Title: Statistical Physics Lecture - 112423

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Landau Theory

- Different approach to mean field theory
- Ignore the details of the interactions and produce symmetry-based qualitive model

Thinking about the free energy:

$$Z = \sum_{\sigma} e^{-\beta E(\sigma, H)} \qquad \sigma = \{\sigma_1, \sigma_2 \ldots, \sigma_N\}$$

$$= \sum_{m} \sum_{\sigma \text{ for } m} e^{-\beta E(\sigma, H)}$$

$$\approx \sum_{m} e^{-\beta N f(m,H)}$$

Landau free energy

as $N \to \infty$ become exactly equal to the term with the smallest f

Thinking about the free energy:

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$$= \sum_{m} \sum_{\sigma \text{ for } m} e^{-\beta E(\sigma, H)}$$

$$\approx \sum_{m} e^{-\beta N f(m, H)}$$

Landau free energy per particle

as $N \to \infty$

become exactly equal to the term with the smallest f

- Landau theory defines a model by its free energy.

For the Ising model, we assume $f(m, H)$, and expect:

- $f(m, 0) = f(-m, 0)$

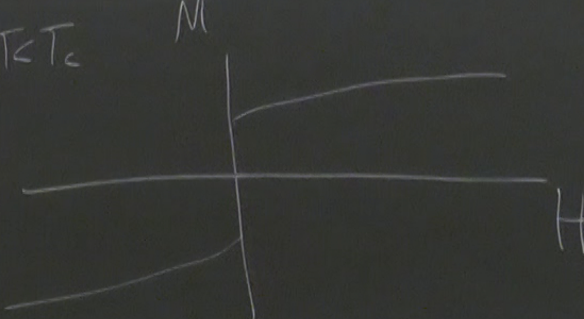
  because $E = \sum_{\langle ij \rangle} \sigma_i \sigma_j$

  $\sigma_i \to -\sigma_i$ for all i

- when H and $T - T_c$ are small, $f - f_0$ is small

- M varies smoothly with T

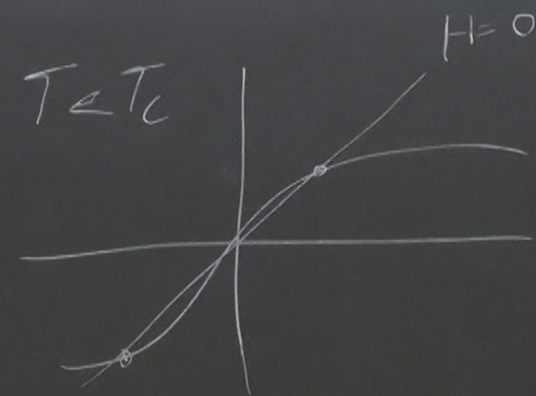
$T < T_c$



T<

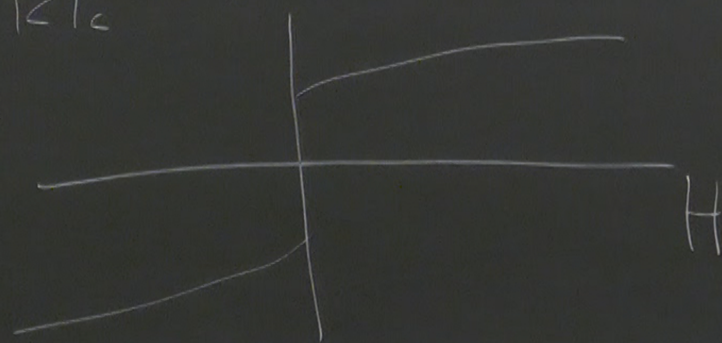
model by its free energy.

$f(m, H)$, and expect:

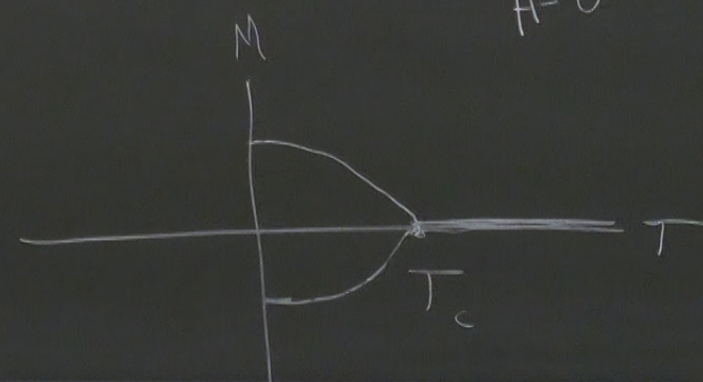


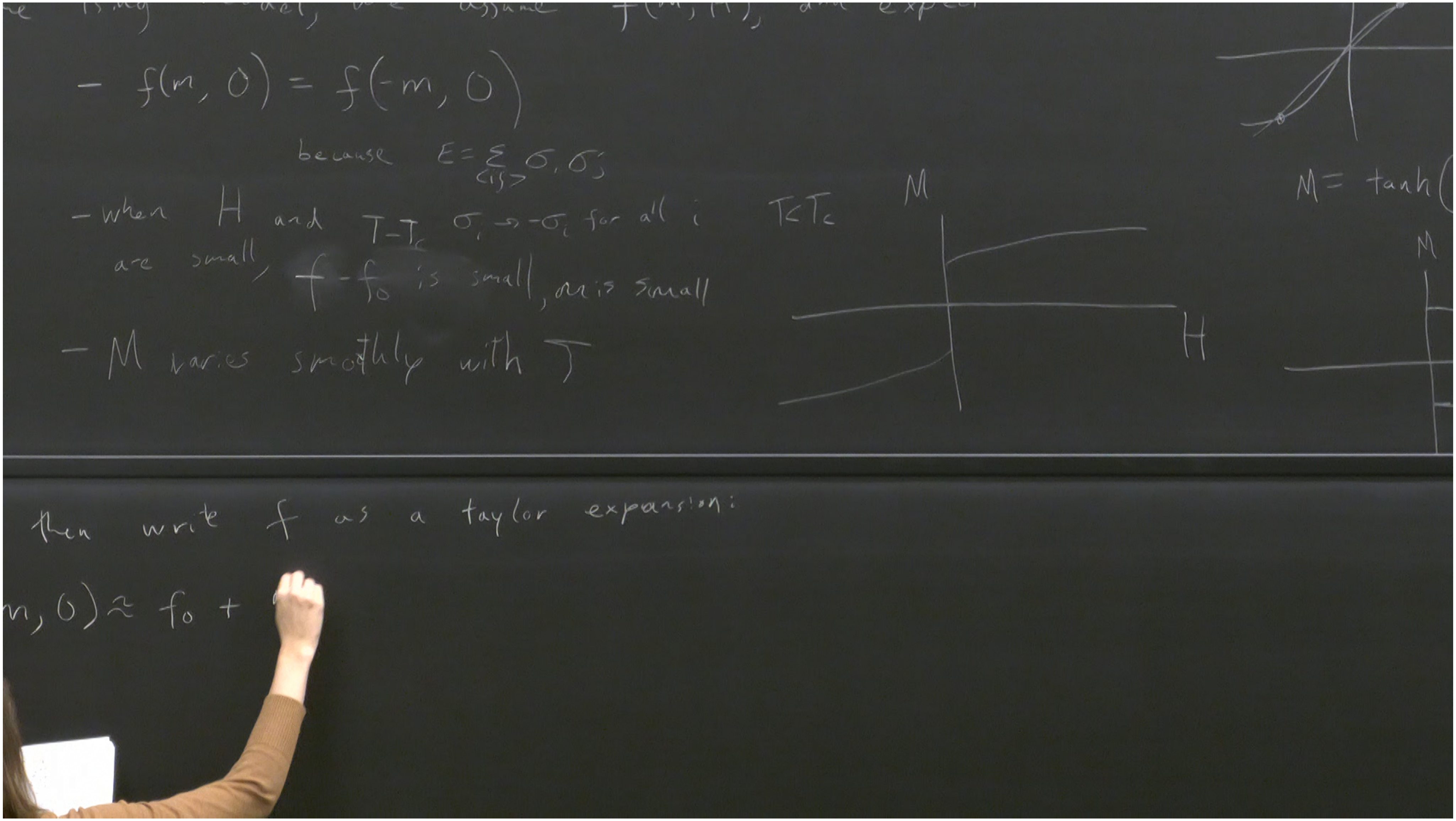
σ_j

for all i $\quad T < T_c$



$$M = \tanh(a M)$$




- f(m, 0) = f(-m, 0)
because E = Σ_{<ij>} σ_i σ_j
σ_i → -σ_i for all i
- when H and T-T_c are small, f - f_0 is small, m is small
T<T_c
M
H
M = tanh(
M
- M varies smoothly with T
then write f as a taylor expansion:
m, 0) ≈ f_0 +

- M varies smoothly with T

- We can then write f as a taylor expansion: u_0?

