Title: Topological Quantum Field Theories Lecture 20231124

Speakers: Lukas Mueller

Collection: Topological Quantum Field Theories - mini-course

Date: November 24, 2023 - 2:00 PM

URL: https://pirsa.org/23110037

4) Dijkgraaf-Witten theories

- Let G be a finite group.

$\Longrightarrow$

- The Lie algebra $\underline{g} = 0$
- All G-bundles carry a unique (flat) connection.

$$Z_{DW}(M) = \int_{Bun_G(M)/Gau} e^{\boxed{iS}} \qquad S = 0$$

$$Z_{DW}(M) = \int_{\mathrm{Bun}_G(M)/\mathrm{Gau}} e$$

Def. Let M be an n-dimensional manifold. $\mathrm{Bun}_G(M)$ is the groupoid with:

Obj: Principal G-bundles on M

Mor: Gauge transformations

$$P \circlearrowleft G \text{ free}$$

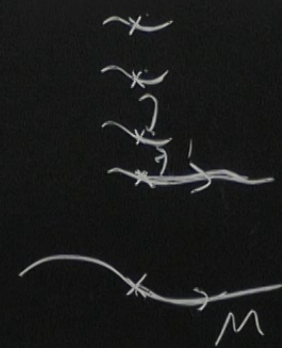
$$\downarrow$$

$$M \cong P/G$$

$$\mathbb{Z}_2 \curvearrowright S_1 = \mathbb{R}/\mathbb{Z}$$

$$x \longmapsto x + \tfrac{1}{2}$$

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[dr] & P' \arrow[d] \\ & M \end{tikzcd}$$



$$S_1/\mathbb{Z}_2 \cong S_1$$

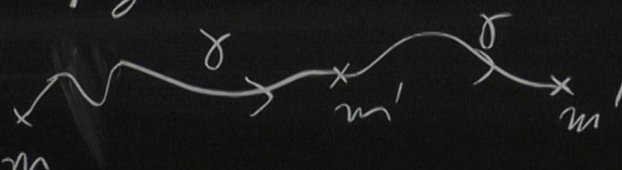
$$e(p \triangleleft g) = e(p) \triangleleft' g$$

Def.: The fundamental groupoid of M

$\Pi_1(M)$:

Obj: points $m \in M$

Mor: homotopy class of paths from m to m'



Composition: composition of paths

$$B\pi_1(M, m) \longrightarrow \Pi_1(M)$$

$$* \longmapsto m$$

$$\gamma : m \to m \longmapsto \gamma$$

Prop.

- $\mathrm{Bun}_G(M \amalg N) = \mathrm{Bun}_G(M) \times \mathrm{Bun}_G(N)$
- $\mathrm{Bun}_G(M) \cong \mathrm{Fun}(\Pi_1(M), BG) \underset{M \text{ connected}}{\cong} \mathrm{Grp}(\pi_1(M), G)/\!/G$

Corollary:

If M is compact then $\mathrm{Bun}_G(M)$ has finite number of objects up to isomorphisms.

Proof: $M \Longrightarrow \pi_1(M)$ is finitly generated.

$\Longrightarrow$ There only finitly many group homomorphisms $\pi_1(M) \longrightarrow G$. $\square$

$$Z(M) = \sum_{\varphi : \pi_1(M) \to G} 1 \, \frac{1}{|G|}$$

connected

$$Z_{DW}(\Sigma) \text{ " }= \mathrm{Fun}(\mathrm{Bun}_G(\Sigma), \mathbb{C}) \cong L^2(-, \mathbb{C}) \text{ "}$$

Def Let G be a groupoid. A function on G with values in a vector space V is a map $f : \mathrm{Obj}(G) \longrightarrow V$ $\quad f(x) = f(y)$ if x and y are isomorphic

Ex.:

$\Pi_1(M) \longrightarrow \mathbb{C} \iff$ locally constant function on M

$\iff \pi_0(M) \longrightarrow \mathbb{C}$

Let X be a set with G-action

$X /\!/ G$:

Obj: $x \in X$ Mor $(g, x) : x \longrightarrow x \triangleleft g$

$$f : X /\!/ G \longrightarrow \mathbb{C} \iff f : \underbrace{X/G}_{= X_G} \longrightarrow \mathbb{C}$$

Def. $\mathcal{G}$ is essentially finite if

- all automorphism groups are finite:

  $g \in \mathcal{C}$ $\quad |\mathrm{End}(g)| < \infty$

- $\mathcal{G}$ has finitely many objects up to isomorphism.

