Title: Statistical Physics Lecture - 112923

Speakers: Emilie Huffman

Collection: Statistical Physics 2023/24

Date: November 29, 2023 - 10:45 AM

URL: https://pirsa.org/23110033

Percolation

$p < p_c$	finite clusters
$p > p_c$	infinite cluster and complementary finite clusters
$p = p_c$:	scale invariance, clusters of all sizes (power law distribution)

Ising

$T > T_c$	small aligned sea o
$T < T_c$	small i down (up) of u
$T = T_c$	scale in cluste

Percolation

$p < p_c$ finite clusters

$p > p_c$ infinite cluster
and complementary
finite clusters

$p = p_c$ scale invariance,
clusters of all sizes

(power law distribution)

Ising

$T > T_c$

Small is
aligned s
sea of

$T < T_c$

Small isl
down (up)
of up

$T = T_c$

Scale invar
clusters

zes

bution)

Ising

$T > T_c$ Small islands of aligned spins in a sea of disorder

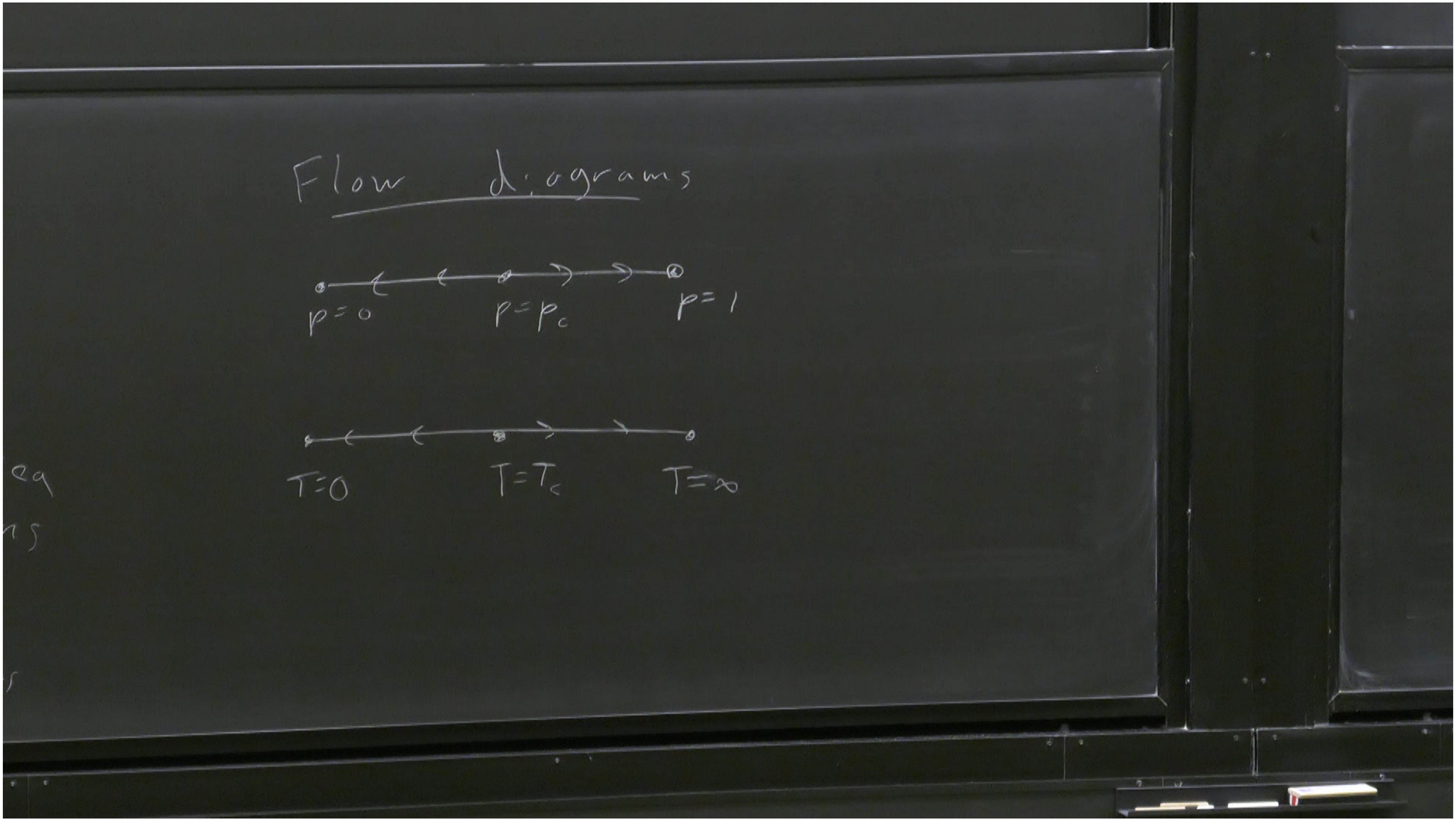
$T < T_c$ Small islands of down (up) spins in a sea of up (down) spins