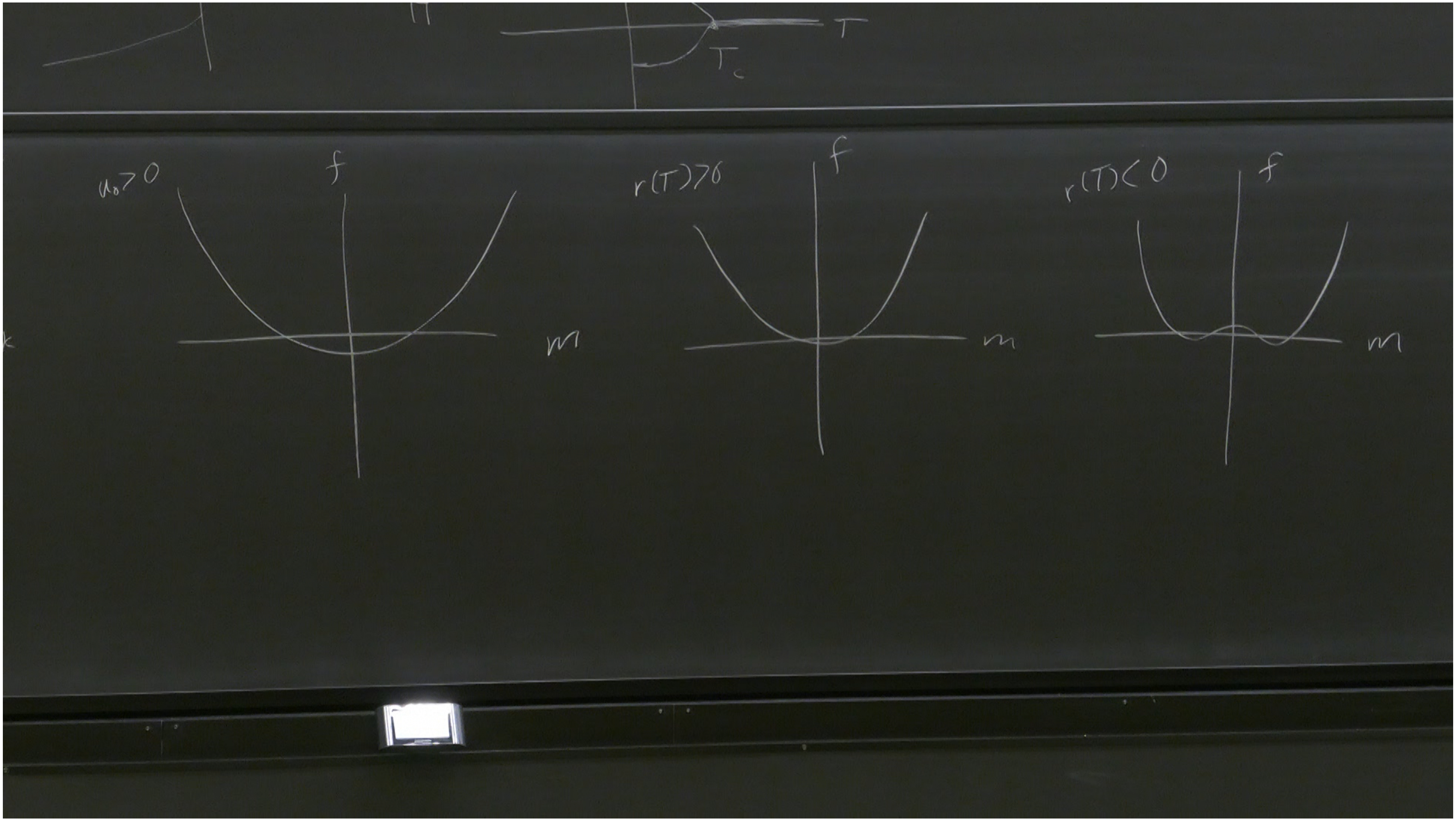
$$f(m,0) \approx \cancel{f_0}^{\,0} + \frac{r(T)}{2} m^2 + \frac{u(T)}{4!} m^4$$

Expand around $t = T_c - T$

$$\frac{r(T)}{2} = \sum_k r_k t^k \quad , \quad \frac{u(T)}{4!} = \sum_k u_k t^k$$

- We'll be minimizing f

$$\rightarrow \frac{u(T)}{4!} = u_0 > 0$$


T
T_c
$u_0 > 0$
f
m
$r(T) > 0$
f
m
$r(T) < 0$
f
m

- We can then write f as a taylor expansion:

