Title: Topological Quantum Field Theories Lecture 20231103

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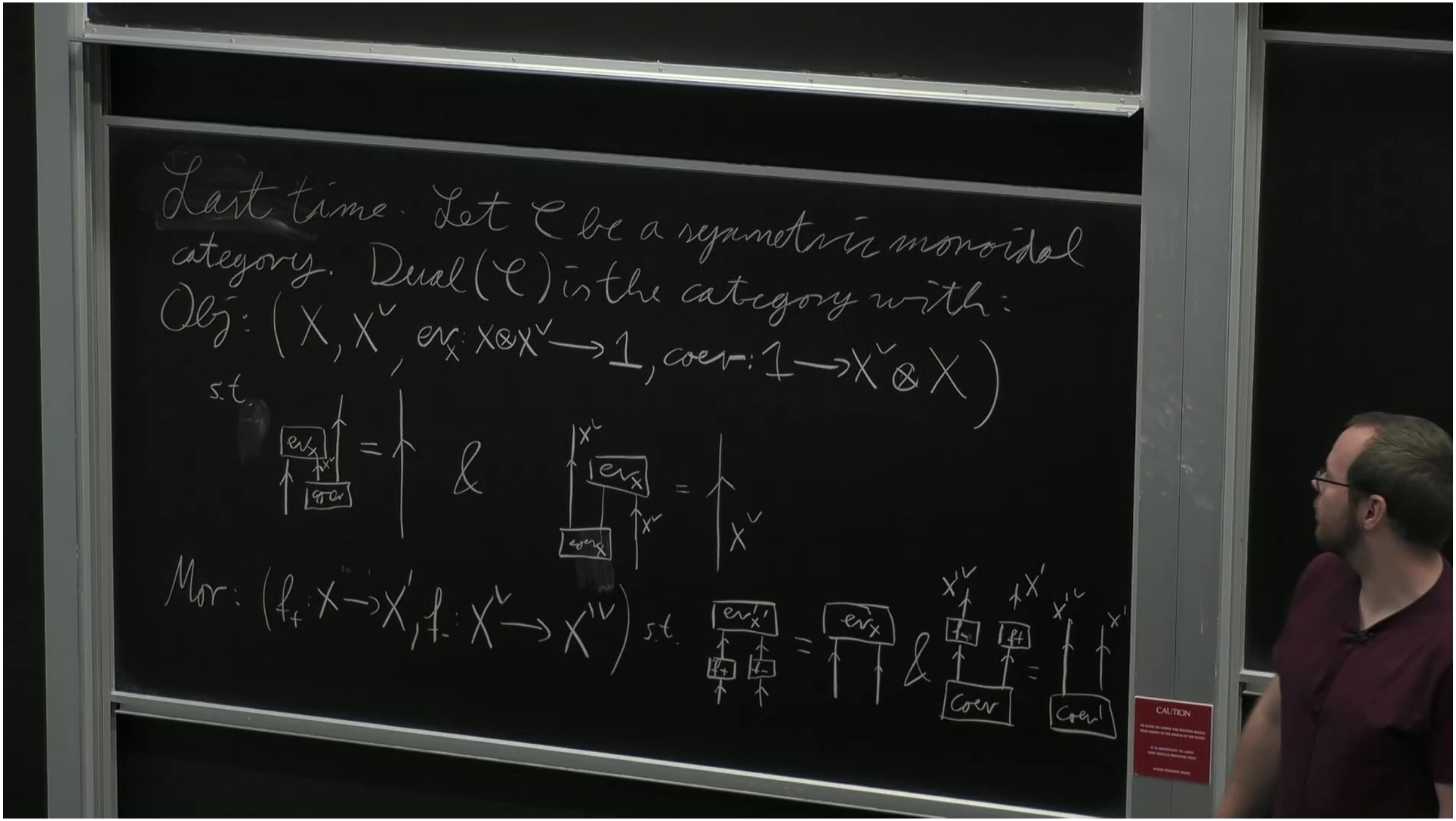
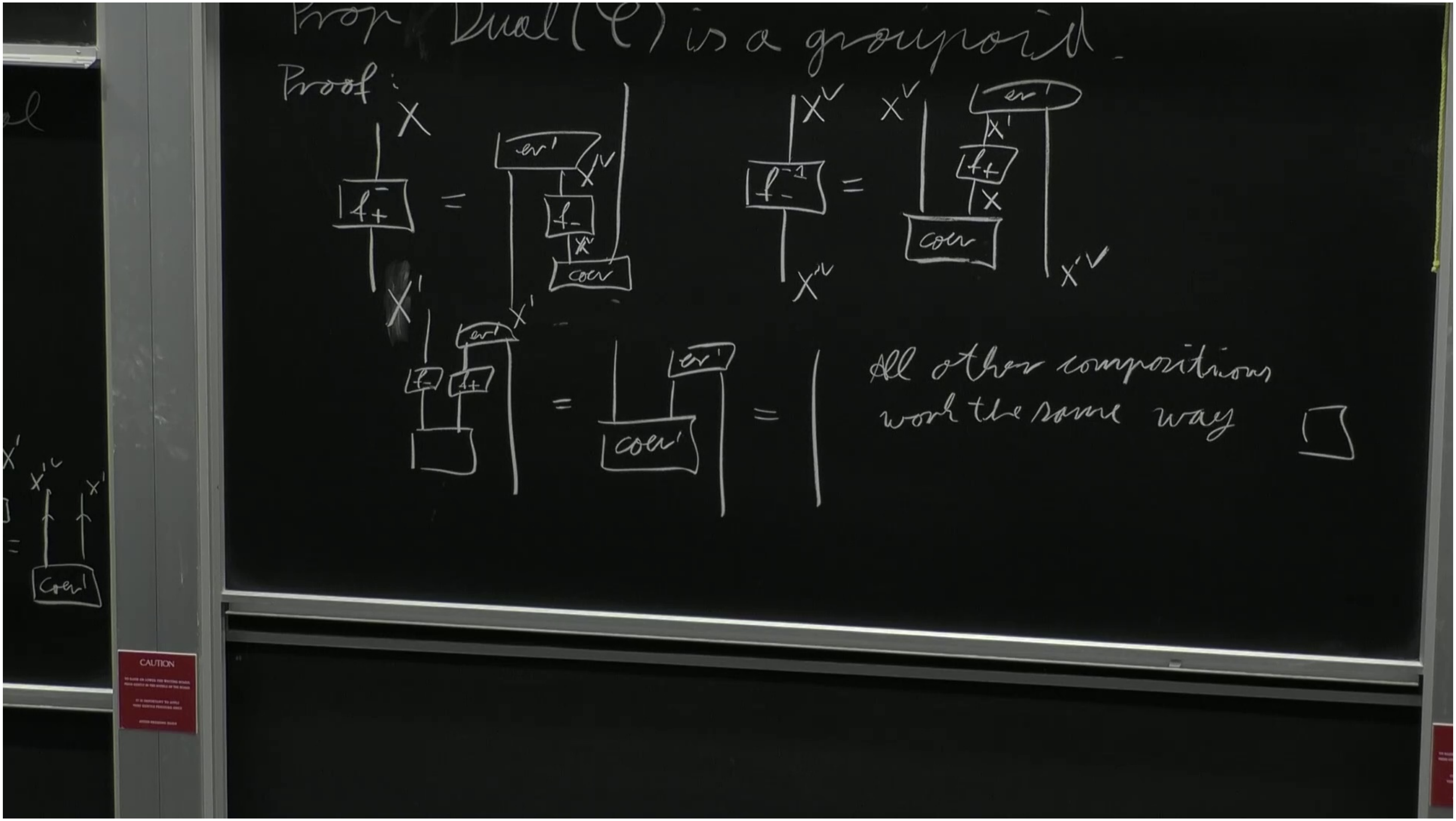
Collection: Topological Quantum Field Theories - mini-course

