantum systems. Thus, it is quite appealing to identify their characteristic features and universal properties, which is quite a challenging mathematical problem. Though the nonlinear vortex equations may look deceptively simple, their analytic solution is not available. In this talk I demonstrate how by borrowing the asymptotic methods of fluid dynamics such a solution can be found in the large-n limit. I then construct a systematic expansion in inverse powers of the topological charge about this asymptotic solution which works amazingly well all the way down to the elementary vortex with n=1. I use this result to study the Majorana zero modes bound to giant vortices. The resulting local density of states has a number of features which give remarkable signatures for an experimental observation of the "Majorana fermions" in two dimensions.

Zoom link https://pitp.zoom.us/j/98994856372?pwd=ZFBOemRZQS9WbHAzMTN6R2lKZEdXQT09

What Becomes of Vortices When They Grow Giant

Alexander Penin

University of Alberta

Perimeter Institute

November 6th, 2023

Context

- Condensed matter
  - *superconductors, superfluids, ...*
  - *topological insulators, Majorana states, ...*
- Particle/field theory
  - *superconducting cosmic strings*
- Theory of solitons
  - *asymptotic methods, effective field theory, ...*

Topics discussed

- Introduction and Motivation
  - *Ginzburg-Landau/Abelian Higgs model*
  - *topology and general structure of quantum vortices*
  - *giant vortices in superconductors and BEC*
- Theory of giant vortices
  - *the main idea (expansion in large winding number n)*
  - *anatomy of critical vortices*
  - *large-n asymptotic solution and finite-n corrections*
  - *integrable limits*
- *Majorana modes of giant vortices*

Based on

A.A. Penin, Q. Weller, Phys.Rev.Lett. **125**, *251601 (2020)*

A.A. Penin, Q. Weller, JHEP **08**, *056 (2021)*

L. Gates, A.A. Penin, Phys.Rev. B ***107***, *125418 (2023)*

L. Gates, A.A. Penin, JHEP **06**, *072 (2023)*

Ginzburg-Landau model

- Free energy functional *(NP 2003)*

$$\mathcal{F} = \int \left[\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 + \frac{\lambda}{2}\left(|\phi|^2 - \eta^2\right)^2\right] \mathrm{d}^3x$$

$$D_\mu = \partial_\mu + ieA_\mu$$

  - *Cooper pair density* $|\phi|^2$ *magnetic field* $B_\lambda = -\frac{1}{2}\varepsilon_{\lambda\mu\nu}F_{\mu\nu}$

  ↪ *stationary state of charged BEC in magnetic field*
- Ground state: $\mathcal{F} = 0$, $\phi = \eta e^{i\alpha}$, $A_\mu = -\partial_\mu\alpha/e$,
  - *mass spectrum:* $m_\phi = \sqrt{2\lambda}\eta$, $m_A = \sqrt{2}e\eta$
- Quantum vortices *(A. Abrikosov (NP 2003))*
  - *string-like solutions of finite tension* $T = \mathcal{F}/L$

Quantum vortices at large

- **Vortex topology**
  - *finite tension* ⇨ $\phi \to \eta e^{i\alpha}$ at $|x| \to \infty$
  - *continuous field* ⇨ $\alpha(\theta + 2\pi) = \alpha(\theta) + 2\pi n$

  ↪ *winding number* n

S_1		S_1
$0 \le \theta \le 2\pi$	$\longrightarrow$	$0 \le \alpha \le 2\pi$
polar angle		*phase*

  - *homotopy group* $\pi_1(S_1) = Z_n$

  ↪ *fields with* $T < \infty$ *represent homotopy classes*

Quantum vortices at large

- *winding number ➪ 1st Chern number*

$$2\pi n = \int_0^{2\pi} (\partial\alpha/\partial\theta)\mathrm{d}\theta = \oint \partial_\mu\alpha\,\mathrm{d}x^\mu = -e\oint A_\mu \mathrm{d}x^\mu = \int -\frac{e}{2}\varepsilon^{\mu\nu}F_{\mu\nu}\mathrm{d}^2\boldsymbol{r}$$