$$f(m,0) \simeq \cancel{f_0}^{\,0} + \frac{r(T)}{2} m^2 + \frac{u(T)}{4!} m^4$$

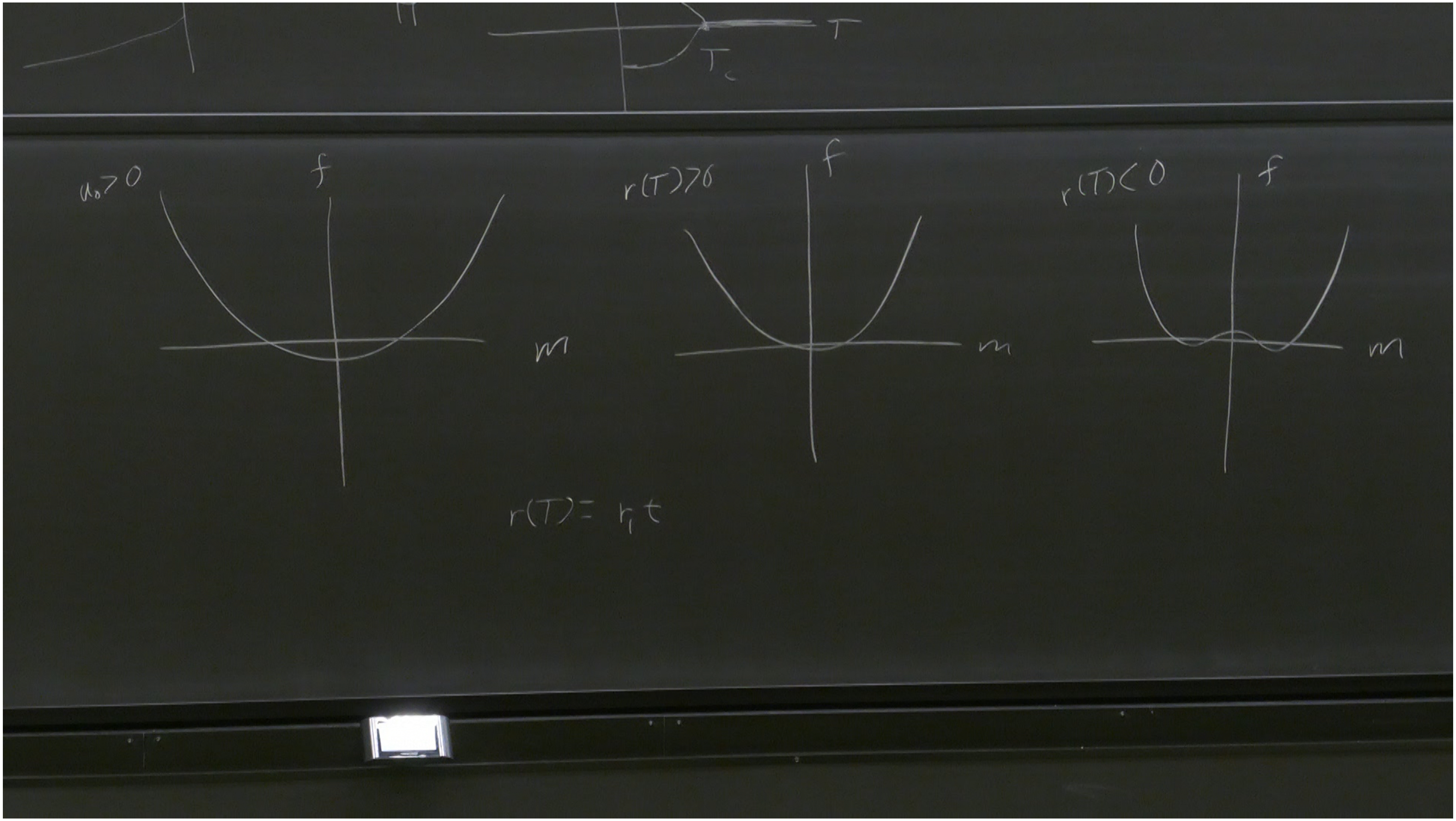
Expand around $t = T - T_c$

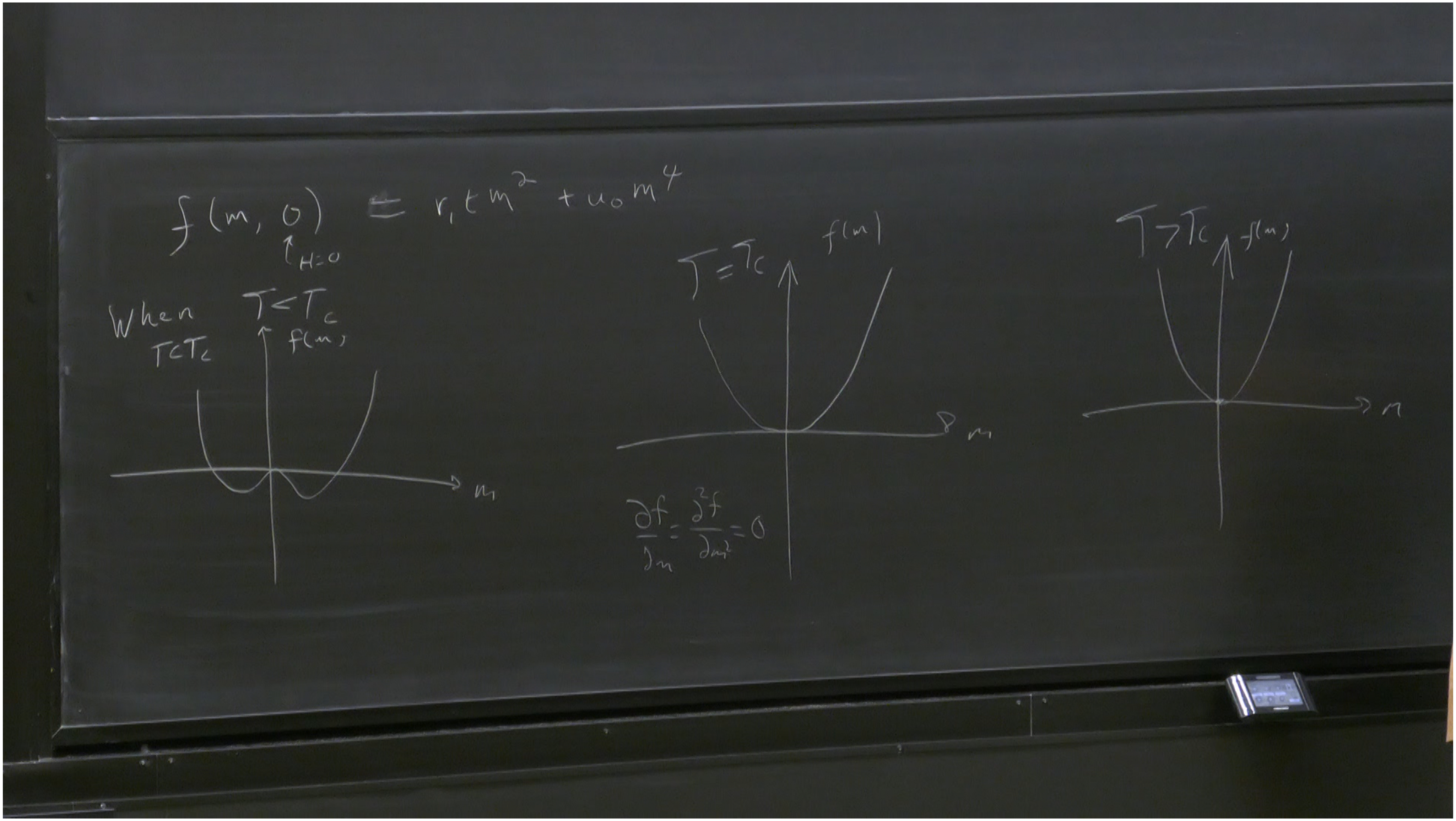
$$\frac{r(T)}{2} = \sum_k r_k t^k \quad , \quad \frac{u(T)}{4!} = \sum_k u_k t^k$$

- We'll be minimizing f.

$$\rightarrow \frac{u(T)}{4!} = u_0 > 0$$

$$\rightarrow \frac{r(T)}{2} = r_1 t$$


T
T_c
$u_0 > 0$
f
m
$r(T) > 0$
f
m
$r(T) < 0$
f
m
$r(T) = r_1 t$


$f(m, 0) = r_1 t m^2 + u_0 m^4$
$H=0$
When $T<T_c$
$T<T_c$
$f(m)$
m
$T=T_c$
$f(m)$
m
$\frac{\partial f}{\partial m} = \frac{\partial^2 f}{\partial m^2} = 0$
$T>T_c$
$f(m)$
m

Minimize the free energy:

$$\frac{\partial f}{\partial m} = 2 r_1 t m + 4 u_0 m^3 = 0$$

$$m(2 r_1 t + 4 u_0 m^2) = 0$$

$$m = 0, \quad 4 u_0 m^2 = -2 r_1 t$$

$$m = \left[\frac{-r_1 t}{2 u_0}\right]^{1/2}$$

when $t < 0$, $T - T_c < 0$

two more real roots

$$m \sim t^{1/2}$$

$$\boxed{\beta = \frac{1}{2}}$$

What about α?

$$C = \frac{\partial \langle E \rangle}{\partial T} \sim |t|^{\alpha}$$

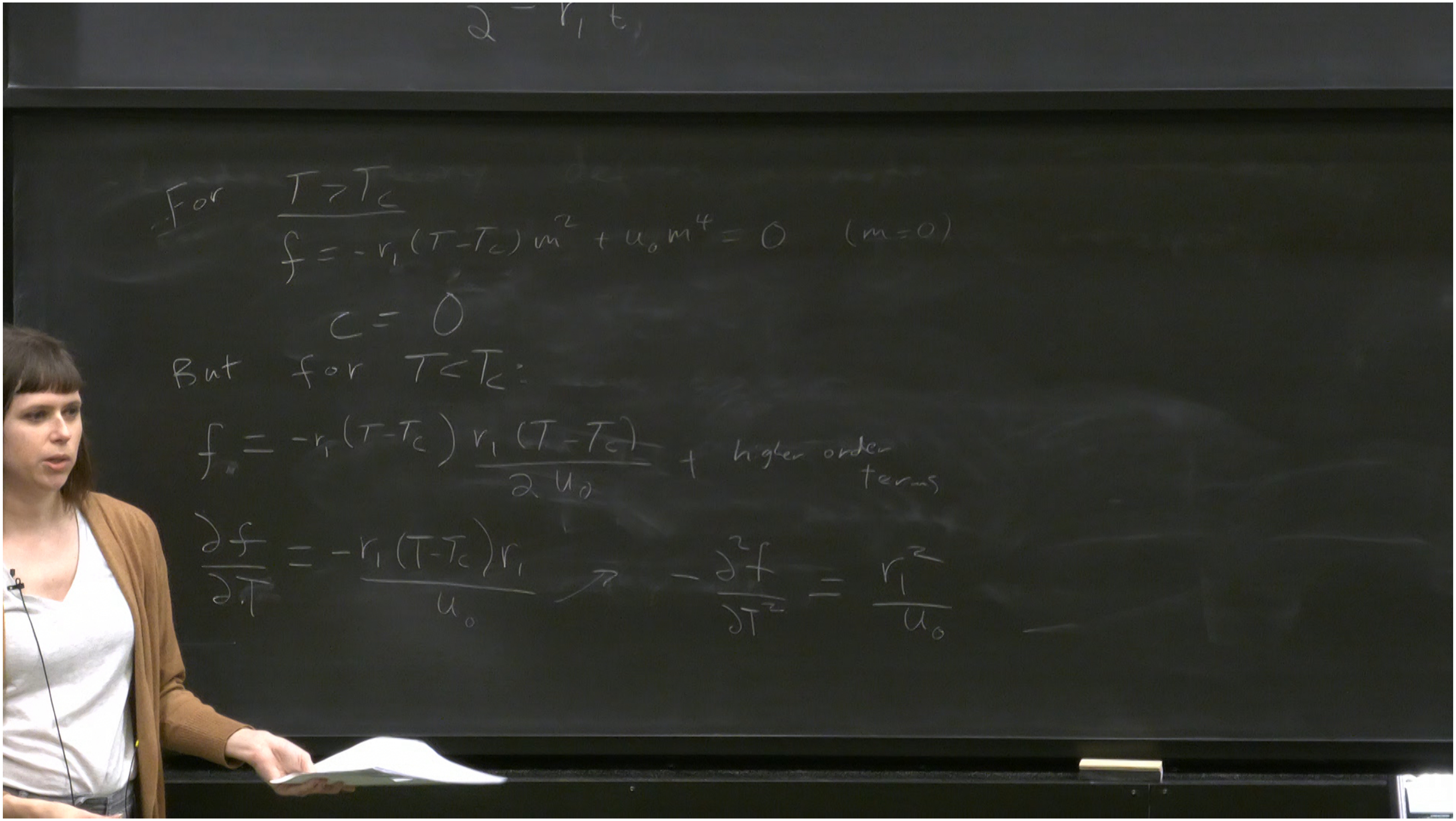
$$Z = \sum_E e^{-\beta E}, \quad \text{so} \quad -\frac{\partial \log Z}{\partial \beta} = \frac{\sum_E E e^{-\beta E}}{Z} = \langle E \rangle$$