Date: November 03, 2023 - 2:00 PM

URL: https://pirsa.org/23110034


Last time. Let C be a symmetric monoidal category. Dual(C) is the category with:
Obj: (X, X^∨, ev_X: X⊗X^∨ → 1, coev: 1 → X^∨⊗X)
s.t.
ev_X
X^∨
coev
=
&
X^∨
ev_X
X^∨
coev_X
=
X^∨
Mor: (f_+: X → X', f_-: X^∨ → X'^∨) s.t.
ev_X'
f_+
f_-
=
ev_X
&
X'^∨
X'
f_-
f_+
coev
=
X'^∨
X'
coev'

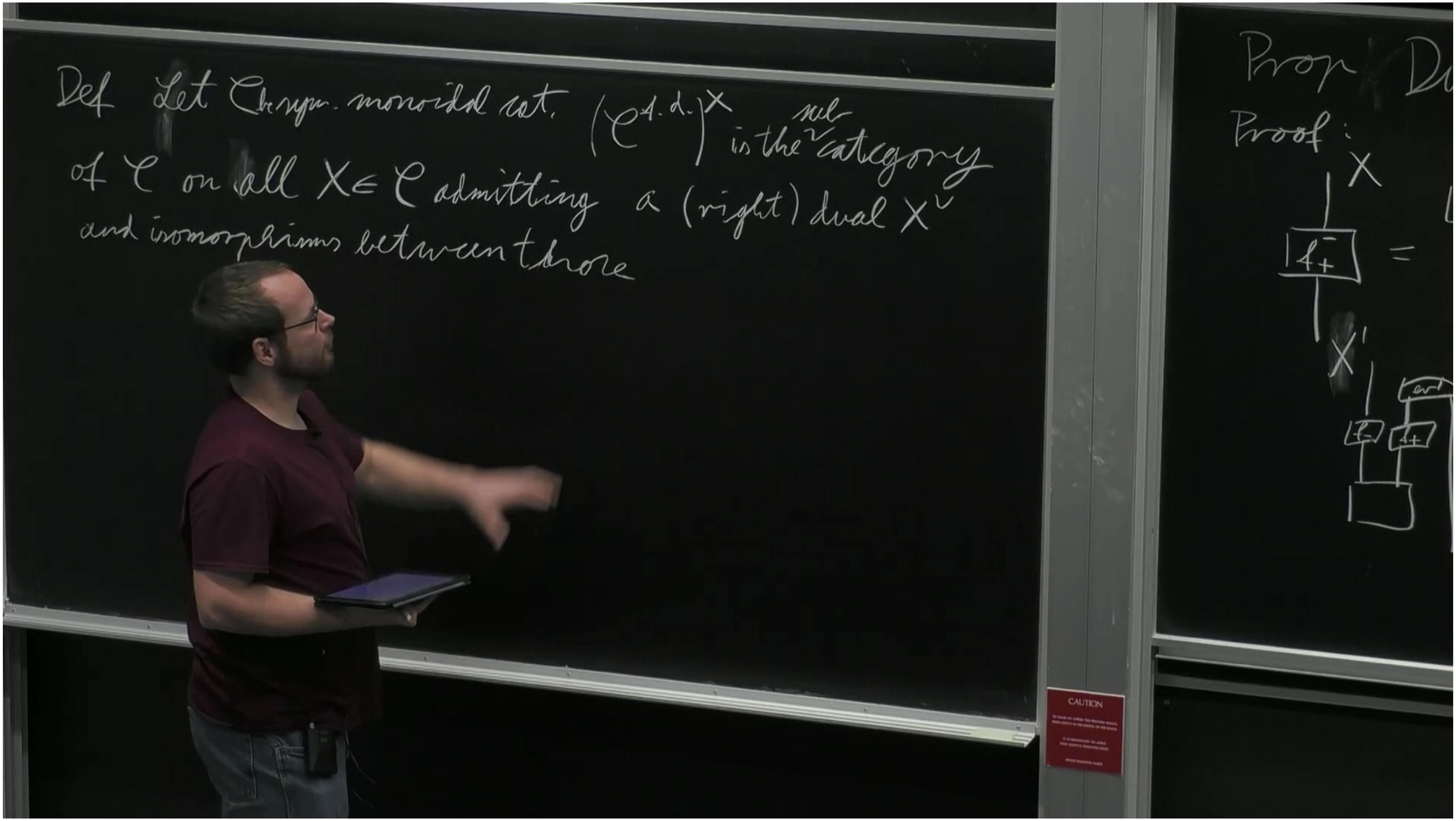

Prop: Dual(C) is a groupoid.
Proof:
All other compositions work the same way
ev'
coev
coev'
CAUTION

Cor Let $\eta : Z_1 \Rightarrow Z_2$ be a sym monoidal natural transformation between n-dim TQFTs. Then η is a natural isomorphism.

Proof $\Sigma \in \mathrm{Bord}_n$ $\mathrm{Red}_\Sigma \eta : \mathrm{Red}_\Sigma Z_1 \Rightarrow \mathrm{Red}_\Sigma Z_2$

$(\eta_\Sigma, \eta_{\bar{\Sigma}})$ is a morphism in $\mathrm{Dual}(\mathrm{Vect}) \Rightarrow \eta_\Sigma$ is an iso.

$\Rightarrow \eta$ is natural iso $\square$


Def Let C braided monoidal cat. $(\mathcal{C}^{r.d.})^\times$ is the sub category
of $\mathcal{C}$ on all $X \in \mathcal{C}$ admitting a (right) dual $X^\vee$
and isomorphisms between those
Prop
Proof:
X
f_+^-
=
X'
ev
f_-
f_+

Def Let $\mathcal{C}$ be sym. monoidal cat. $(\mathcal{C}^{f.d.})^{\times}$ is the (sub)category of $\mathcal{C}$ on all $X \in \mathcal{C}$ admitting a (right) dual $X^\vee$ and isomorphisms between those as morphisms.

Prop. The functor

$$\begin{aligned} \mathrm{Dual}(\mathcal{C}) &\longrightarrow (\mathcal{C}^{f.d.})^{\times} \\ (X, X^\vee, ev, coev) &\longmapsto X \\ (f_+, f_-) &\longmapsto f_+ : X \to X' \end{aligned}$$

is an equivalence of categories.

Prop Du

Proof:

X

$\boxed{f_+^-} =$

X'

Exercise: A functor $G: \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence iff:

- G is essentially surjective: $\forall d \in \mathcal{D}\ \exists c \in \mathcal{C}$, s.t. $G(c) \cong d$
- G is fully faithfull: The maps $G: \mathrm{Hom}_{\mathcal{C}}(c, c') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(G(c), G(c'))$ is a isomorphism.

Proof:

Exercise: A functor $G: \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence iff:

