Title: Statistical Physics Lecture - 112223

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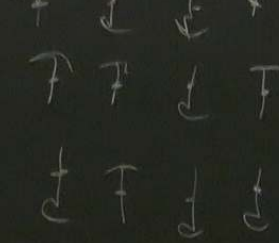
The Ising Model

- From before: we were looking for a model to simulate a ferromagnet:

$\sigma = \{\sigma_1, \sigma_2, \ldots \sigma_N\}$

$$Z = \sum_{\sigma} \exp(-\beta E(\sigma))$$

$\sigma_i = \pm 1$



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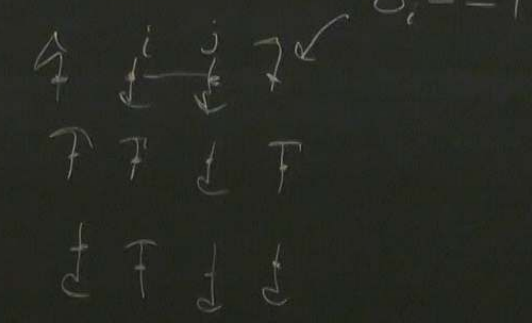


- The ferromagnetic phase transitions can be simulated with the Ising model:

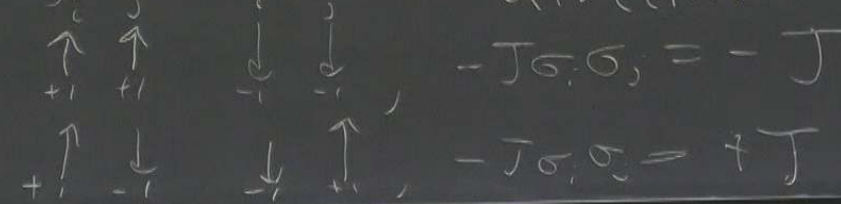
$$E(\sigma) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i$$