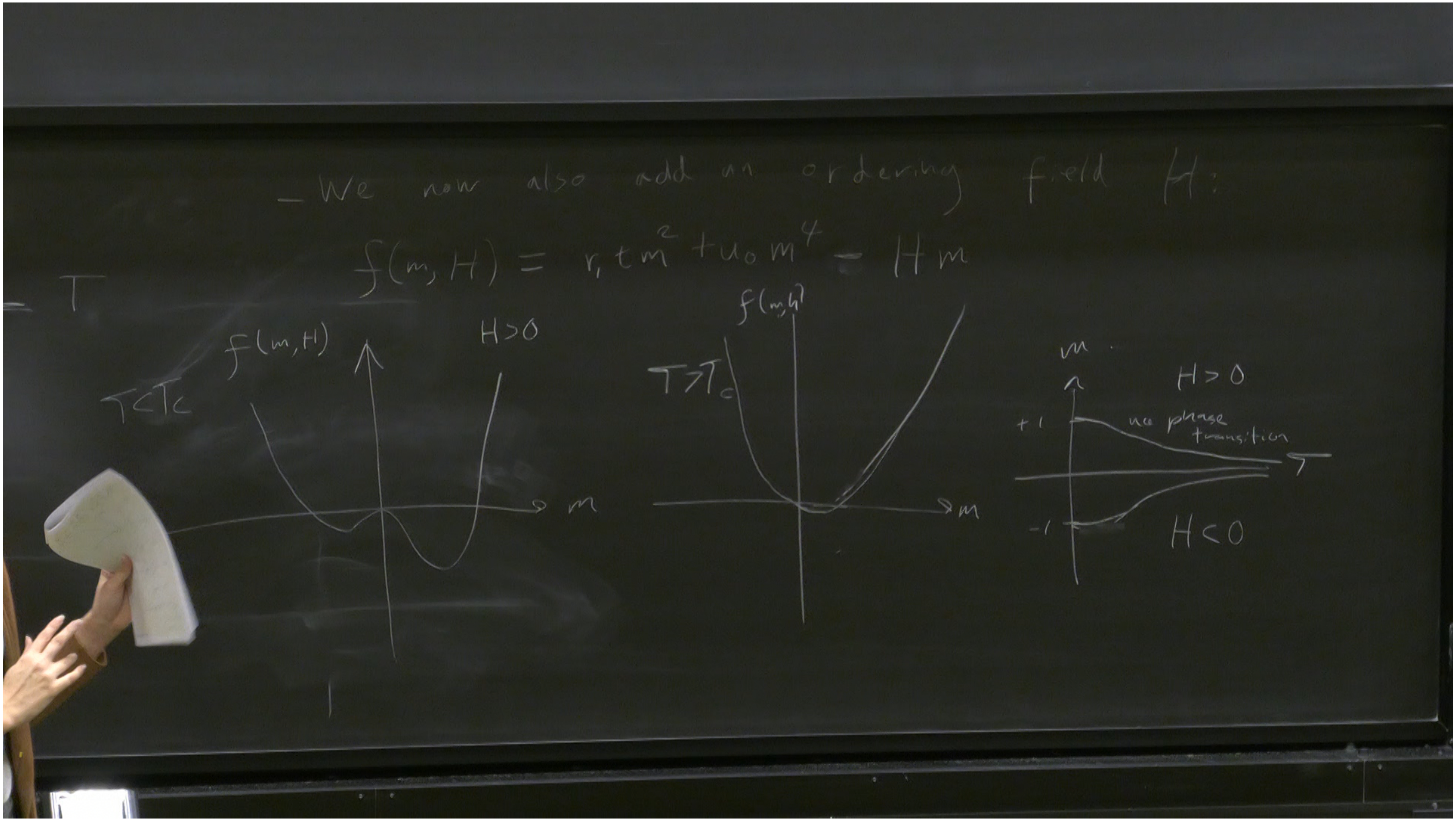
thus $$C = \frac{\partial}{\partial T}\left(-\frac{\partial \log Z}{\partial \beta}\right)$$

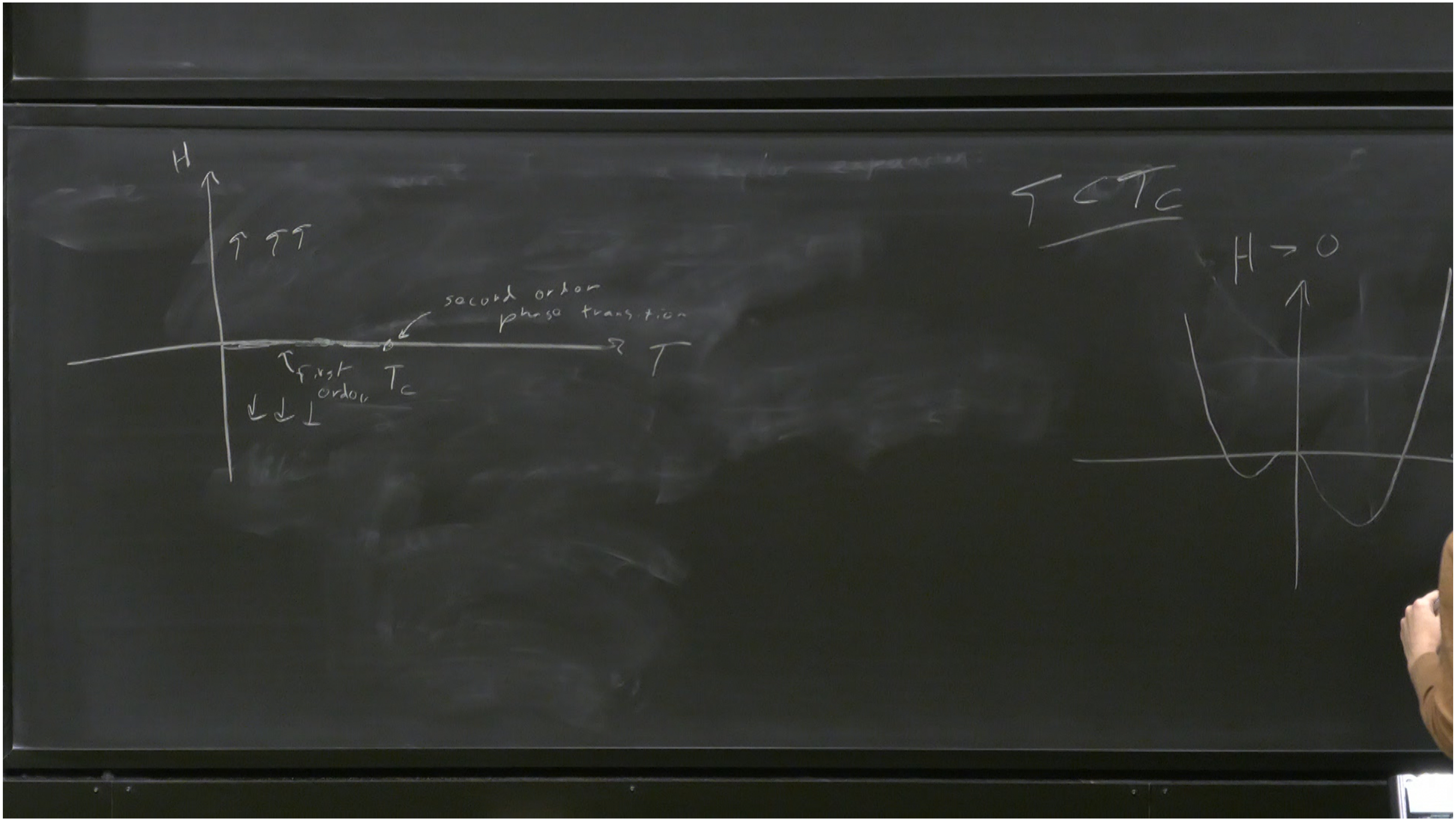
$$\beta = \frac{1}{kT} \qquad d\beta = -\frac{1}{k}\frac{1}{T^2}\, dT$$