$T = T_c$ Scale invariance; clusters of all sizes

Flow

$p = 0$

$T = 0$


Flow diagrams
p=0
p=p_c
p=1
T=0
T=T_c
T=∞

(power law distribution)

From Ising model to a field theory (in momentum space)

Gaussian Integrals

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\, dr\, d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2} r\, dr$$

$u = r^2 \qquad du = 2r\, dr$

$$I^2 = 2\pi \frac{1}{2} \int_0^{\infty} e^{-u} du$$

$$= \pi \left[-e^{-u} \right]_0^{\infty} = \pi$$

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} dx = \sqrt{\frac{2\pi}{a}}$$

(power law distribution)

From Ising model to a field theory (in momentum space)

Gaussian Integrals

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2}\, r\, dr$$

$u = r^2 \qquad du = 2r\,dr$

$$I^2 = 2\pi \, \frac{1}{2} \int_0^{\infty} e^{-u}\, du$$

$$= \pi \left[-e^{-u}\right]_0^{\infty} = \pi$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}\, dx = \sqrt{\frac{2\pi}{a}}$$

$$e^{-\frac{1}{2}w^T A w} = \sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}} d^n y \, e^{-\frac{1}{2}y^T A y + y^T A w}$$

vector

$n \times n$ real, positive definite, symmetric matrix $(\lambda_i > 0)$

Start with

$$\int_{\mathbb{R}} d^n y \, e^{-\frac{1}{2}y^T A y + y^T A W}$$

orthogonal

$O A O^T = a$ diagonal matrix

$$\sum_{ij} y_i A_{ij} y_j = \sum_{i,k,j} \overbrace{y_i}^{y'} O_{ik} a$$

$$= \int_{\mathbb{R}} d^n y'$$

Jacobian is 1

$$e^{-\frac{1}{2}w^T A w} = \sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2}y^T A y + y^T A w}$$

vector

$n \times n$

real, positive definite, symmetric matrix $(\lambda_i > 0)$

Start with

$$\int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2}y^T A y + y^T A w}$$

orthogonal

$$O A O^T = a$$

diagonal matrix

$$\sum_{ij} y_i A_{ij} y_j = \sum_{i,k,j} \overbrace{y_i O_{ik}}^{y'} a_k$$

$$= \int_{\mathbb{R}} d^n y' \prod_i e^{-\frac{1}{2}y_i'^2 a_i + \sum_j y_i' a_i O^T_{ij} w_j}$$