- Magnetic flux quantization ☞ $-\frac{e}{2}\varepsilon^{\mu\nu}F_{\mu\nu} = e\,(\boldsymbol{\nabla}\times\boldsymbol{A})_\perp = eB_\perp$

$$\Phi = \int B_\perp \mathrm{d}^2\boldsymbol{r} = 2\pi n/e$$

- Universal vortex features
  - *inner region or core (symmetric phase):* $\phi \approx 0,\, B_\perp \neq 0$
  - *outer region (broken symmetry phase):* $|\phi| \approx \eta,\, B_\perp \approx 0$

Vortex equation

- *Axially symmetric vortices*

$$\phi(r,\theta) = f(r)e^{in\theta}, \qquad A_\theta = -na(r)/e, \qquad A_r = 0$$

- *Abrikosov-Nielsen-Olesen vortex equations* $(e \neq 0)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \left[\lambda(f^2-1) + \frac{n^2}{r^2}(1-a)^2\right] f = 0$$
$$r\frac{d}{dr}\left(\frac{1}{r}\frac{da}{dr}\right) + 2(1-a)f^2 = 0$$

  - new units $e\eta r \to r$, $f/\eta \to f$, $\lambda/e^2 \to \lambda$ ➪ $e = \eta = 1$
  - boundary conditions $a(0) = f(0) = 0,\ a(\infty) = f(\infty) = 1$

- *Ginzburg-Pitaevskii vortex equation* $(e = 0$ ➪ $a(r) = 0, \lambda = 1)$
  - *massless Goldstone boson*
  - *logarithmically divergent rotational energy* $|\partial_\mu \phi|^2 \to \eta^2 n^2/r^2$

Vortex equation

- Famous solutions
  - *type-II superconductors* $n = 1, \lambda \to \infty$ ➪ *approximate solution*
    A. Abrikosov
  - *critical coupling* $\lambda = 1, n = 1$ ➪ *useless Taylor series*
    H. J. de Vega, F. A. Schaposnik, Phys. Rev. D **14**, 1100 (1976)
  - *critical coupling, hyperbolic plane, all* n ➪ *nice analytic solution*
    E. Witten, Phys. Rev. Lett. **38**, 121 (1977)
- *What about flat-space vortices with* $n \gg 1$*?*
  - *"giant" or "multi" vortices observed in real BECs*
  - *crucial for observation of "Majorana fermions"*

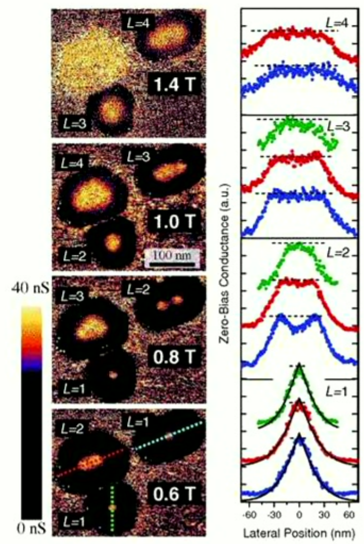
Giant vortices in condensed matter

- Quite nontrivial to cook up
  - *quasistable (decay into multiple* $n = 1$ *states)*
- Currently observed
  - *mesoscopic superconductors up to* $n = 4$
  - *cold atoms BEC up to* $n = 60$
  - *superfluid* ^{4}*He up to* $n = 365$
- Suggested experiments
  - *polariton condensates in optical media*
  - *screw dislocations, heterosructures*

Giant vortices in condensed matter

- *Mesoscopic superconductors*

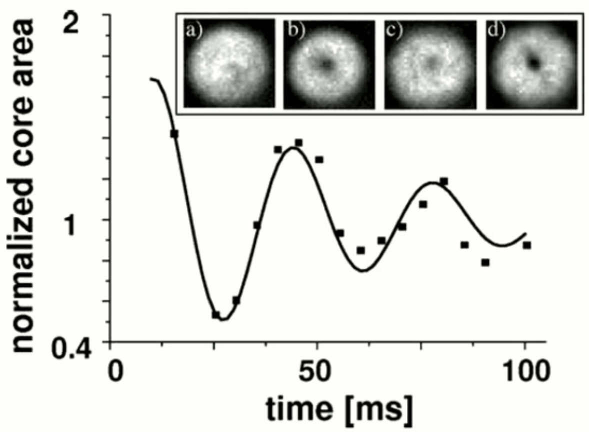
T. Cren *et al.*, Phys. Rev. Lett. **107**, 097202 (2011)