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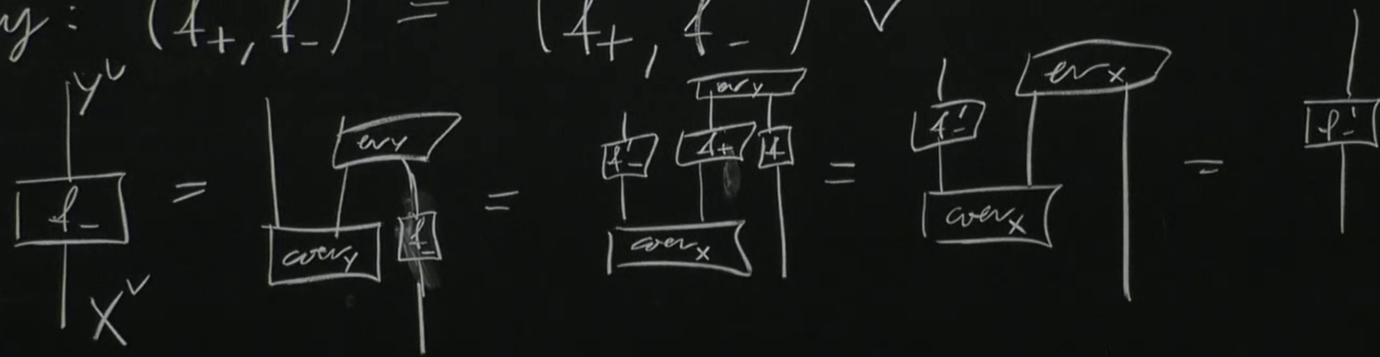
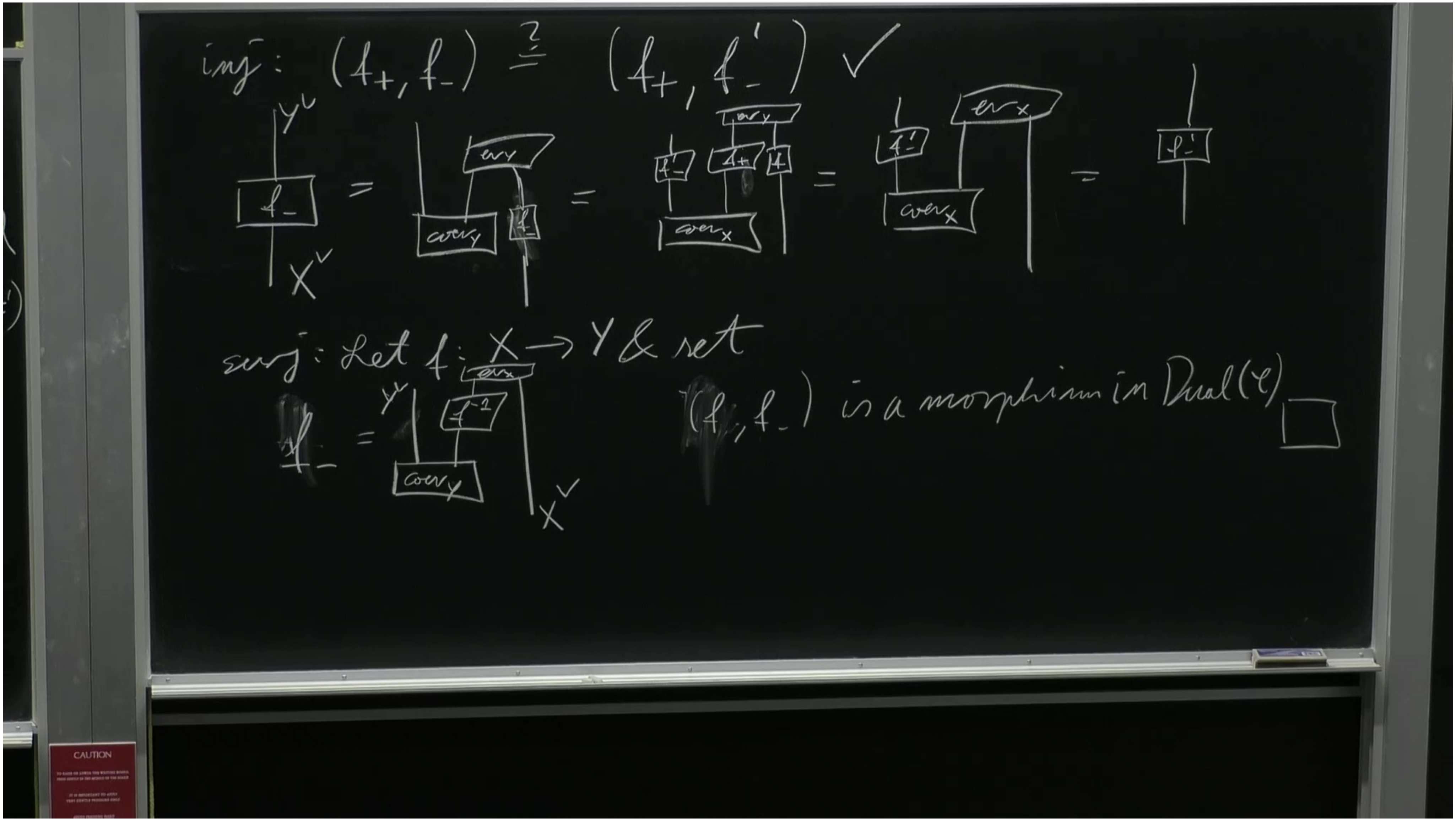
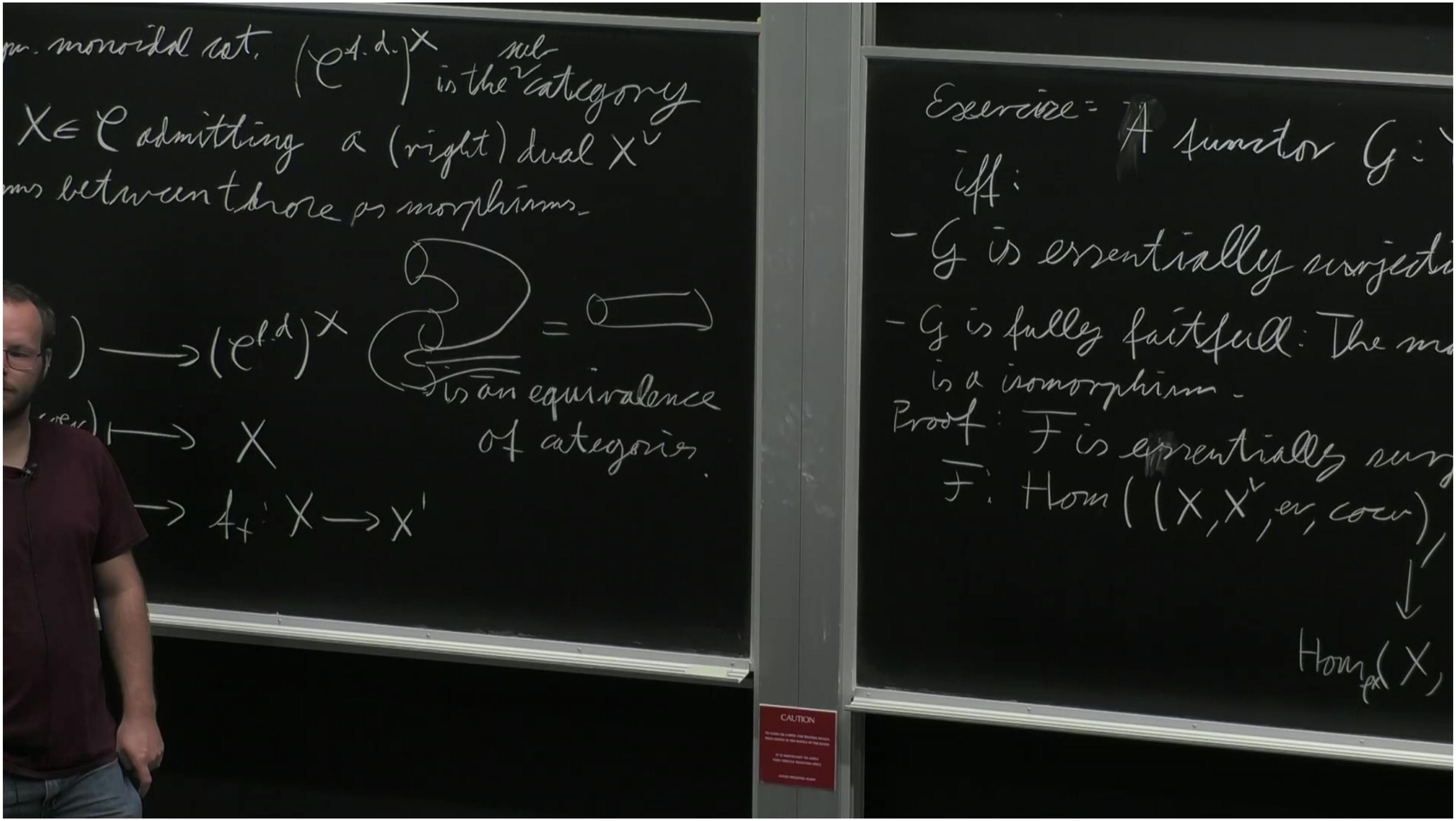
Proof: F is essentially surjective by construction.