± 1

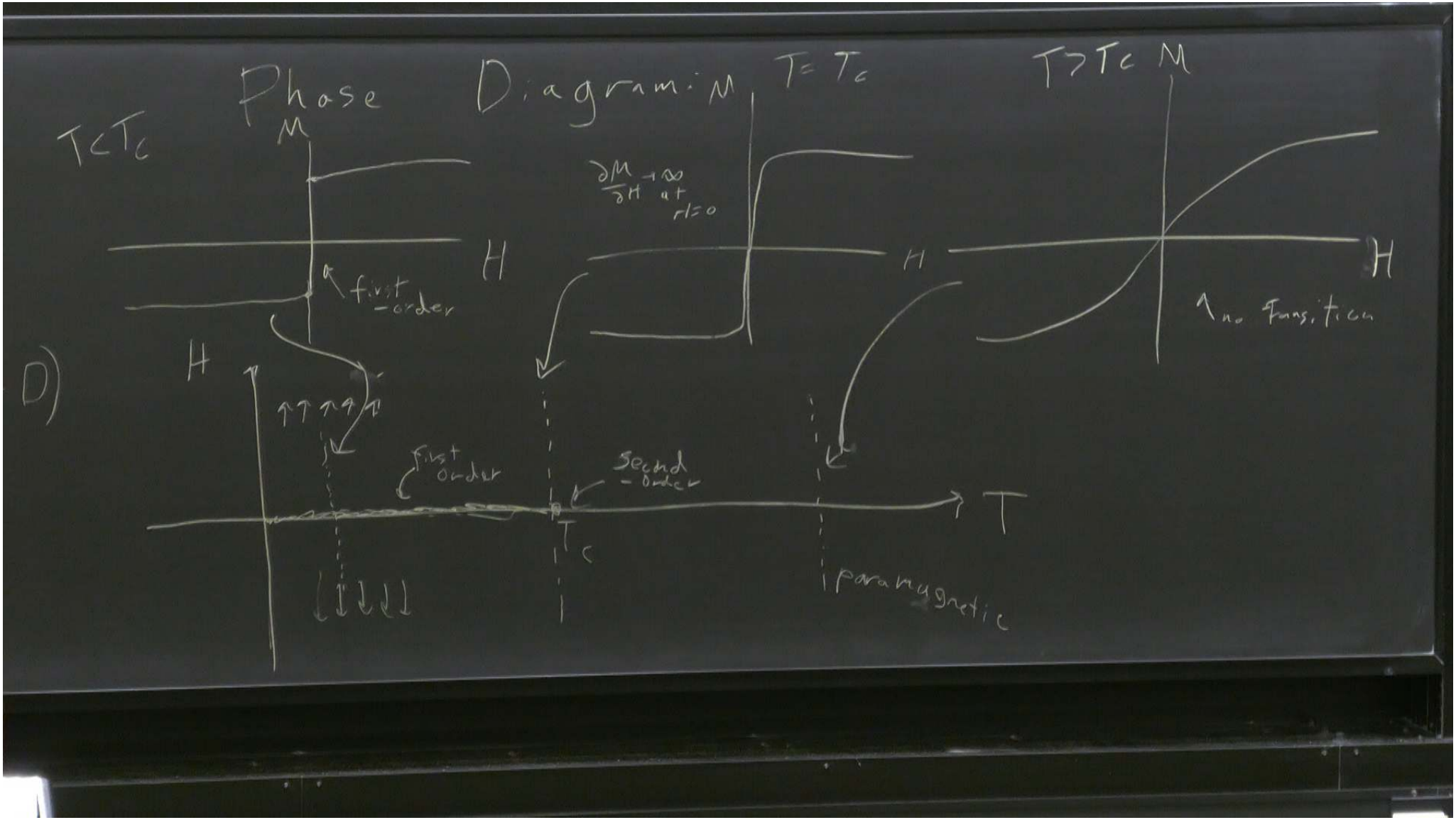
$\sigma_i = +1$ for $\uparrow$

$\sigma_i = -1$ for $\downarrow$

- The interaction term $-J\sigma_i\sigma_j$ favors nearest neighbor spins pointing in the same direction



- The $-H\sum_i \sigma_i$ term is the magnetic field (external), which we had before for the paramagnet ($J=0$).
- Order parameter: $M = \frac{1}{N}\langle \sum_i \sigma_i \rangle$
- In 1D there is no finite-temperature phase transition ($T_c = 0$)


Phase Diagram: M
T<Tc
M
H
first-order
T=Tc
∂M/∂H → ∞ at r|=0
H
T>Tc M
H
no transition
D)
H
first order
second-order
T
Tc
paramagnetic

- In 1D and 2D, the Ising model is exactly solvable.
  1D: tutorial; 2D: ↑ Onsager
- In 3D, it's still an open question
- In 4D and higher, we can use mean field theory

Mean Field Theory

- We'll take an approach that leads to qualitatively correct results in more than one dimension.
- We rewrite the model as:

$$E(\sigma) = -\frac{J}{2}\sum_i \sum_{j \in n.n.(i)} \sigma_i \sigma_j - H\sum_i \sigma_i$$

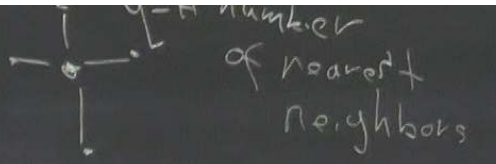
$q \equiv$ number of nearest neighbors

But, each bond is counted twice

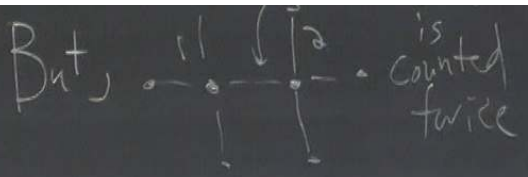
- So we have a factor of $\frac{1}{2}$ to fix overcounting.
- Now let's think from the perspective of spin 0.