Jacobian is 1

Complete the square:

$$= \int_{\mathbb{R}} d^n y' \prod_i e^{-\frac{1}{2} a_i \left(y'_i - \sum_j O^T_{ij} w_j\right)^2 + \frac{1}{2} a_i \sum_{j,k} O^T_{ij} w_j \overset{O_{ki}}{O^T_{ik}} w_k}$$

$$u_i = y'_i - (Ow)_i \qquad du_i = dy'_i$$

$$= \prod_i \left[\left(\sqrt{\frac{2\pi}{a_i}} \right) e^{\frac{1}{2} \sum_{j,k} w_k \underbrace{O_{ki} a_i O^T_{ij}}_{A} w_j} \right]$$

$$= \sqrt{\frac{(2\pi)^n}{\det A}} \, e^{\frac{1}{2} w^T A w}$$

$e^{\frac{1}{2} w^T A w}$

$$e^{\frac{1}{2} w^T A w} = \sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2} y^T A y + y^T A w}$$

Back to the Ising model:

$$E(\sigma) = -\frac{1}{2} \sum_{\langle i,j \rangle} \sigma^T J \sigma + \beta \tilde{H}^T \sigma$$

vector matrix vector

$$\boxed{e^{\frac{1}{2} w^T A w} = \sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2} y^T A y + y^T A w}}$$

Back to the Ising model:

$$E(\sigma) = -\frac{1}{2} \underset{\text{vector}}{\sigma^T} \underset{\text{matrix}}{J} \underset{\text{vector}}{\sigma} - \beta \tilde{H}^T \sigma$$

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta}{2} \sigma^T J \sigma + \beta \tilde{H}^T \sigma}$$

J is a matrix $\quad J_{ij}$ connect... neigh...

$$\sigma = \begin{pmatrix} +1 \\ +1 \\ -1 \\ \vdots \\ \vdots \end{pmatrix} \qquad J_{ij} = \ldots$$

$\tilde{H}$ is also a vector

$$\tilde{H} = \begin{pmatrix} H \\ H \\ \vdots \\ H \end{pmatrix}$$

$$\frac{\ \ +A}{(2\pi)^n} \int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2} y^T A y + y^T A w}$$

model:

$$\sigma^T J \sigma + \beta \tilde{H}^T \sigma$$

vector   matrix   vector

$$+ \beta \tilde{H}^T \sigma$$

J is a matrix J_{ij} connects nearest neighbors, symmetric

$$\sigma = \begin{pmatrix} +1 \\ +1 \\ -1 \\ \vdots \\ \vdots \end{pmatrix}$$

$$J_{ij} = J_{ji}$$

$\tilde{H}$ is also a vector

$$\tilde{H} = \begin{pmatrix} H \\ H \\ \vdots \\ H \end{pmatrix}$$

$a_i > 0$

$$\boxed{\sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}} d^n y \; e^{-\frac{1}{2} y^T A y + y^T A w}}$$

Ising model:

$$\underset{}{\frac{1}{2}} \underbrace{\sigma^T}_{\text{vector}} \underbrace{J}_{\text{matrix}} \underbrace{\sigma}_{\text{vector}} + \tilde{H}^T \sigma$$

$$^T \underbrace{J}_{\text{matrix}} \sigma + \beta \tilde{H}^T \sigma$$

s not positive definite.

J is a matrix J_{ij} connects nearest neighbors, symmetric

$$\sigma = \begin{pmatrix} +1 \\ +1 \\ -1 \\ \vdots \end{pmatrix} \qquad J_{ij} = J_{ji}$$