$$F: \mathrm{Hom}((X, X^\vee, ev, coev), (Y, Y^\vee, ev_Y, coev_Y))$$
$$\downarrow$$
$$\mathrm{Hom}_{px}(X, Y)$$

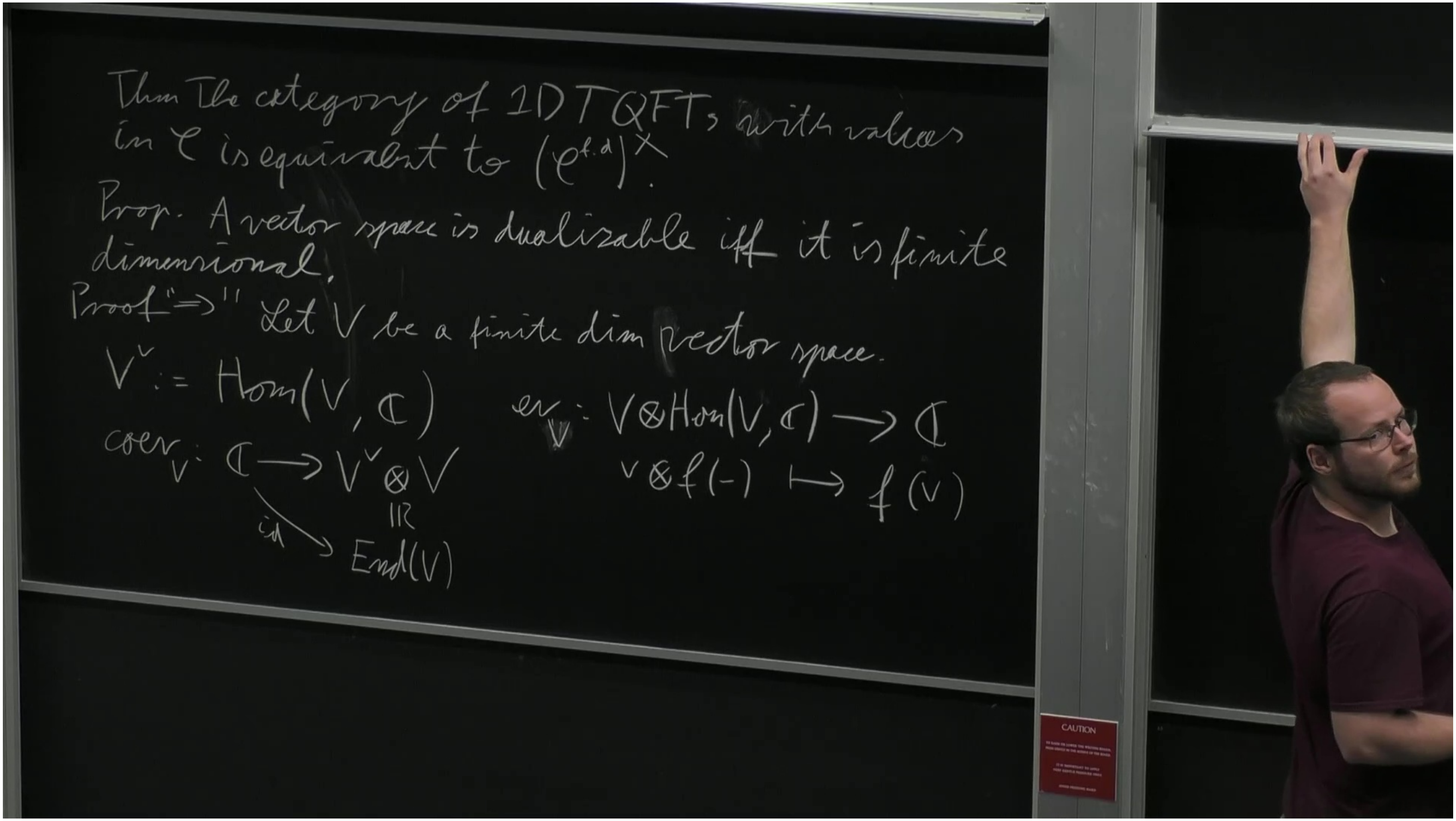
$$\text{inj}: \ (f_+, f_-) \overset{?}{=} (f_+, f'_-) \ \checkmark$$




inj: (f+, f-) =? (f+, f'-) ✓
surj: Let f: X → Y & set
(f+, f-) is a morphism in Dual(C)
evY
coevY
evX
coevX



monoidal cat. $(\mathcal{C}^{f.d.})^X$ is the sub category
$X \in \mathcal{C}$ admitting a (right) dual $X^\vee$
between those as morphisms.
$) \longrightarrow (\mathcal{C}^{f.d})^X$
is an equivalence of categories
$\longmapsto X$
$\longrightarrow f_+ : X \longrightarrow X'$
Exercise: A functor G:
iff:
- G is essentially surjecti
- G is fully faitfull: The m
is a isomorphism.
Proof: F is essentially sur
F: Hom((X, X^\vee, ev, coev),
$\mathrm{Hom}_{ex}(X,$
CAUTION



Thm The category of 1D TQFTs with values in $\mathcal{C}$ is equivalent to $(\mathcal{C}^{f.d})^{\times}$.
Prop. A vector space is dualizable iff it is finite dimensional.
Proof "⇒" Let V be a finite dim vector space.
$V^\vee := \mathrm{Hom}(V, \mathbb{C})$
$ev_V : V \otimes \mathrm{Hom}(V, \mathbb{C}) \longrightarrow \mathbb{C}$
$v \otimes f(-) \longmapsto f(v)$
$coev_V : \mathbb{C} \longrightarrow V^\vee \otimes V \cong \mathrm{End}(V)$
id
CAUTION

coev: $\mathbb{C} \longrightarrow V^\vee \otimes V$

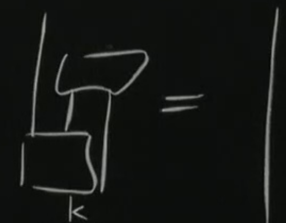
$$1 \longmapsto \sum_i e_i^* \otimes e_i = \text{"} \sum |e_i\rangle\langle e_i| \text{"}$$

$$v \longmapsto v \otimes e_i^* \otimes e_i \longmapsto \sum_i e_i^*(v)\, e_i = v$$

$\Leftarrow$ Let $V^\vee$ be a dual to V.

coev$_V$: $\mathbb{C} \longrightarrow V^\vee \otimes V$

$$1 \longmapsto \sum_{i=1}^{k} w_i \otimes v_i$$



$$v = \sum_{i=1}^{k} ev(v, w_i)\, v_i$$

$\Rightarrow \{v_i\}_{i=1,..k}$ is generating set for V. $\Rightarrow$ V is finite dim $\square$

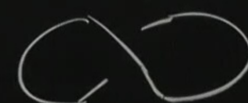
Cor. The state spaces of any n-dim TQFT is f. d.