$$F = kT \log Z$$

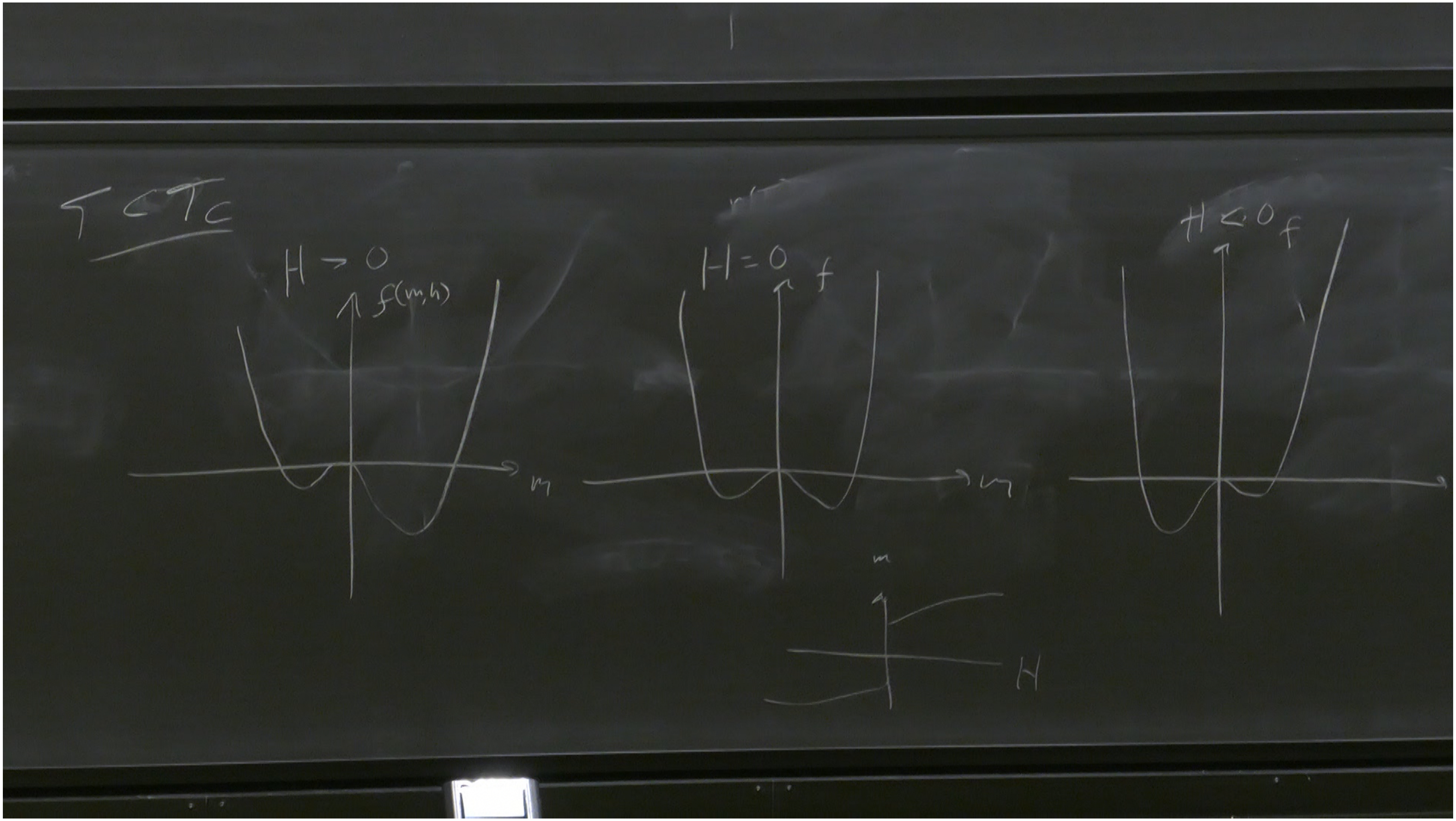
$$C = \frac{\partial^2 F}{\partial T^2}$$


For $\underline{T > T_c}$
$f = -r_1(T-T_c)m^2 + u_0 m^4 = 0 \quad (m=0)$
$c = 0$
But for $T < T_c$:
$f = -r_1(T-T_c)\dfrac{r_1(T-T_c)}{2u_0}$ + higher order terms
$\dfrac{\partial f}{\partial T} = -\dfrac{r_1(T-T_c)r_1}{u_0} \nearrow -\dfrac{\partial^2 f}{\partial T^2} = \dfrac{r_1^2}{u_0}$