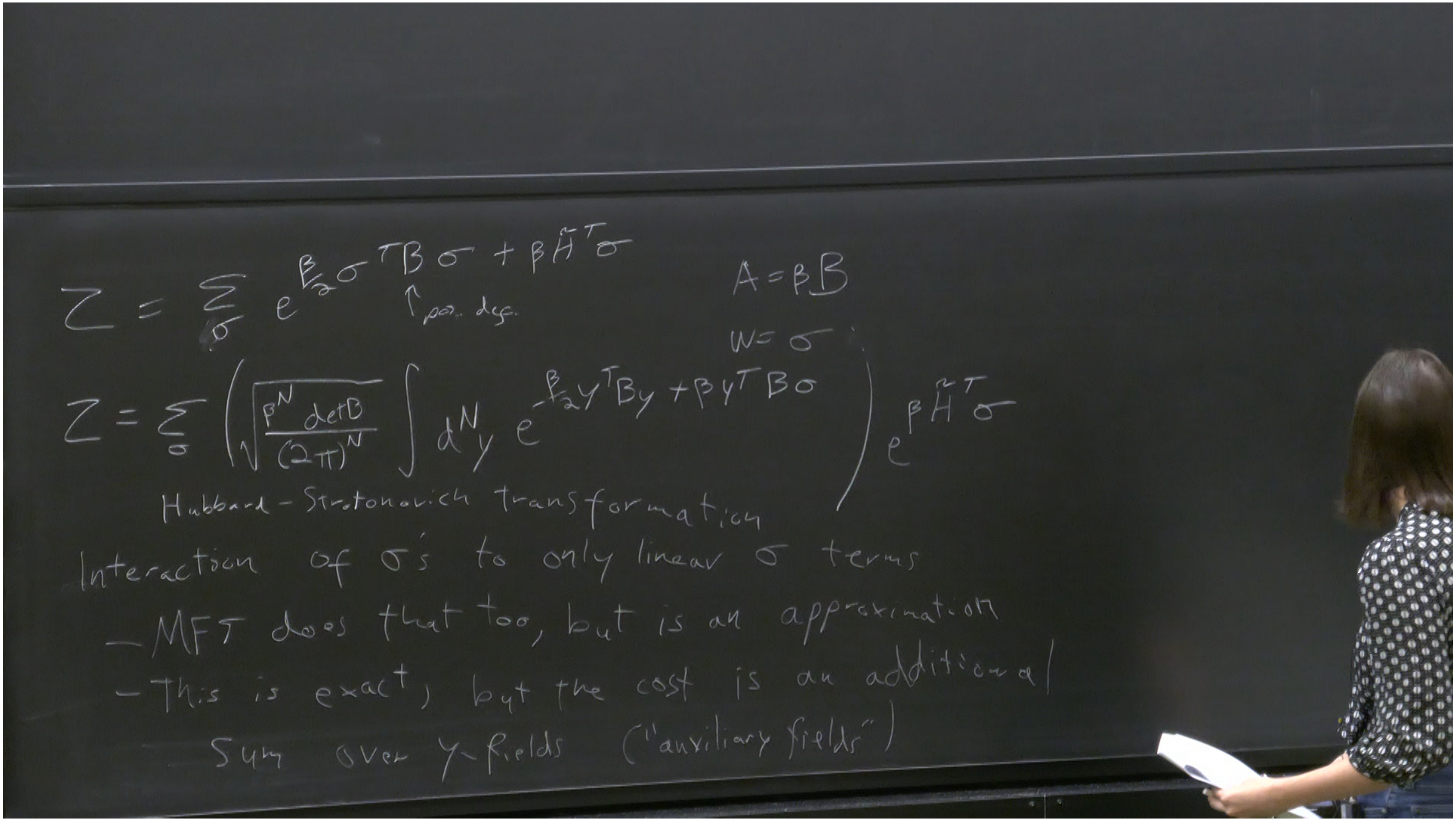
$\tilde{H}$ is also a vector

$$\tilde{H} = \begin{pmatrix} H \\ H \\ \vdots \\ H \end{pmatrix}$$

Introduce $B = J + \lambda \mathbb{1}$

This doesn't change any observables.

$$\langle O \rangle_{\lambda \neq 0} = \frac{\sum_{\{\sigma\}} O(\sigma)\, e^{-\beta\left(E(\sigma) + \frac{\lambda N}{2}\right)}}{\sum_{\{\sigma\}} e^{-\beta\left(E(\sigma) + \frac{\lambda N}{2}\right)}} = \frac{\cancel{e^{-\frac{\beta \lambda N}{2}}} \sum_{\{\sigma\}} O(\sigma)\, e^{-\beta(E(\sigma))}}{\cancel{e^{-\frac{\beta \lambda N}{2}}} \sum_{\{\sigma\}} e^{-\beta(E(\sigma))}} = \langle O \rangle_{\lambda = 0}$$