Giant vortices in condensed matter

- *Cold atoms BEC*

P. Engels *et al.*, Phys. Rev. Lett. **90**, 170405 (2003)



A Theory of Giant Vortices

The key idea

- "Find a small parameter" *(Lev Landau)*

$$\frac{1}{n^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \left[\frac{\lambda}{n^2}(f^2-1) + \frac{(1-a)^2}{r^2}\right] f = 0$$
$$r\frac{d}{dr}\left(\frac{1}{r}\frac{da}{dr}\right) + 2(1-a)f^2 = 0$$

- Expand in $1/n$, but *how?*
  - *vanishing coefficient of the highest derivative*
  - *singular perturbation, asymptotic series*

↪ Boundary layer theory *(Ludwig Prandtl)*
  - *expansion parameter = dynamical scale ratio*
  - *different approximations in different regions*

Effective field theory in coordinate space

- Expansion in inverse powers of n
  - *associate winding number with a scale ratio*
  - *separate scales and expand as in effective field theory*
- Universal vortex structure
  - *core* $\phi \approx 0, B_\perp \approx \text{const}$ | *boundary layer* | *tail* $|\phi| \approx \text{const}, B_\perp \approx 0$
- Scales hierarchy ($\lambda = \mathcal{O}(1)$)
  - *range of nonlinear interaction (BL depth)* $1/m_{A,\phi} = \mathcal{O}(1)$
  - *core size* $r_n \propto \sqrt{n}$ *to accommodate the flux* $\Phi \propto n$

↪ *ratio of dynamical scales* $(r_n m_{A,\phi})^{-1} \propto 1/\sqrt{n}$

Effective field theory in coordinate space

- Example: critical scalar self-coupling $\lambda = 1$
  - *hidden $N = 2$ SUSY, "self-dual" solution*
  - *first order Bogomolny equations*

$$(D_1 + iD_2)\,\phi = 0$$

$$F_{12} - |\phi|^2 + 1 = 0$$

  - *string tension* $\propto$ *topological/central charge* $T = 2\pi n$

E. B. Bogomolny, Sov. J. Nucl. Phys. **24**, 449 (1976)

Effective field theory in coordinate space

- Vortex equations

$$\frac{df}{dr} - \frac{n}{r}(1-a)f = 0$$
$$\frac{da}{dr} + \frac{r}{n}(f^2-1) = 0$$

- Regions with different dynamics ($r_n = \sqrt{2n}$)
  - *core:* $r \lesssim r_n$
  - *boundary layer:* $r - r_n = \mathcal{O}(1)$
  - *tail:* $r \gtrsim r_n$

Effective field theory in coordinate space

- Vortex equations

$$\frac{df}{dr} - \frac{n}{r}(1-a)f = 0$$
$$\frac{da}{dr} + \frac{r}{n}(f^2-1) = 0$$

- Core region
  - $r \to 0$ *asymptotic:* $f(r) \propto r^n$, $a(r) \sim r^2/r_n^2$
  - ↪ *expand in* f
  - *linear massless equations with a source*

$$\frac{df}{dr} - \frac{n}{r}\left(1 - \frac{r^2}{r_n^2}\right) f = 0, \qquad \frac{da}{dr} - \frac{r}{n} = 0$$

  - *core solution*

$$f(r) = F \exp\left[\frac{n}{2}\left(\ln\left(\frac{r^2}{r_n^2}\right) - \frac{r^2}{r_n^2} + 1\right)\right], \qquad a(r) = \frac{r^2}{r_n^2}$$

  - ↪ *accommodate all vortex energy and flux*

Effective field theory in coordinate space

- Vortex equations

$$\frac{df}{dr} - \frac{n}{r}(1-a)f = 0$$
$$\frac{da}{dr} + \frac{r}{n}(f^2-1) = 0$$