  $\pi_0(\mathcal{G}) = \{$isomorphism classes of objects$\}$

Def $\mathcal{G}$ e.f. and $f : \mathcal{G} \longrightarrow V$

$$\int_{g\in\mathcal{G}} f(g) = \sum_{g\in\pi_0(\mathcal{C})} \frac{f(g)}{|\mathrm{Aut}(g)|}$$

The groupoid cardinality $\mathcal{G}$

$$|\mathcal{G}| = \int_{\mathcal{G}} 1 = \sum_{g\in\pi_0} \frac{1}{|\mathrm{Aut}(g)|}$$

Ex:

- $|X /\!/ G| = \dfrac{|X|}{|G|}$

- $|\mathrm{Bun}_G(M)| = |\mathrm{Hom}(\pi_1(M), G) /\!/ G| = Z_{DW}(M)$

$$\|$$

$$\int_{\mathrm{Bun}_G(M)} 1$$

$$\cdot\ |X/\!/G| = \frac{|X|}{|G|}$$

$$\cdot\ |\mathrm{Bun}_G(M)| = |\mathrm{Hom}(\pi_1(M), G)/\!/G| = Z_{DW}(M)$$

$$\| $$

$$\int_{\mathrm{Bun}_G(M)} 1$$

$$\begin{tikzcd} \Sigma_1 \arrow[r, hook] & M & \Sigma_2 \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Bun}_G(M) \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ \mathrm{Bun}_G(\Sigma_1) & & \mathrm{Bun}_G(\Sigma_2) \end{tikzcd}$$

$$Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$$

$$\mathrm{Fun}(\mathrm{Bun}_G(\Sigma_1), \mathbb{C}) \longrightarrow \mathrm{Fun}(\mathrm{Bun}_G(\Sigma_2), \mathbb{C})$$

$$f(-) \longmapsto \left(P_{\Sigma_2} \longmapsto \int_{(P,h) \in r_2^{-1}[P_{\Sigma_2}]} f(r_1(P)) \right)$$

$$Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$$

$$\mathrm{Fun}(\mathrm{Bun}_G(\Sigma_1), \mathbb{C}) \longrightarrow \mathrm{Fun}(\mathrm{Bun}_G(\Sigma_2), \mathbb{C})$$

$$f(-) \longmapsto \left(P_{\Sigma_2} \longmapsto \int_{(P,h) \in r_2^{-1}[P_{\Sigma_2}]} f(r_1(P)) \right)$$

Def: Let $\lambda : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a functor and $g_2 \in \mathcal{G}_2$. Then $\lambda^{-1}[g_2]$ is the groupoid with:

Obj: $\left(g_1 \in \mathcal{G}_1,\ h : \lambda(g_1) \xrightarrow{\sim} g_2 \right)$

Mor: $f : (g_1, h) \longrightarrow (g_1', h')$ is a morphism

$f : g_1 \longrightarrow g_1' \in \mathcal{G}_1$, s.t.

$$\begin{tikzcd}
\lambda(g_1) \arrow[rr, "\lambda(f)"] \arrow[dr, "h"'] & & \lambda(g_1') \arrow[dl, "h'"] \\
& g_2 &
\end{tikzcd}$$

commutes.