$Z = \sum_\sigma e^{\frac{\beta}{2}\sigma^T B \sigma + \beta \tilde{H}^T \sigma}$
pos. def.
$A = \beta B$
$W = \sigma$
$Z = \sum_\sigma \left(\sqrt{\frac{\beta^N \det B}{(2\pi)^N}} \int d^N y \, e^{-\frac{\beta}{2} y^T B y + \beta y^T B \sigma} \right) e^{\beta \tilde{H}^T \sigma}$
Hubbard-Stratonovich transformation
Interaction of σ's to only linear σ terms
- MFT does that too, but is an approximation
- This is exact, but the cost is an additional
sum over y-fields ("auxiliary fields")

... over y-fields ("auxiliary fields")

To simplify the ϕ-terms $\quad \phi = \frac{1}{A}\left(y + B^{-1}\tilde{H}\right) \qquad \beta y + \tilde{H} = BA\phi$

constant number — to get dimensions right

$$Z = \sqrt{\frac{\beta^N A^{2N} \det B}{(2\pi)^N}} \int d^N\phi \left(e^{-\frac{\beta}{2}\left(A^2 \phi^T B \phi + \tilde{H}^T B^{-1} \tilde{H} - 2A\tilde{H}^T \phi\right)} \sum_{\sigma} e^{\beta A \sigma^T B \phi} \right)$$

Now, let's go ahead and do the σ sum

$$\sum_{\sigma} e^{\beta A \sigma^T B \phi} = \sum_{\sigma} \prod_i e^{\beta A\left(\sigma_i \sum_j B_{ij} \phi_j\right)}$$

$$= \prod_i \sum_{\sigma_i = \pm 1} e^{\beta A\left(\sigma_i (B\phi)_i\right)}$$

$$= \prod_i 2\cosh\left(\beta A (B\phi)_i\right)$$

$$= 2^N e^{\sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right)}$$

$$Z = \sqrt{\frac{\beta^N A^{2N} 2^N \det B}{\pi^N}}\; e^{-\frac{\beta}{2}\tilde{H}^T B^{-1}\tilde{H}} \int d^N\phi \; e^{-S(\phi)}$$

$$S(\phi) = \frac{\beta}{2} A^2 \phi^T B \phi - \beta A \tilde{H}^T \phi - \sum_i \ln(\cosh(\beta A (B\phi)_i))$$

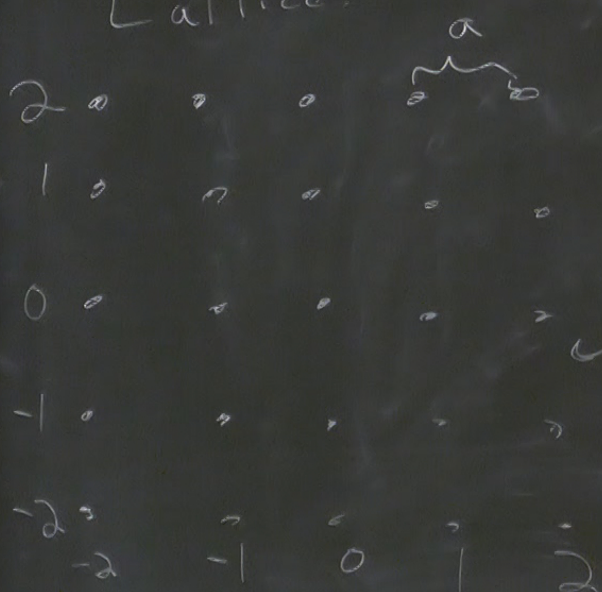
ϕ_i is a scalar field $\in \mathbb{R}$

$\langle \phi_i \rangle$ is related to $\langle \sigma_i \rangle$ (tutorial)