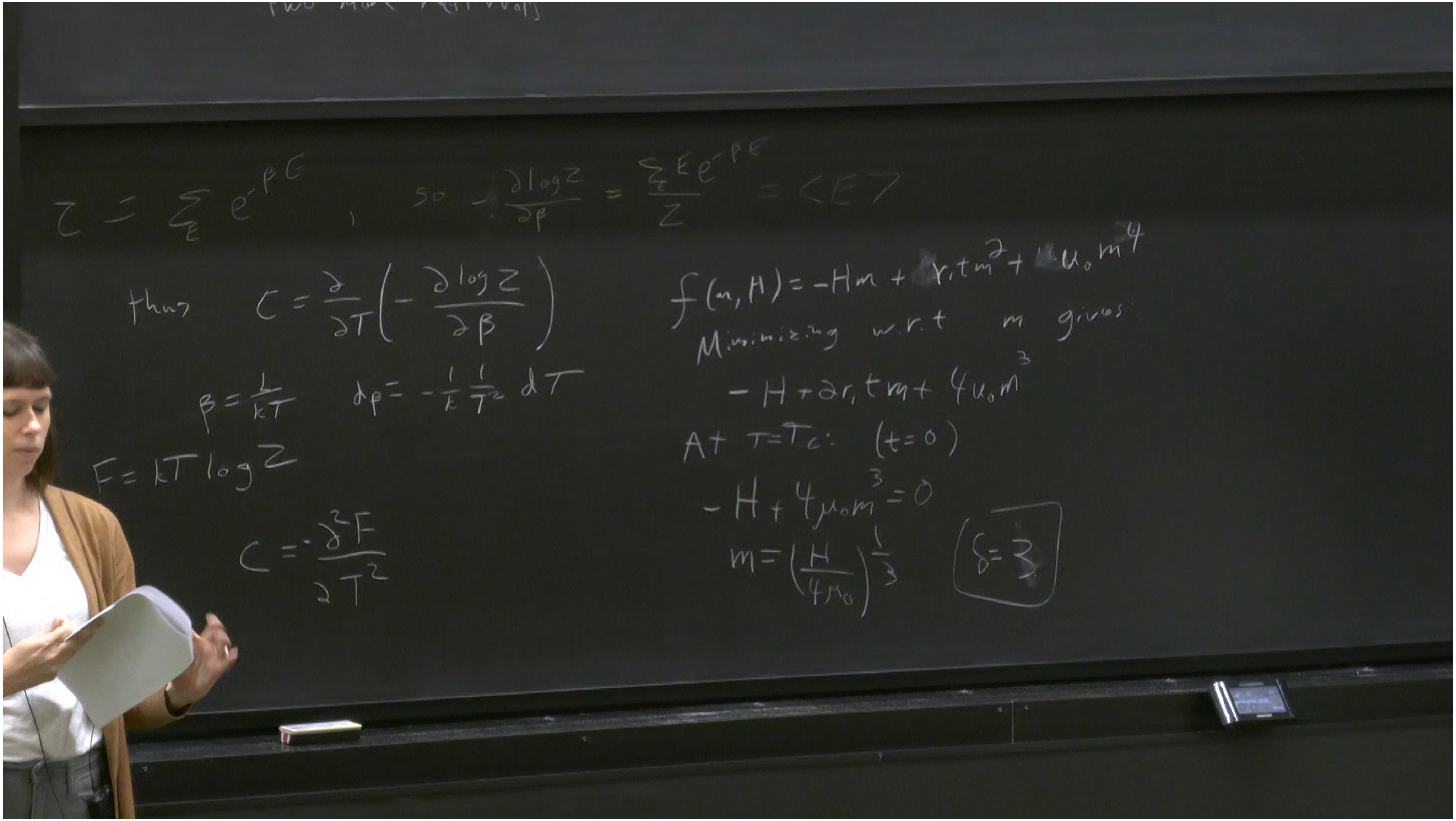

- We now also add an ordering field H:
f(m, H) = r, t m^2 + u_0 m^4 - H m
T
T<Tc
f(m,H)
H>0
m
f(m,H)
T>Tc
m
m
H>0
+1
no phase transition
T
-1
H<0



H
T
second order phase transition
first order
T_c
$T < T_c$
$H \to 0$



T < T_c
H > 0
f(m,h)
m
H = 0
f
m
m
H
H < 0
f


$Z = \sum_E e^{-\beta E}$, so $-\frac{\partial \log Z}{\partial \beta} = \frac{\sum_E E e^{-\beta E}}{Z} = \langle E \rangle$
thus $C = \frac{\partial}{\partial T}\left(-\frac{\partial \log Z}{\partial \beta}\right)$
$\beta = \frac{1}{kT}$ $d\beta = -\frac{1}{k}\frac{1}{T^2}\, dT$
$F = kT \log Z$
$C = -\frac{\partial^2 F}{\partial T^2}$
$f(m,H) = -Hm + r_0 t m^2 + u_0 m^4$
Minimizing w.r.t m gives:
$-H + 2r_0 t m + 4u_0 m^3$
At $T=T_c$: $(t=0)$
$-H + 4\mu_0 m^3 = 0$
$m = \left(\frac{H}{4\mu_0}\right)^{\frac{1}{3}}$
$\boxed{\delta = 3}$

Minimize the free energy:

$$\frac{\partial f}{\partial m} = 2r_1 t m + 4u_0 m^3 = 0$$

$$m(2r_1 t + 4u_0 m^2) = 0$$

$m = 0$, $\quad 4u_0 m^2 = -2r_1 t$

$$m = \left[\frac{-r_1 t}{2u_0}\right]^{1/2}$$

when $t<0$, $T-T_c<0$

two more real roots

$m \sim t^{1/2}$

$$\boxed{\beta = \tfrac{1}{2}}$$

What about α?

$$C = \frac{\partial \langle E \rangle}{\partial T} \sim |t|^{\alpha}$$

$$\chi \sim |t|^{-\gamma}$$

$$\chi = \left(\frac{\partial H}{\partial m}\right)^{-1}$$

$$H = 2r_1 t m + 4u_0 m^3$$

$$\frac{\partial H}{\partial m} = 2r_1 t + 12u_0 m^2 = \chi^{-1}$$

$$\chi = \frac{1}{2r_1 t + 12u_0 m^2}$$

0 at criticality

$\chi \sim t^{-1}$

$$\boxed{\gamma = 1}$$

Landau-Ginzburg Theory

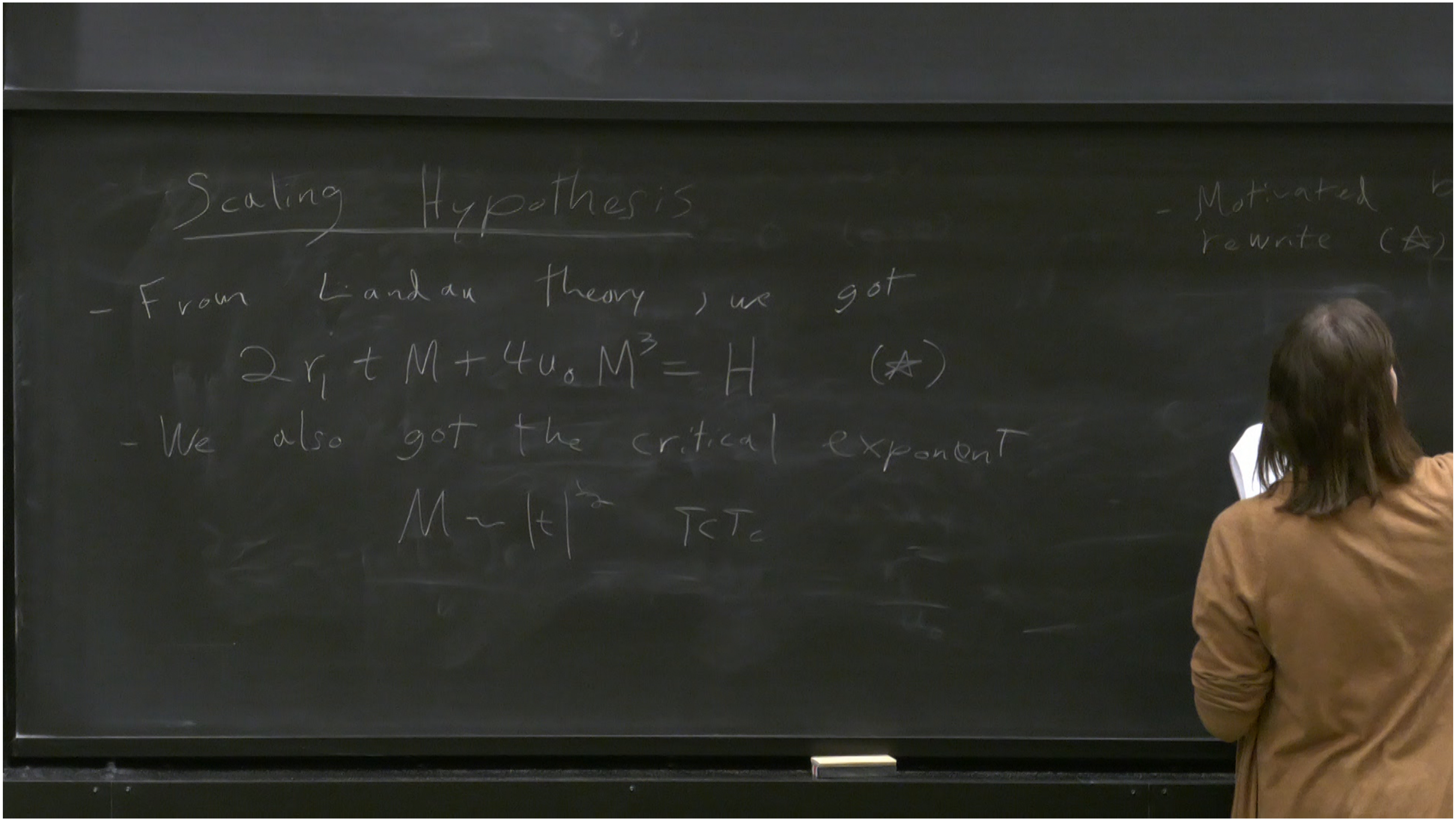
- Promotion of Landau-type assumptions
  to a field theory
- Allows us to model correlations/fluctuations

$$F(m(\vec{r})) = \int d\vec{r} \left\{ \left[a\,(m(\vec{r}))^2 + b\,(m(\vec{r}))^4 + \ldots \right] + \left[\alpha\, |\nabla m(\vec{r})|^2 + \ldots \right] \right\}$$