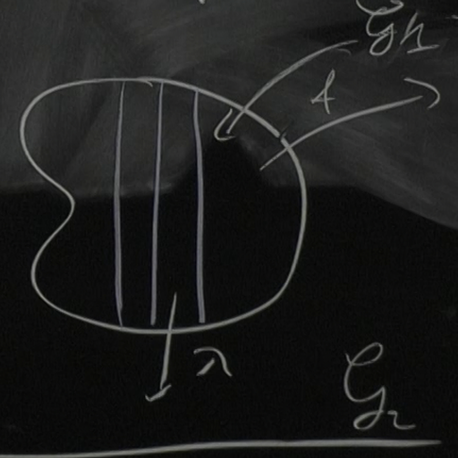
Thm: This assignment extends to a TQFT

$$Z_{DW} : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}.$$

Lemma: $\lambda : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ $\quad f : \mathcal{G}_1 \longrightarrow V$

$$\int_{\mathcal{G}_1} f = \int_{g_2 \in \mathcal{G}_2} \int_{\lambda^{-1}[g_2]} f(g_1)$$

$$\lambda^{-1}[g_2] \ni (g_1, h)$$



Proof:

$$\varepsilon : \mathbb{C} \xrightarrow{\sim} Z_{DW}(\emptyset) = \mathrm{Fun}(*, \mathbb{C})$$

$$\lambda \longmapsto (* \longmapsto \lambda)$$

$$\mu : Z_{DW}(\Sigma) \otimes Z_{DW}(\Sigma') \xrightarrow{\sim} Z_{DW}(\Sigma \amalg \Sigma') = \mathrm{Fun}(\mathrm{Bun}_G(\Sigma \amalg \Sigma'), \mathbb{C})$$

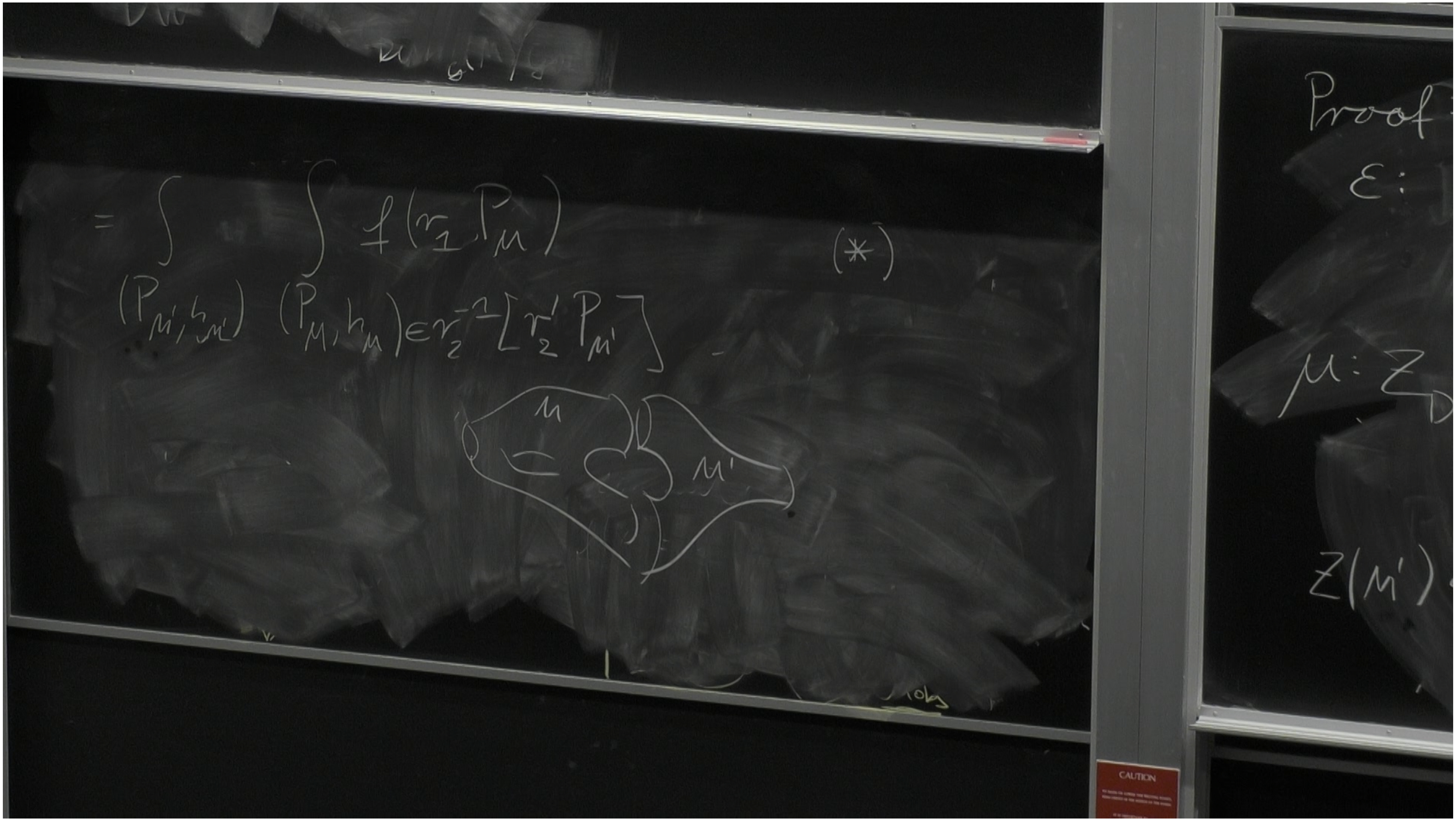
$$f(-) \otimes g(-) \longmapsto f((-)|_{\Sigma}) \cdot g((-)|_{\Sigma'})$$