$$E_0(\sigma) \approx -J\sigma_0 q\bar{\sigma} - \frac{1}{N}\sum_i \sigma_i$$

$$\approx -J\sigma_0 q M$$



But,



$\approx -J\sigma_0 q M$

$E_0(\sigma) \approx \underbrace{(-Jq\bar{M} - H)}_{\leftarrow H_{eff}} \sigma_0$

Every spin feels this "effective" magnetic field, and it now looks like a bunch of independent paramagnets, so let's recall:

From before: $E = -\sum_{\sigma_i} \mu h \sigma_i$

normalized $\rightarrow M = -\frac{1}{N}\left(\frac{\partial F}{\partial h}\right)_T = \mu \tanh \beta \mu h$

So, by taking $\mu \to 1$, $h \to -Jq\bar{M} - H$, we have that

$$\bar{M} = \tanh\left(\beta(Jq\bar{M} + H)\right)$$

- This leads to quantitatively correct answers for $D \geq 4$!

From before: $E = -\sum_{\sigma_i} \mu h \sigma_i$

normalized → $M = -\frac{1}{N}\left(\frac{\partial F}{\partial h}\right)_T = \mu \tanh \beta \mu h$

correct answers for $D \geq 4$!

Warning: A partition function based on $E = \sum_i E_i(\sigma)$ would lead to inconsistent results. (calculate F from Z, minimize $F \longrightarrow M$ would be different)

(we can obtain from writing $\sigma_i = \sigma_i - M + M$)

$E = \sum_i E_i(\sigma) = -J_q M \sum_i \sigma_i - H \sum_i \sigma_i$

"over counts the energy"