- can get η and ν

$$Z = \int \mathcal{D}m(\vec{r})\, e^{-\beta F[m(\vec{r})]}$$

- constant $m(\vec{r}) \rightarrow$ reduces to Landau theory



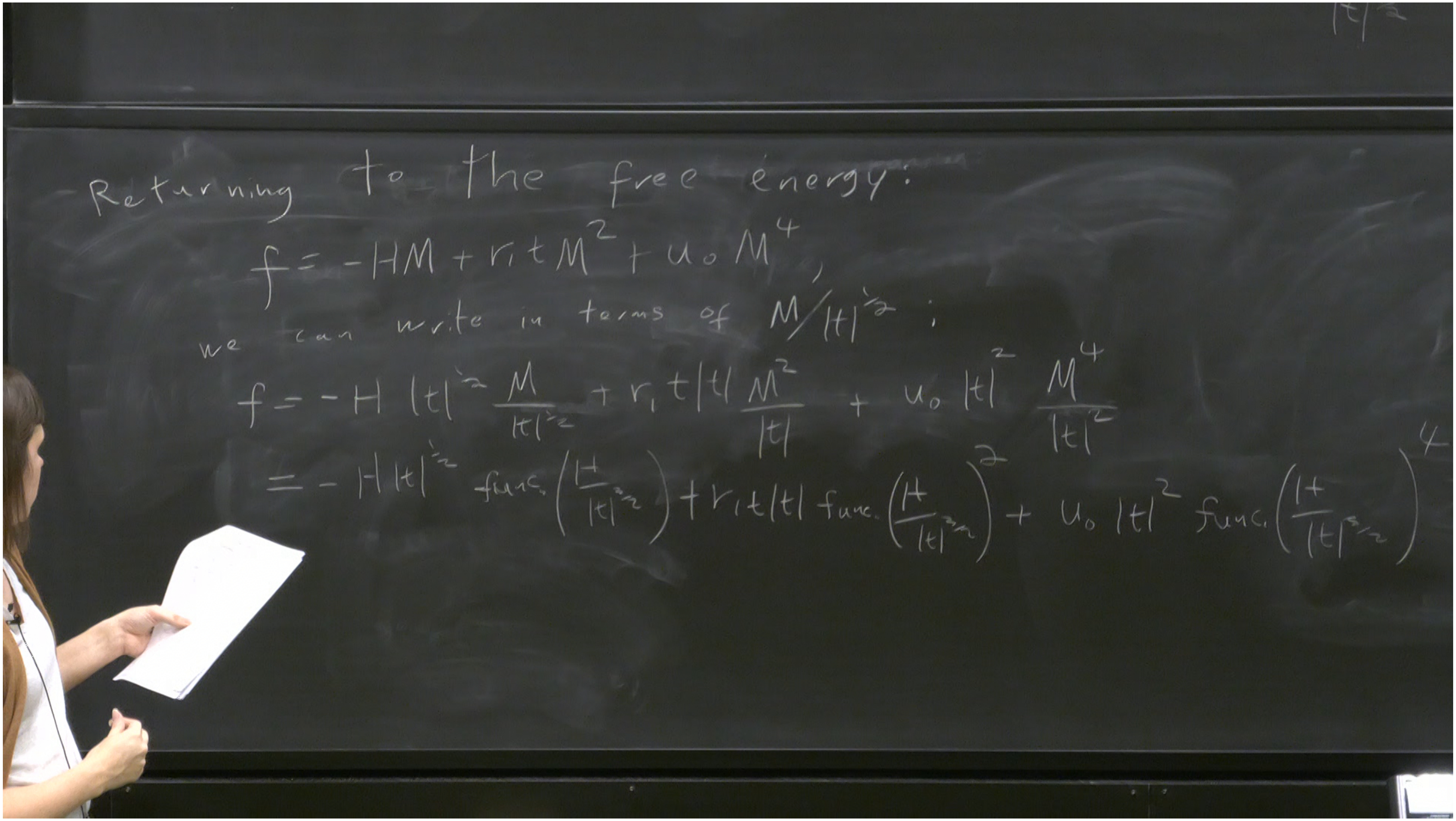
Scaling Hypothesis
- From Landau theory, we got
$2 r_1 t M + 4 u_0 M^3 = H$ (☆)
- We also got the critical exponent
$M \sim |t|^{1/2}$ $T < T_c$
- Motivated b
rewrite (☆)

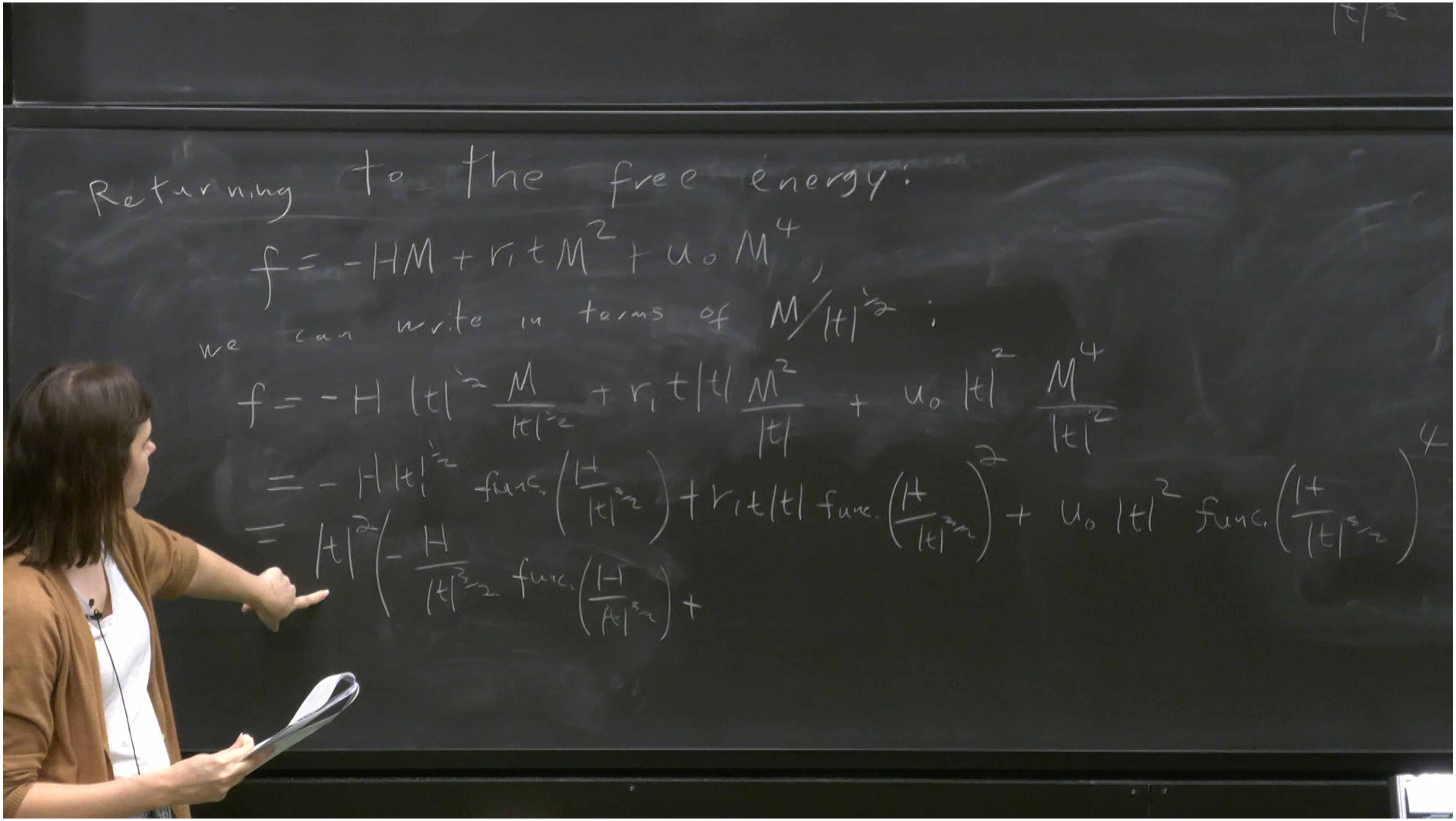
- Motivated by the fact that M scales in this way, let's rewrite $(\star)$ by dividing by $|t|^{3/2}$:

$$\frac{H}{|t|^{3/2}} = 2r_0 \frac{M}{|t|^{1/2}} \overset{\text{sign}(t)}{\frac{t}{|t|}} + 4u_0 \left(\frac{M}{|t|^{1/2}}\right)^3$$