$$Z(M') \circ Z(M) = Z(M' \circ M)$$

$$\mathrm{Bun}_G(\Sigma_1) \xleftarrow{r_1} \mathrm{Bun}_G(M) \xrightarrow{r_2} \mathrm{Bun}_G(\Sigma_2) \xleftarrow{r_2'} \mathrm{Bun}_G(M') \xrightarrow{r_3'} \mathrm{Bun}_G(\Sigma_3)$$

$$\int_{(P_{M'},h)\in r_3^{-1}[P_{\Sigma_3}]} Z_{DW}(M)(f)[r$$

$$= \int_{(P_{M'}, h_{M'})} \int_{(P_M, h_M) \in r_2^{-1}[r_2' P_{M'}]} f(r_1 P_M) \qquad (*)$$



$= \int_{(P_{M'}, h_{M'})} \int_{(P_M, h_M) \in r_2^{-1}[r_2' P_{M'}]} f(r_1 P_M)$ (*)
M
M'
Proof
ε:
μ: Z
Z(M')

$$Z(M'\circ M)[f] = \int_{[P_{M'\circ M},h] \in r_3^{M'\circ M^{-1}}[P_{\Sigma_3}]} f\left(r_1^{M'\circ M}(P)\right) \qquad (**)$$

$$r_3^{M'\circ M^{-1}}[P_{\Sigma_3}] \xrightarrow{F} (r_3'^{-1})[P'_{\Sigma_3}]$$