Proof: $\mathrm{Red}_{\Sigma} Z : \mathrm{Bord}_1 \to \mathrm{Vect}$

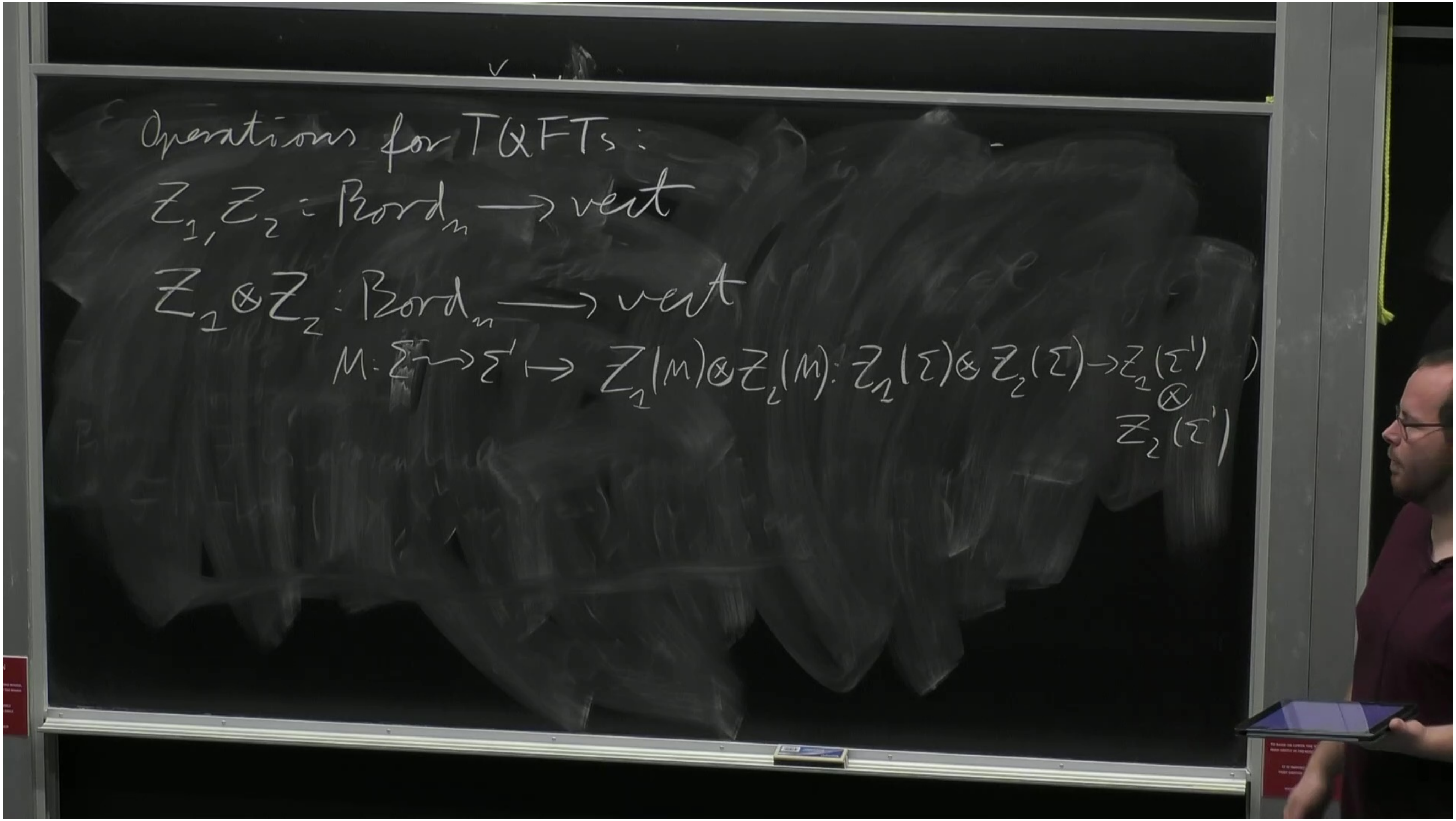
$pt^+ \mapsto Z(\Sigma)$ $\Rightarrow Z(\Sigma)$ is f. d. $\square$

Cor.

$$Z(\Sigma \times S^1) = \dim\left(Z(\Sigma)\right)$$

Proof In 1D $Z(S^1) =$ 

$$1 \mapsto \sum e_i^* \otimes e_i \mapsto \sum e_i \otimes e_i^* \mapsto \sum e_i^*(e_i) = \dim Z(pt)$$


Operations for TQFTs:
$Z_1, Z_2 : \mathrm{Bord}_n \longrightarrow \mathrm{vect}$
$Z_1 \otimes Z_2 : \mathrm{Bord}_n \longrightarrow \mathrm{vect}$
$M : \Sigma \to \Sigma' \mapsto Z_1(M) \otimes Z_2(M) : Z_1(\Sigma) \otimes Z_2(\Sigma) \to Z_1(\Sigma') \otimes Z_2(\Sigma')$

$Z_1, Z_2 : \mathrm{Bord}_n$

$$Z_1 \otimes Z_2 : \mathrm{Bord}_n \longrightarrow \mathrm{vect}$$

$$M : \Sigma \to \Sigma' \mapsto Z_1(M) \otimes Z_2(M) : Z_1(\Sigma) \otimes Z_2(\Sigma) \to Z_1(\Sigma') \otimes Z_2(\Sigma')$$

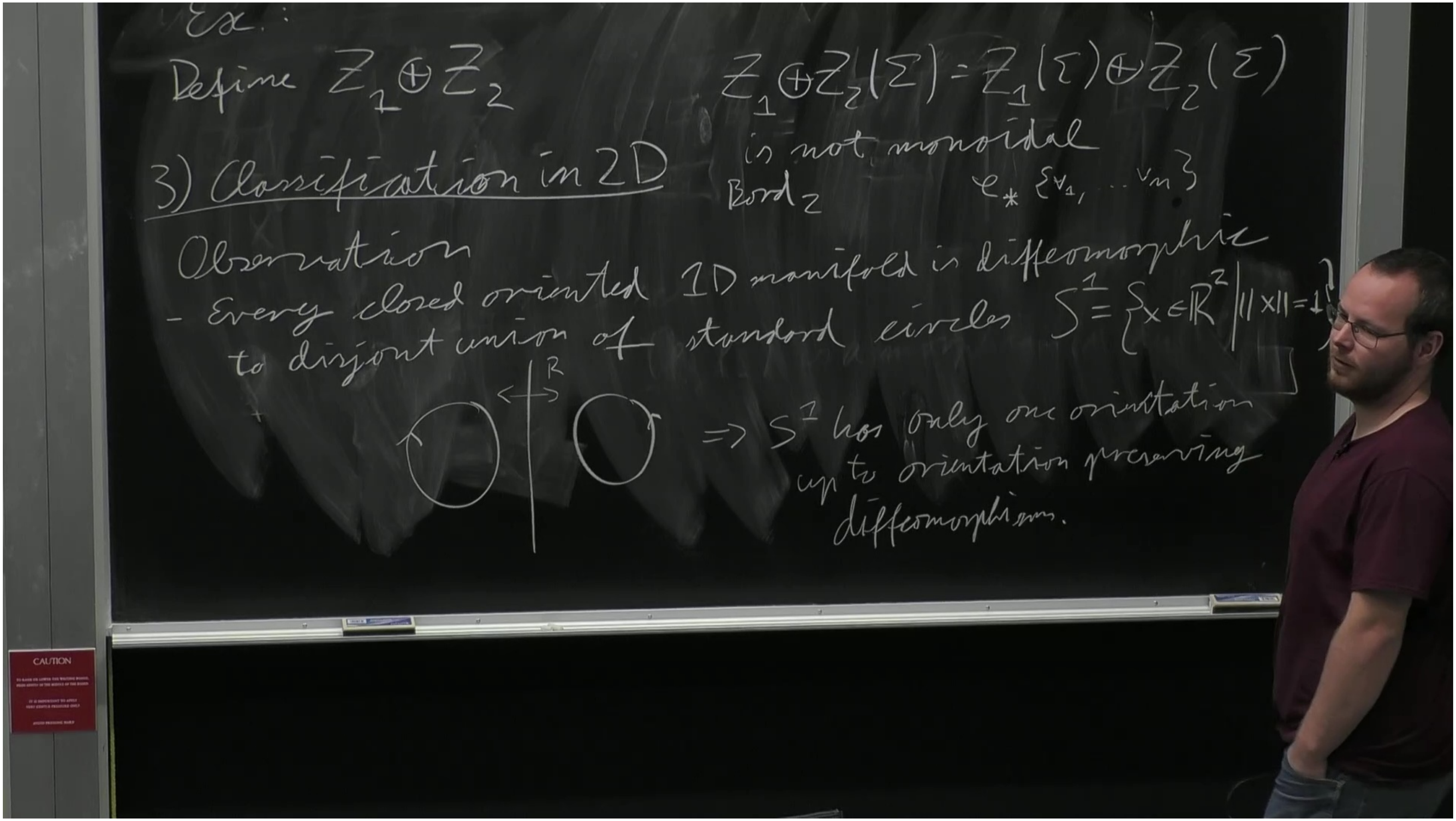
$$1 : \mathrm{Bord}_n \longrightarrow \mathrm{vect}$$