- Boundary layer
  - *nonlinear dynamics at the scale* $r - r_n \equiv x = \mathcal{O}(1)$
  - ↪ *expand in* x/r_n
  - *1-dim equations for a domain wall separating two phases*

$$w' + \gamma = 0\,, \qquad \gamma' - 1 + e^{2w} = 0$$

where $f(r_n + x) = e^{w(x)}$, $\gamma(x) = n\,(a(r_n + x) - 1)\,/r_n$

↪ $w'' + 1 - e^{2w} = 0$ *with the first integral* $w'^2 - e^{2w} + 2w = -1$

Effective field theory in coordinate space

- Vortex equations

$$\frac{df}{dr} - \frac{n}{r}(1-a)f = 0$$
$$\frac{da}{dr} + \frac{r}{n}(f^2-1) = 0$$

- Boundary layer
  - *boundary layer solution*

$$\int_{w_0}^{w(x)} \frac{\mathrm{d}w}{(e^{2w}-2w-1)^{1/2}} = x$$
$$\gamma(x) = -(e^{2w(x)} - 2w(x) - 1)^{\frac{1}{2}}$$

where $w_0 = -0.2997174398\ldots$

  - *asymptotic behavior* $(x \to \infty)$: $w(-x) \sim -x^2/2 - 1/2, \;\; w(x) \sim w_\infty e^{-\sqrt{2}x}$

where $w_\infty = w_0 \exp\left[\int_{w_0}^{0} \left(\sqrt{2}/(e^{2w}-2w-1)^{1/2} + 1/w\right) \mathrm{d}w\right] = -0.331186\ldots$

Effective field theory in coordinate space

- **Vortex equations**

$$\frac{df}{dr} - \frac{n}{r}(1-a)f = 0$$
$$\frac{da}{dr} + \frac{r}{n}(f^2-1) = 0$$

- **Tail region**
  - *small deviation from vacuum*
  - ↪ *expand in* $f-1$ *and* $a-1$
  - *free massive fields* $(f-1)'' + (f-1)'/r - 2(f-1) = 0$
  - *tail solution* $f(r) \sim 1 + \frac{\nu}{2\pi} K_0(\sqrt{2}r), \qquad a(r) \sim 1 + \frac{\mu}{2\pi}\sqrt{2}rK_1(\sqrt{2}r)$

the field of scalar charge ν and magnetic dipole moment μ

Asymptotic solution

- Core $$f(r) = F \exp\left[\frac{n}{2}\left(\ln\left(\frac{r^2}{r_n^2}\right) - \frac{r^2}{r_n^2} + 1\right)\right], \qquad a(r) = \frac{r^2}{r_n^2}$$

- Boundary layer $$f(r) = e^{w(x)}, \qquad a(r) = 1 - \sqrt{\frac{2}{n}\left(e^{2w(x)} - 2w(x) - 1\right)}$$

- Tail $$f(r) \sim 1 + \frac{\nu}{2\pi} K_0(\sqrt{2}r), \qquad a(r) \sim 1 + \frac{\mu}{2\pi}\sqrt{2}rK_1(\sqrt{2}r)$$