The "corrected" energy expression is

$E = -J_q M \sum_i \sigma_i + \frac{J_q M^2 N}{2} - H \sum_i \sigma_i$

$\frac{\partial F}{\partial M} = 0$ from $F = -kT \ln Z$

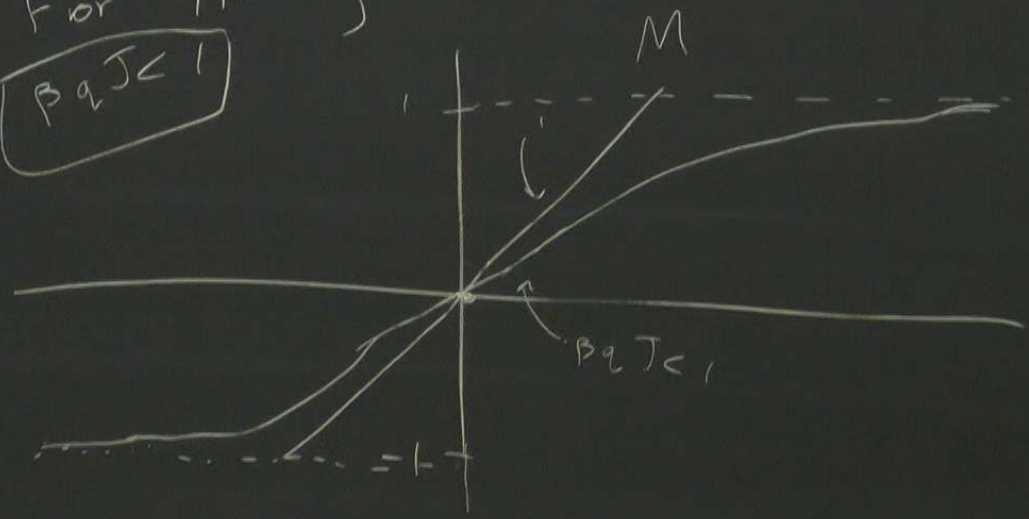
$M = \tanh(\beta(J_q M + H))$

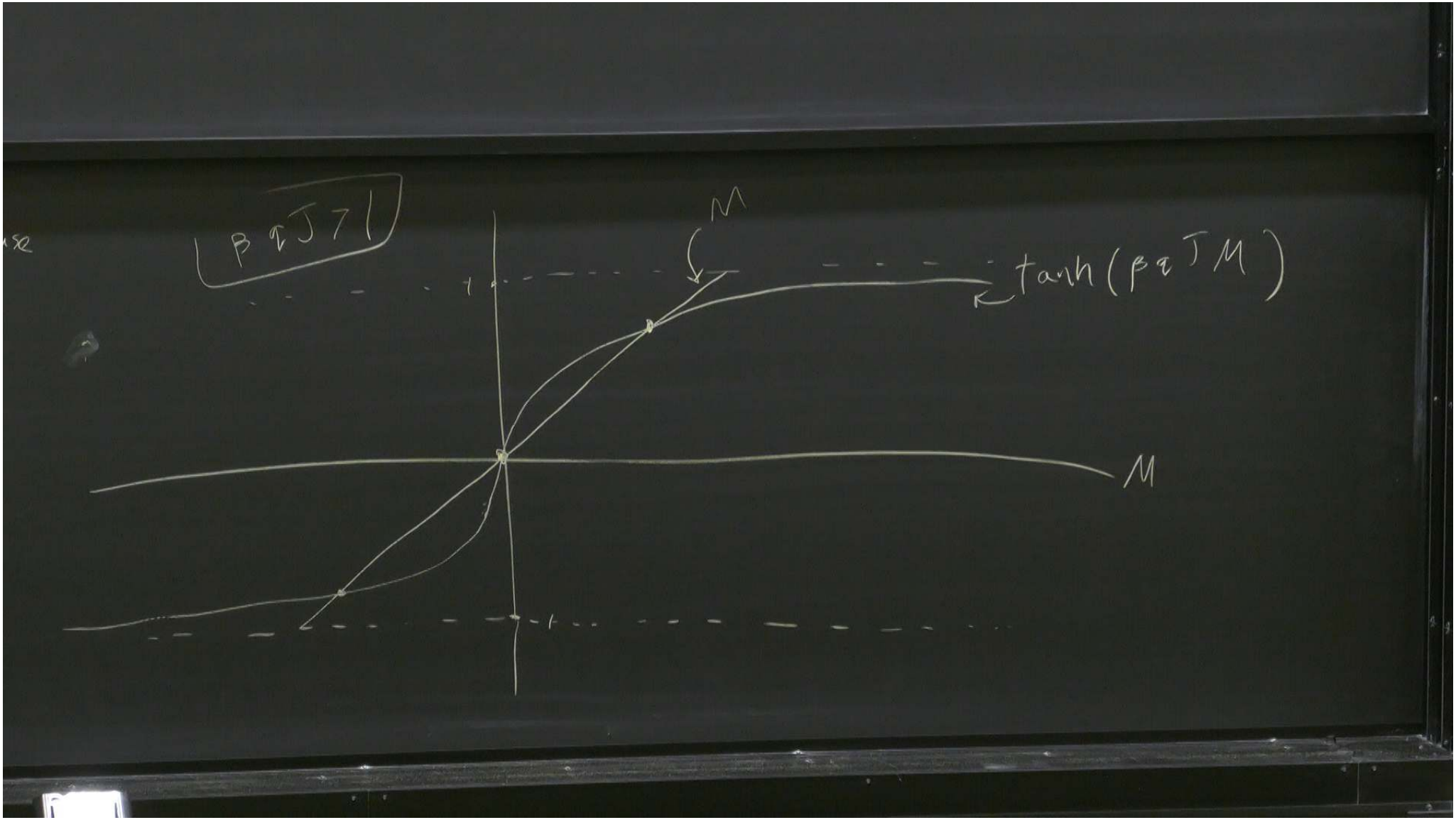
When $\beta J_q M$ is small, we can use

$\tanh x \approx x$

For $H=0$,

$\beta_q J < 1$




βzJ7l
M
tanh(βzJM)
M

$$\mathcal{L} = \sum_i \frac{Jq M \sigma_i}{2} + \frac{Jq M N}{2} - H \sum_i \sigma_i$$