$$M : \Sigma \to \Sigma' \mapsto \mathrm{id} : \mathbb{C} \to \mathbb{C}$$

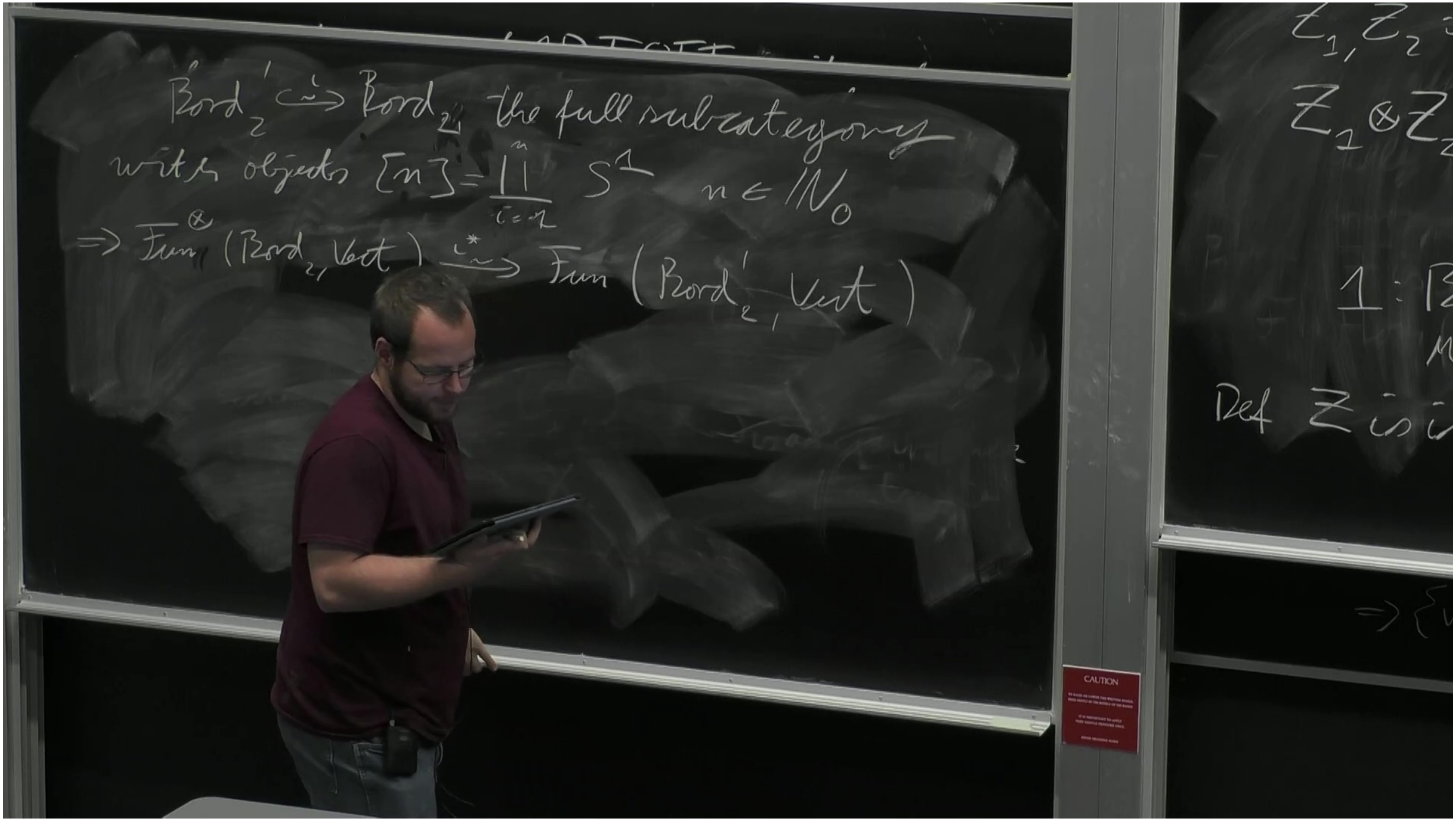
Def Z is invertible if $\exists\, Z^{-1}$, s.t. $Z \otimes Z^{-1} \cong 1$

$$Z^{-1} \otimes Z \cong 1$$

$\Rightarrow \{v_i\}_{i=1,\dots,k}$ is generating set for V. $\Rightarrow$ V is finite dim $\square$



Ex.:
Define $Z_1 \oplus Z_2$
$Z_1 \oplus Z_2(\Sigma) = Z_1(\Sigma) \oplus Z_2(\Sigma)$
is not monoidal
3) Classification in 2D
Bord_2
$\mathcal{C}_* \{v_1, \dots v_n\}$
Observation
- Every closed oriented 1D manifold is diffeomorphic to disjoint union of standard circles $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$
$\mathbb{R}$
$\Rightarrow S^1$ has only one orientation up to orientation preserving diffeomorphism.


Bord₂' ↪ Bord₂ the full subcategory
with objects [n] = ∐ᵢ₌₁ⁿ S¹ n ∈ ℕ₀
⇒ Fun^⊗(Bord₂, Vect) ≅ Fun(Bord₂', Vect)
Def

Operations for TQFTs:

$Z_1, Z_2 : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$

$Z_1 \otimes Z_2 : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$

$M : \Sigma \to \Sigma' \mapsto Z_1(M) \otimes Z_2(M) : Z_1(\Sigma) \otimes Z_2(\Sigma) \to (Z_1(\Sigma') \otimes Z_2(\Sigma'))$

$1 : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$

$M : \Sigma \to \Sigma' \mapsto \mathrm{id} : \mathbb{C} \to \mathbb{C}$

$\Sigma \xrightarrow{e} \Sigma'$

$\Sigma \xrightarrow{e} \Sigma' \times [0,1] \hookleftarrow \Sigma$

Def Z is invertible if $\exists\, Z^{-1}$, s.t. $Z \otimes Z^{-1} \cong 1$

$Z^{-1} \otimes Z \cong 1$

$\Rightarrow \{v_i\}_{i=1,\dots,k}$ is generating set for V. $\Rightarrow V$ is finite dim $\square$

with objects $[n] = \coprod_{i=1}^{n} S^1 \quad n \in \mathbb{N}_0$

$$\Rightarrow \mathrm{Fun}^{\otimes}(\mathrm{Bord}_{2,1}, \mathrm{Vect}) \xrightarrow[\sim]{\iota^*} \mathrm{Fun}(\mathrm{Bord}'_{2,1}, \mathrm{Vect})$$

Fact 3.2 Two connected compact oriented 2D manifolds Σ & Σ' are diffeomorphic if and only if they have the same number of boundary components k and the same genus g