This equation relates $H/|t|^{3/2}$ to $\frac{M}{|t|^{1/2}}$, so $H/|t|^{3/2}$ is some function of $M/|t|^{1/2}$

$$\frac{M}{|t|^{1/2}} = \text{func.}\left(\frac{H}{|t|^{3/2}}\right)$$


Returning to the free energy:
$f = -HM + r_1 t M^2 + u_0 M^4$,
we can write in terms of $M/|t|^{1/2}$:
$f = -H\,|t|^{1/2} \frac{M}{|t|^{1/2}} + r_1 t|t| \frac{M^2}{|t|} + u_0 |t|^2 \frac{M^4}{|t|^2}$
$= -H|t|^{1/2}\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right) + r_1 t|t|\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0 |t|^2\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^4$



$|t|^{1/2}$
Returning to the free energy:
$f = -HM + r_1 t M^2 + u_0 M^4$,
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$f = -H\,|t|^{1/2} \frac{M}{|t|^{1/2}} + r_1 t|t| \frac{M^2}{|t|} + u_0 |t|^2 \frac{M^4}{|t|^2}$
$= -H|t|^{1/2}\ \text{func.}\left(\frac{H}{|t|^{3/2}}\right) + r_1 t|t|\ \text{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0 |t|^2\ \text{func.}\left(\frac{H}{|t|^{3/2}}\right)^4$
$= |t|^2\Big(-\frac{H}{|t|^{3/2}}\ \text{func.}\left(\frac{H}{|t|^{3/2}}\right) +$

Returning to the free energy:

$$f = -HM + r_1 t M^2 + u_0 M^4,$$

we can write in terms of $M/|t|^{1/2}$:

$$f = -H\,|t|^{1/2}\,\frac{M}{|t|^{1/2}} + r_1 t|t|\,\frac{M^2}{|t|} + u_0\,|t|^2\,\frac{M^4}{|t|^2}$$

$$= -H|t|^{1/2}\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right) + r_1 t|t|\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0\,|t|^2\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^4$$

$$= |t|^2\left(-\frac{H}{|t|^{3/2}}\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right) + r_1 \overset{\text{sign}(t)}{\frac{t}{|t|}}\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^4\right)$$

king about the free energy:

$$= \sum_{\sigma} e^{-\beta E(\sigma, H)} \qquad \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$$

$$= \sum_{m} \sum_{\sigma \text{ for } m} e^{-\beta E(\sigma, H)}$$

Landau free energy per particle

$$\approx \sum_{m} e^{-\beta N f(m, H)}$$

as $N \to \infty$ become exactly equal to the term with the smallest f

Scaling Hypothesis

$$f = |t|^2 \, \text{func.}\left(\frac{H}{|t|^{3/2}}\right)$$

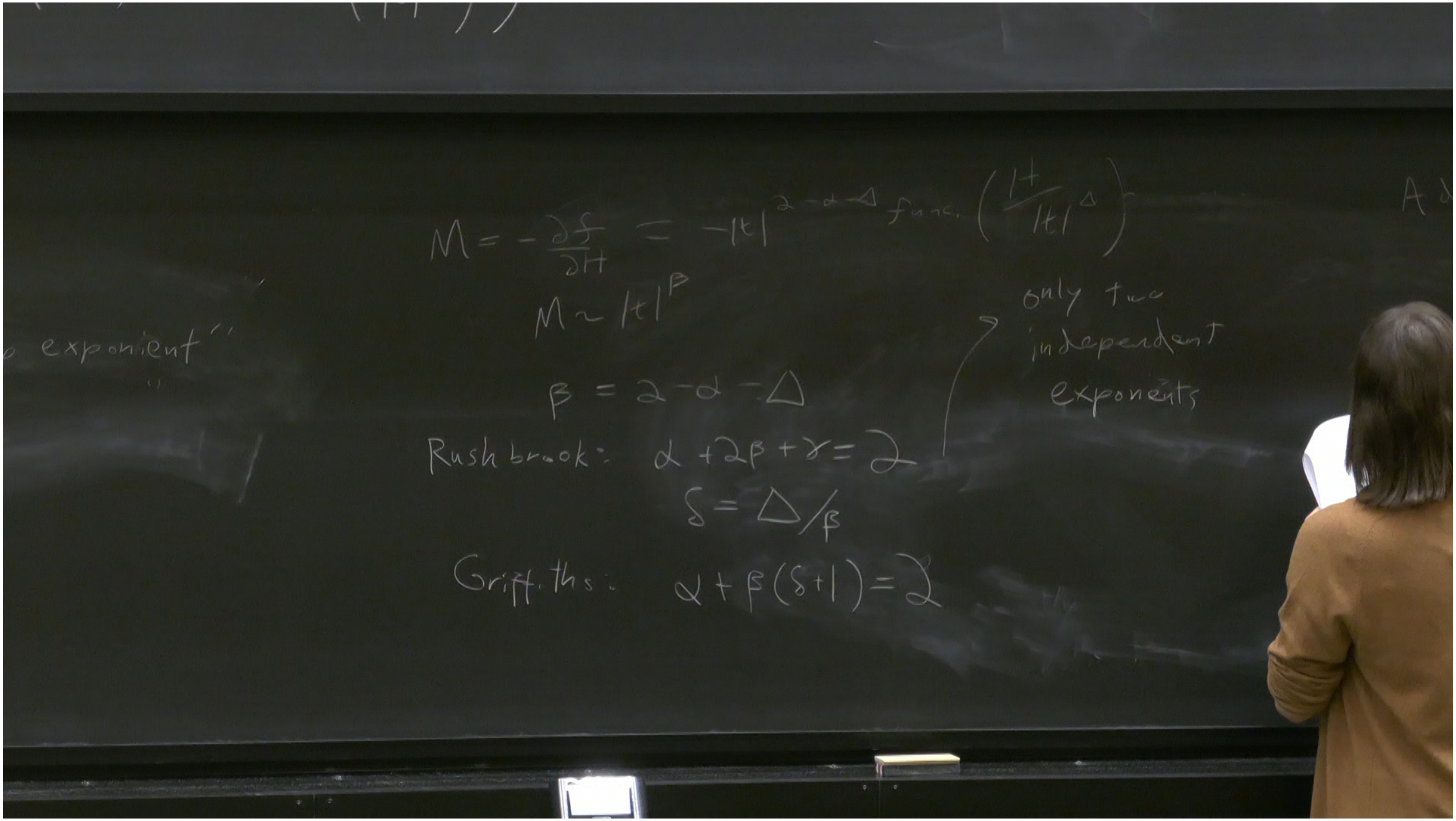
$$M = -\frac{\partial f}{\partial H}$$

we generalize it to: "gap exponent"

$$f = |t|^{2-\alpha} \, \text{func.}\left(\frac{H}{|t|^{\Delta}}\right)$$

$$C \sim |t|^{-\alpha}$$

$$C \sim (1+t)\frac{\partial^2 f}{\partial t^2}$$



exponent"
$M = -\frac{\partial f}{\partial H} = -|t|^{2-\alpha-\Delta} \text{func.}\left(\frac{H}{|t|^{\Delta}}\right)$
$M \sim |t|^{\beta}$
$\beta = 2-\alpha-\Delta$
Rushbrook: $\alpha + 2\beta + \gamma = 2$
$\delta = \Delta/\beta$
Griffiths: $\alpha + \beta(\delta+1) = 2$
only two independent exponents

$f / H = -|t|^{2-\alpha-\Delta}\ \text{func.}\left(\frac{H}{|t|^{\Delta}}\right)$

$\sim |t|^{\beta}$

$= 2-\alpha-\Delta$

k: $\alpha + 2\beta + \gamma = 2$

$\delta = \Delta/\beta$

s: $\alpha + \beta(\delta+1) = 2$

only two independent exponents

Adding a hyperscaling assumption:

$L^{\rho} = \xi^{\rho}$

Fisher: $\gamma = \nu(2-\eta)$

Josephson: $\alpha = 2 - \nu d$

Tong, SFT 3.2.1