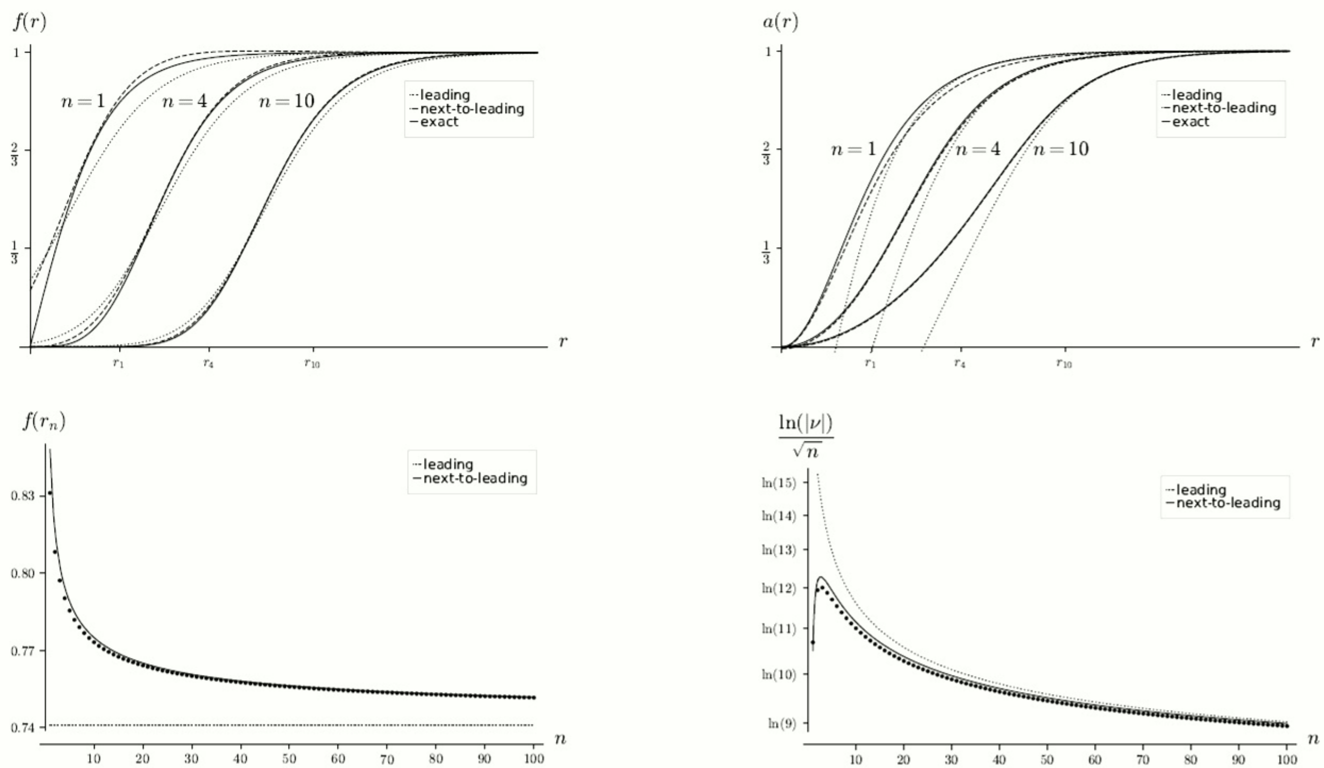
- Matching at $1 \ll |x| \ll r_n$: $$F = 1/\sqrt{e}, \qquad \nu = \mu = 4w_\infty\sqrt{\pi}e^{2\sqrt{n}+\ln(n)/4}$$

Finite-n corrections

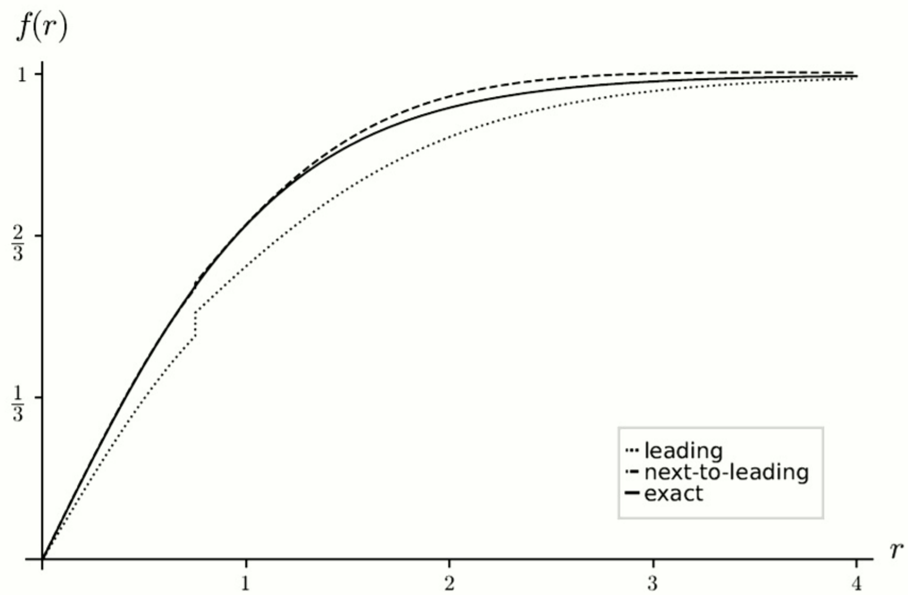
- Core and tail regions
  - *neglected terms are exponentially small*
  - ↪ *only matching corrections*
- Boundary layer
  - *write* $r = r_n(1 + x/\sqrt{2n})$, $f = \exp(w + \delta w/\sqrt{2n})$ *and expand in* $1/\sqrt{n}$
  - ↪ *linear inhomogeneous equation* $\delta w'' - 2e^{2w}\delta w = -w'$
  - *next-to-leading correction* $\delta w(x) = w'(x)\left(C + \int_0^x \int_z^\infty \frac{w'^2(y)}{w'^2(z)} \mathrm{d}y\mathrm{d}z\right)$