$$g = \frac{2+k-\chi(\Sigma)}{2}$$

$\overset{\mathrm{id}}{\longrightarrow} \mathrm{End}(V)$

Operation

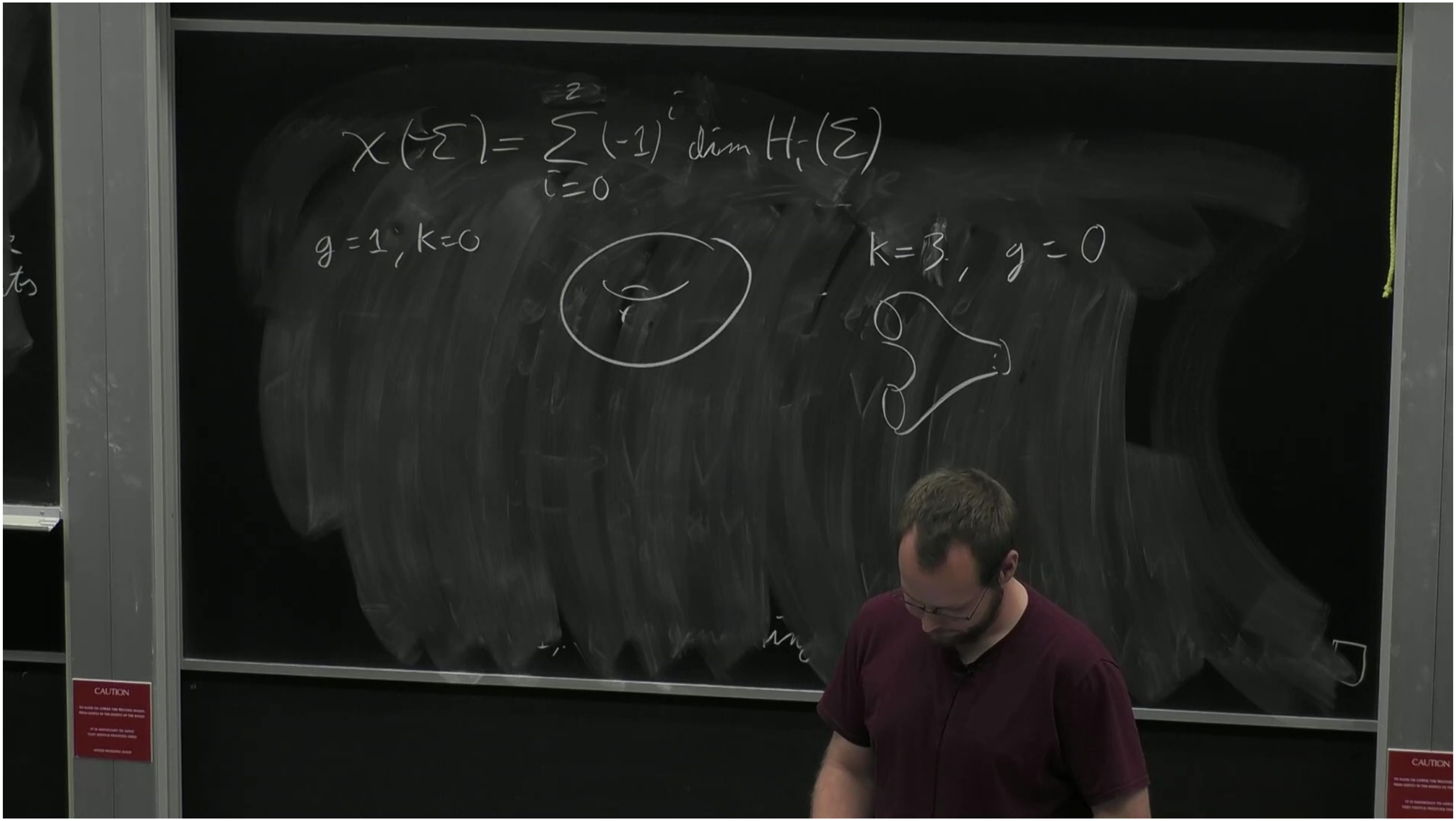
$Z_1, Z_2 :$

$Z_1 \otimes Z_2$

$1 : B$

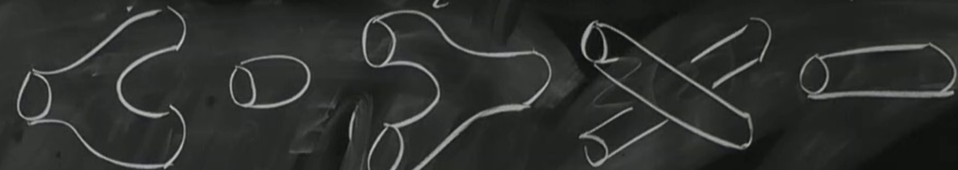
Def Z is in

$\Rightarrow \{v_i$


$\chi(\Sigma) = \sum_{i=0}^{2} (-1)^i \dim H_i(\Sigma)$
$g = 1, k = 0$
$k = 3, g = 0$
CAUTION

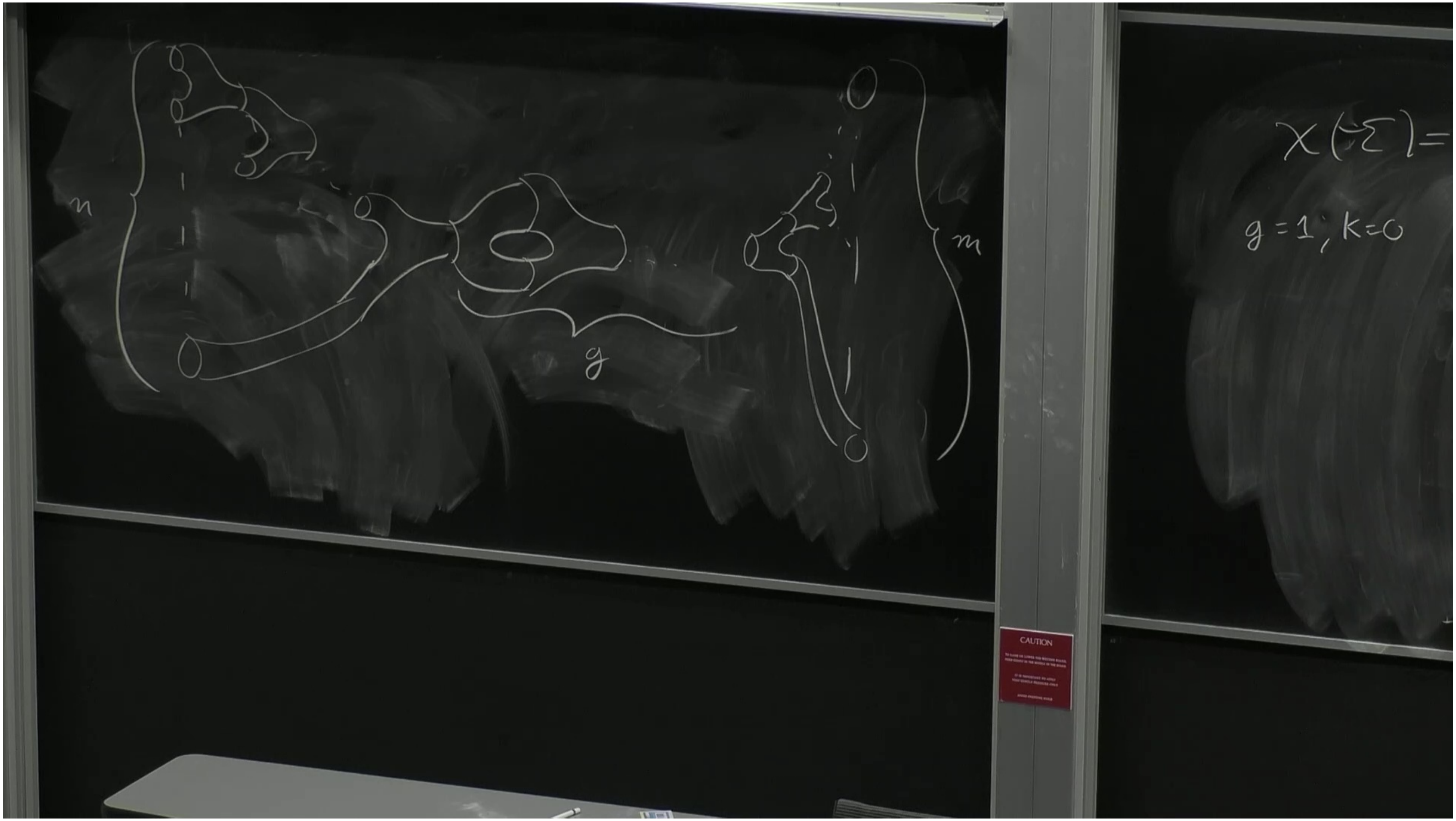
Prop Every morphism in Bord_2' can be built from