∴ We can get nonzero M, $M = \pm m$

when $\beta q J > 1$

$\beta \propto \frac{1}{kT}$

$\frac{qJ}{k} > T$

$$\boxed{T < \frac{qJ}{k} \equiv T_c}$$

In 2D,

$$T_c \approx \frac{2.27\,J}{k}$$

Different (smaller) than

MFT $T_c = \frac{4J}{k}$

Critical Exponents

$M = \tanh(\beta(JqM + H))$

$\beta JqM + H = \tanh^{-1}(M)$

$\tanh^{-1} x \approx x + \frac{1}{3}x^3$

$M + \frac{H}{kT_c} \approx M + \frac{1}{3}M^3$

$\rightarrow H \approx \frac{kT}{3} M^3$

$\rightarrow \boxed{\delta = 3}$

$$M \propto \frac{T}{T_c}\sqrt{3\left(1-\frac{T}{T_c}\right)} \rightarrow \boxed{\beta = \frac{1}{2}}$$

$$\chi = \frac{\partial M}{\partial H} \qquad \chi \sim \begin{cases} \frac{1}{k}(T-T_c)^{-1}, & T \geq T_c, \quad H \to 0 \\ \frac{1}{2k}(T_c-T)^{-1}, & T \leq T_c, \quad H \to 0 \end{cases}$$

$$\boxed{\gamma = 1}$$

Notes: Due to RG, the critical behavior is the same for:

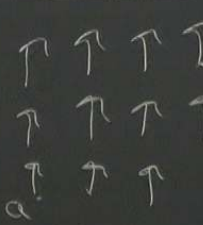
- some disorder ($J \to J_{ij}$ with same fluctations)
- longer-range interactions, as long as they decay fast enough

η and ν exponents

- There are two critical exponents I haven't discussed yet. They are both related to correlations.

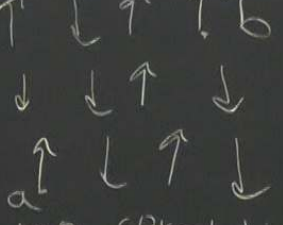
Correlation length (ξ)

ordered



a is correlated

disordered



no correlation between a and b — small ξ

- The ferromagnetic phase transitions can be simulated with the Ising model:

- Measurements of correlation functions

$$g_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

For a continuous phase transition,

$$g(r) \sim |r|^{-\tau} e^{-|r|/\xi}$$

(r: distance between spins)

At $T = T_c$, correlation length becomes infinite:

$$\xi(T) \sim |t|^{-\nu}$$

At $T = T_c$,

$$g(r) \sim |r|^{2-D-\eta}$$

(η: anomalous dimension)

a ↑ ↑ ↑ ↑

a is correlated

a ↑ ↓ ↑ ↓

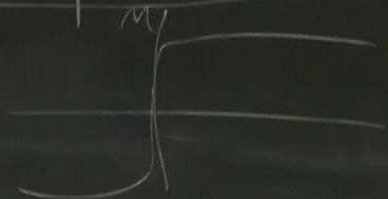
no correlation between a and b — small ξ

- Mean field theory, just took the average of all the spins, no interactions used, and so no correlation info.

- We will need more sophisticated methods to get η and ν.

"modern" classification of continuous phase transitions

- divergent susceptibility
- infinite correlation length, ξ
- power-law behavior of g(r)



$\chi \sim (T-T_c)^{-\gamma}$

$\gamma = 1$

$g(r) \sim |r|^{2-D-\eta}$