where $C = \int_{-\infty}^0 \left(\frac{z}{3} + \int_z^\infty \frac{w'^2(y)}{w'^2(z)}\mathrm{d}y\right) dz = 0.529935\ldots$

Convergence of large-n expansion



Convergence for elementary vortex



Noncritical vortices

- Vortex equations $(e \neq 0)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \left[\lambda(f^2-1) + \frac{n^2}{r^2}(1-a)^2\right] f = 0$$

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{da}{dr}\right) + 2(1-a)f^2 = 0$$

- *core size* $r_n = \sqrt{2n}/\lambda^{1/4}$, *string tension* $T = 2\pi\sqrt{\lambda}n$
- *boundary layer (domain wall) equation*

$$f'' - \left[\lambda\left(f^2-1\right) + \gamma^2\right] f = 0$$

$$\gamma'' - 2\gamma f^2 = 0$$

Integrable limits

$$f'' - \left[\lambda\left(f^2 - 1\right) + \gamma^2\right] f = 0$$
$$\gamma'' - 2\gamma f^2 = 0$$

- *Large scalar self-coupling* $\lambda \gg 1$
  - *expansion in* $1/\lambda$
- *Small scalar self-coupling* $\lambda \ll 1$
  - *expansion in* λ

Large scalar self-coupling

- $m_\phi \to \infty$ *nonpropagating scalar field* ➪ *expand in* f''
- *scalar field asymptotic solution*

$$f(r_n + x) = \left(1 - 2\,\mathrm{sech}\left(\sqrt{2}(x+1) + \mathrm{arcsech}(1)\right)\right)^{\frac{1}{2}}$$

- *finite-n energy correction*

$$T = 2\pi\sqrt{\lambda}n\eta^2\left(1 - \frac{4(\sqrt{2}-1)}{3}\frac{\lambda^{1/4}}{\sqrt{n}} + \mathcal{O}(1/\sqrt{n})\right)$$

  - *expansion parameter* $(r_n m_A)^{-1} \propto \lambda^{1/4}/\sqrt{n}$

  ↪ *breaks down at* $n \lesssim \sqrt{\lambda}$

Small scalar self-coupling

- decoupling of gauge field outside the core $r \gtrsim r_n$
- scalar field asymptotic "half-kink" solution

$$f(r) = \tanh(y)$$

where $y = \sqrt{\lambda/2}(r - r_n)$

- finite-n energy correction

$$T = 2\pi\sqrt{\lambda}n\eta^2 \left(1 + \frac{4}{3\lambda^{1/4}\sqrt{n}} + \mathcal{O}(1/\sqrt{n})\right)$$

  - expansion parameter $(r_n m_\phi)^{-1} \propto (\lambda^{1/4}\sqrt{n})^{-1}$

  ↪ breaks down at $n \lesssim 1/\sqrt{\lambda}$

Small scalar self-coupling

- *decoupling of gauge field outside the core* $r \gtrsim r_n$
- *next-to-leading scalar field solution*

$$f(r) = \tanh(y) + \frac{1}{\sqrt{2n}\lambda^{1/4}} \frac{9 + 12y - 8e^{-2y} - e^{-4y}}{6\sqrt{2}(2 + e^{2y} + e^{-2y})}$$

where $y = \sqrt{\lambda/2}(r - r_n)$

- *finite-n energy correction*

$$T = 2\pi\sqrt{\lambda} n\eta^2 \left(1 + \frac{4}{3\lambda^{1/4}\sqrt{n}} + \mathcal{O}(1/\sqrt{n})\right)$$