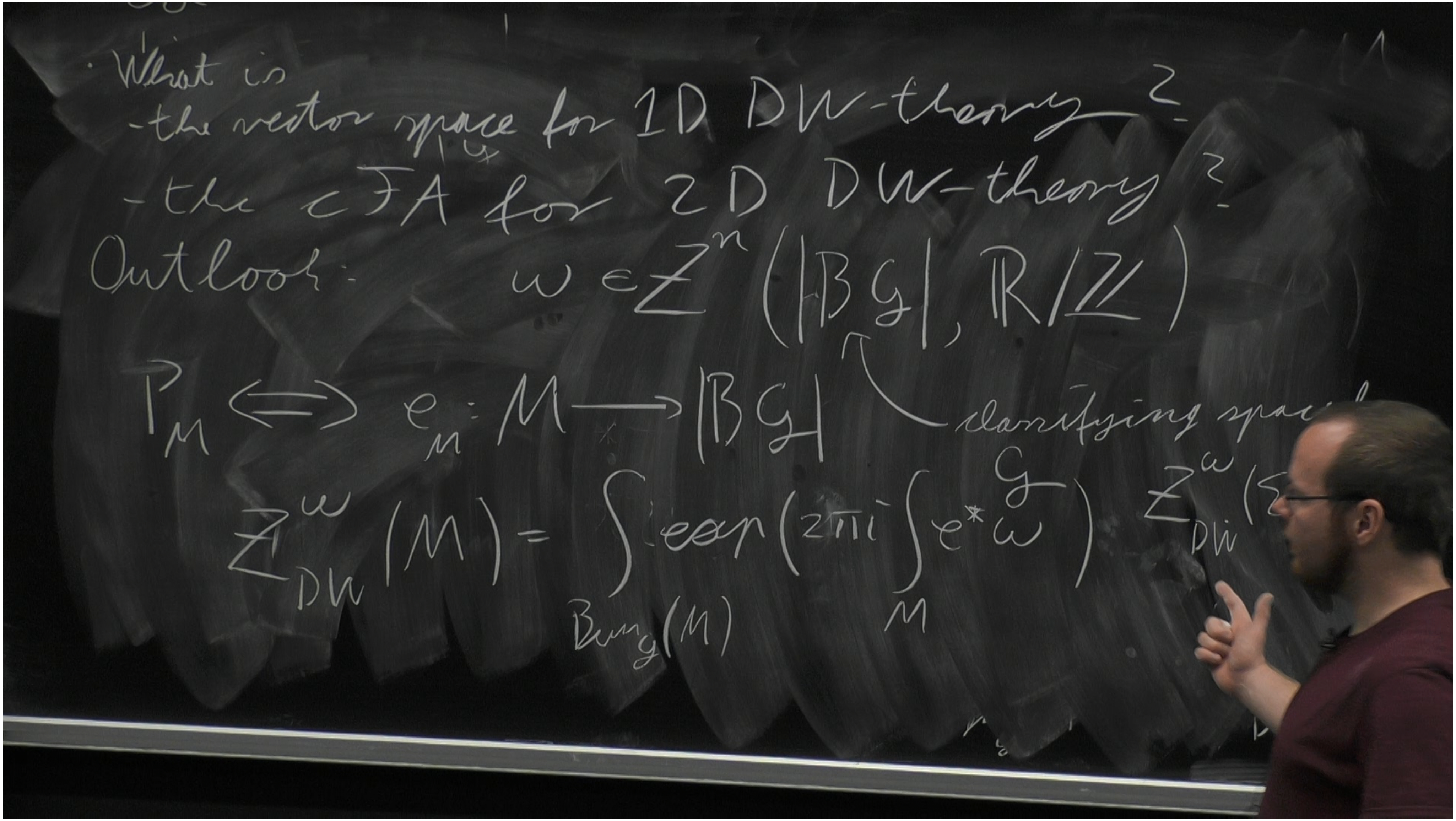
$$r_2^{-1}\left[r_2^{!} P_{M'}\right] \xrightarrow{\ \omega\ } F^{-1}\left[(P_{M'}, h)\right]$$

$$\left(P_{M'},\ h : r_2 P_M \xrightarrow{\sim} r_2^{!} P_{M'}\right) \longmapsto \left(P_M \overset{\circ}{_h} P_{M'},\ h\right).$$

Left to show that ω is an equivalence.

(See notes)

$$\boxed{\mathrm{Bun}_G(M' \circ M) = \mathrm{Bun}_G(M') \overset{h}{\underset{\mathrm{Bun}(\Sigma_2)}{\times}} \mathrm{Bun}(M)}$$



· What is
- the vector space for 1D DW-theory?
- the CFA for 2D DW-theory?
Outlook: $\omega \in Z^n(|BG|, \mathbb{R}/\mathbb{Z})$
$P_M \Longleftrightarrow e_M : M \longrightarrow |BG|$
classifying space
$Z^\omega_{DW}(M) = \int_{Bun_G(M)} \exp\left(2\pi i \int_M e^* \omega\right)$

$BG \rightsquigarrow X$ in a π-finite space

$\pi_i(X) = 0 \quad i >> 0$

$|\pi_i(X)| \leq \infty$

$$Z_X^{\omega}(M) = \sum_{f: M \to X} \exp\left(2\pi i \int_M f^*\omega\right) \frac{|\pi_2(Map(M,X), f)|}{|\pi_1(Map(M,X), f)|} \cdots$$

– Sum over other discrete things (Spin structures orientations)