$\tilde{H} \to 0 \qquad e^{-\beta/2\, \tilde{H}^T B^{-1} \tilde{H}} \longrightarrow 1$

Momentum Space

Lattice:



$(2n+1)^D$ sites

D is dimension

n is the number of sites > 0

Ex: n=2

D=2

$\Lambda = \frac{\pi}{a}$

periodic lattice

$X = \sum_{\mu=1}^{D} a$

lattic[e]
s[p]

periodicity:

$X = X + ($

...tum Space

$(2n+1)^D$ sites

D is dimension

n is the number of sites > 0

Ex: $n=2$

$D=2$

$$\Lambda = \frac{\pi}{a}$$

periodic lattice

$$X = \sum_{\mu=1}^{D} a \, l_\mu \, \vec{e}_\mu$$

integer

lattice spacing

basis

periodicity:

$$X = X + (2n+1)\, a \vec{e}_\mu$$

$i k_\mu$

Lattice Fourier transform:

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot x_i}$$

k lives on a k-space lattice

$$k = \sum_{\mu=1}^{D} k_\mu \vec{e}_\mu \qquad k_\mu = \frac{2m\pi}{(2n+1)a}$$

What is this k?

$$\sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot x_i} = \sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot(x_i + (2n+1)a\vec{e}_\mu)}$$

We know $e^{i2\pi m} = 1$ (m integer)

$$ik_\mu (2n+1)a = i2\pi m \longrightarrow \frac{k_\mu (2n+1)a}{2\pi} = m \Rightarrow k_\mu = \frac{2m\pi}{(2n+1)a}$$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

Inverse?

$$f(x_j) = \sum_k \tilde{f}(k) e^{ik\cdot x_j}$$

$f(x_j) =$

$$\sum_k \tilde{f}(k) e^{ik\cdot x_j} = \sum_{k,i} f(x_i)\, e^{-ik\cdot(x_i - x_j)}$$

$$= \sum_{i=1}^{N} f(x_i) \underbrace{\sum_k e^{-ik\cdot(x_i - x_j)}}_{N\delta_{x_i,x_j}}$$

Identity:

$$\sum_{m=-n}^{n} e^{-ik_m(x_i - x_j)}$$

$$= \prod_\mu \sum_{k_\mu} e^{ik_\mu(l_{i\mu} - l_{j\mu})a}$$

$$= \prod_\mu \left[\delta_{l_{i\mu} - l_{j\mu},0}(2n+1)\right] = N\delta_{x_i,x_j}$$

$$ik_\mu (2n+1)a = i2\pi M$$

$$f(x_j) = \sum_{i=1}^{N} f(x_i) \sum_k e^{-ik\cdot(x_i - x_j)}$$

$$= \sum_{i=1}^{N} f(x_i)\, N\, \delta_{x_i, x_j}$$

$$= N f(x_j)$$

$$e^{-ik\cdot(x_i - x_j)}$$

$$e^{-ik\cdot(x_i - x_j)}$$

$$\mu)a \qquad N\delta_{x_i,x_j}$$

$$= \prod_\mu \left[\delta_{l_{i\mu} - l_{j\mu}, 0}\,(2n+1)\right] = N\delta_{x_i,x_j}$$

$$\tilde{f}(-k) = \sum_{i=1}^{N} f(x_i)\, e^{ik\cdot x_i} = \tilde{f}^{*}(k)$$

$$\tilde{f}(k,q) = \sum_{i,j}^{N} f(x_i, x_j)\, e^{-ik\cdot x_i}\, e^{-iq\cdot x_j}$$

$$\Leftrightarrow \quad f(x_i, x_j) = \frac{1}{N^2} \sum_{k,q} \tilde{f}(k,q)\, e^{ik\cdot x_i}\, e^{iq\cdot x_j}$$