  - *expansion parameter* $(r_n m_\phi)^{-1} \propto (\lambda^{1/4}\sqrt{n})^{-1}$
  - ↪ *breaks down at* $n \lesssim 1/\sqrt{\lambda}$

Majorana Zero Modes

- Topological insulators
  - *Bogoljubov quasiparticles with linear dispersion* $E = v_F|\boldsymbol{p}|$
  - ↪ *effective Dirac fermions*
- TI/SC heterostructures
  - *zero-energy states in vortex background*
    - *topologically protected by index, self-conjugate*
    - ↪ *effective Majorana fermions*
- Elementary vortex
  - *tiny gap, difficult to distinguish from non-zero modes*
  - ↪ *experimental controversy*

Majorana Zero Modes

- Zero mode equation

R. Jackiw, P. Rossi (1984)

$$(D_1 \pm iD_2)\psi^{\pm} + \phi\psi^{*\pm} = 0$$

  - n *normalizable zero modes* $(0 \leq l \leq n/2 - 1)$

$$\xi_l^+ = \frac{1}{\sqrt{2}}\left(e^{il\theta}\psi_l^+ + e^{i(n-1-l)\theta}\psi_{n-1-l}^+\right)$$
$$\eta_l^+ = \frac{i}{\sqrt{2}}\left(e^{il\theta}\psi_l^+ - e^{i(n-1-l)\theta}\psi_{n-1-l}^+\right)$$

  - *partial waves* ψ_l^+ *are real*

↪ real or "Majorana" bound states

Majorana Zero Modes

- Asymptotic solution

$$\psi_l^+(r) \approx \begin{cases} e^{-(r-\bar{r}_l)^2/4}\,, & l < n/2\,, \quad \bar{r}_l = \sqrt{2l/n}\, r_n \\ e^{-3(r-\bar{r}'_l)^2/4}\,, & l > n/2\,, \quad \bar{r}'_l = \sqrt{(2/3)(2-l/n)}\, r_n \end{cases}$$

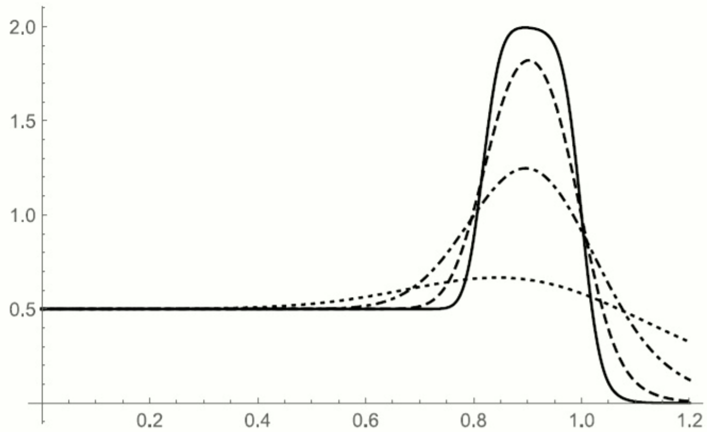
  - *scalar charge $= 2\times$ fermion charge, fermion orbit encircles two flux quanta*

  ↪ *only $n/2$ Landau states are accomodated in the core*

  - *fine interplay between magnatic effects and Andreev reflection*
  - *zero modes ⇨ warped lowest Landau states*
  - *bound states trapped inside the core ⇨ magnetically gapped*

Local density of states

- *for* $n = 4,\ 16,\ 64,\ 256$



- *asymptotic* $n \to \infty$

$$\rho(r) \sim \frac{2n}{r_n^2} \begin{cases} 1/2, & r/r_n < \sqrt{2/3}\,, \\ 2\,, & \sqrt{2/3} < r/r_n < 1\,, \end{cases}$$

Summary

- Expansion in inverse topological charge
  - *topological charge = dynamical scale ratio*
  - ↪ *effective field theory in coordinate space*
- An elegant theory of giant vortices
  - *analytic asymptotic results*
  - ↪ *capture universal properties for large n*

    cf. ADHM construction for 3d magnetic monopoles
  - *finite-n corrections*
  - ↪ *amazing convergence for $n > \sqrt{\lambda},\ 1/\sqrt{\lambda}$*

    a small parameter does not have to be small!
- *New way to observe 2d Majorana fermions.*