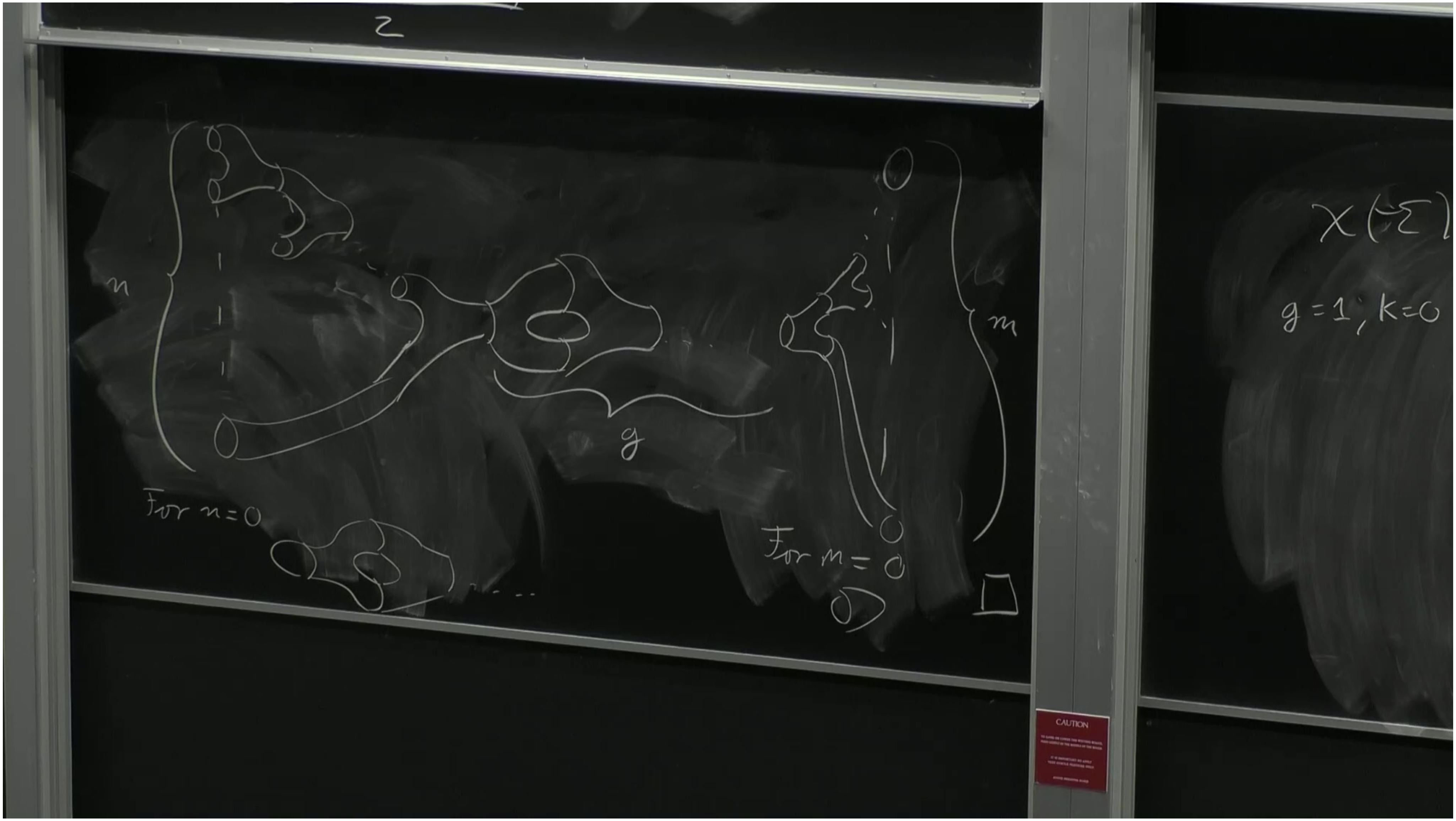
$\emptyset \longrightarrow [1]$

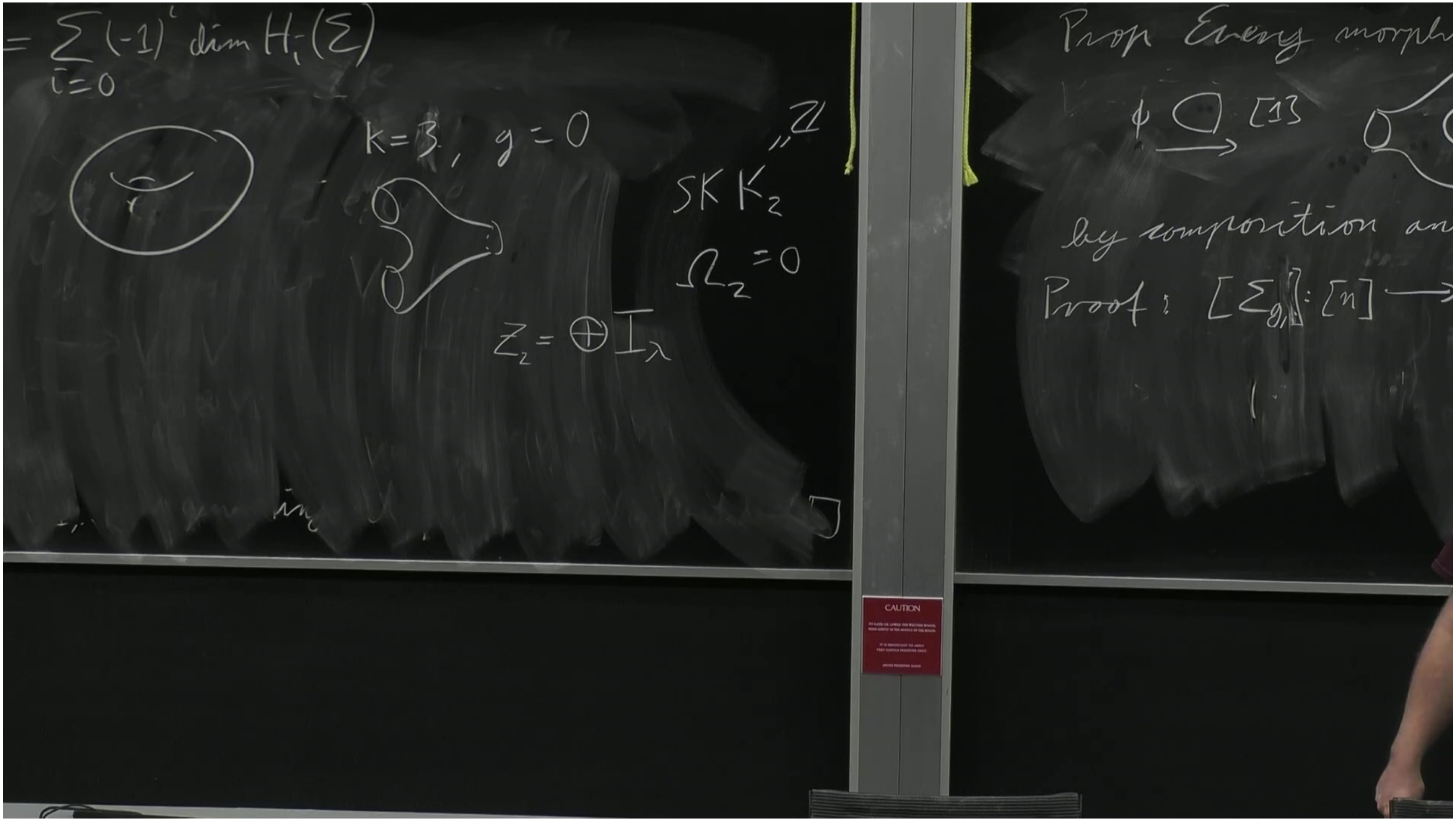


by composition and $\otimes$-products.

Proof: $[\Sigma_{g,1}] : [n] \longrightarrow [m]$


n
g
m
χ(Σ) =
g = 1; k = 0


n
m
g
For n=0
For m=0
g=1, k=0
CAUTION


= Σ(-1)^i dim H_i(Σ)
i=0
k=3, g=0
SKK_2
Ω_2 = 0
Z_2 = ⊕ I_λ
Prop Every morph
[1]
by composition an
Proof: [Σ_g] : [n]
CAUTION