Title: Statistical Physics Lecture - 112223

Speakers: Emilie Huffman

Collection: Statistical Physics 2023/24

Date: November 22, 2023 - 10:45 AM

URL: https://pirsa.org/23110030

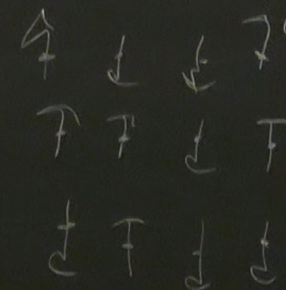
The Ising Model

- From before: we were looking for a model to simulate a ferromagnet:

$$\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$$

$$Z = \sum_{\sigma} \exp(-\beta E(\sigma))$$

$\sigma_i = \pm 1$



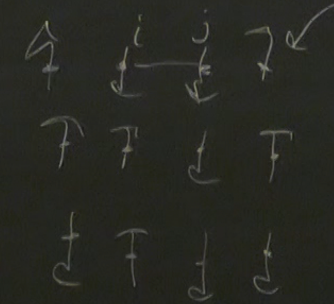
The Ising Model

From before: we were looking for a model to simulate a ferromagnet:

$\sigma = \{\sigma_1, \sigma_2, \ldots \sigma_N\}$

$$Z = \sum_{\sigma} \exp(-\beta E(\sigma))$$

$\sigma_i = \pm 1$



± 1

– The ferromagnetic phase transitions can be simulated with the Ising model:

$$E(\sigma) = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - H\sum_i \sigma_i$$

$\sigma_i = +1$ for $\uparrow$

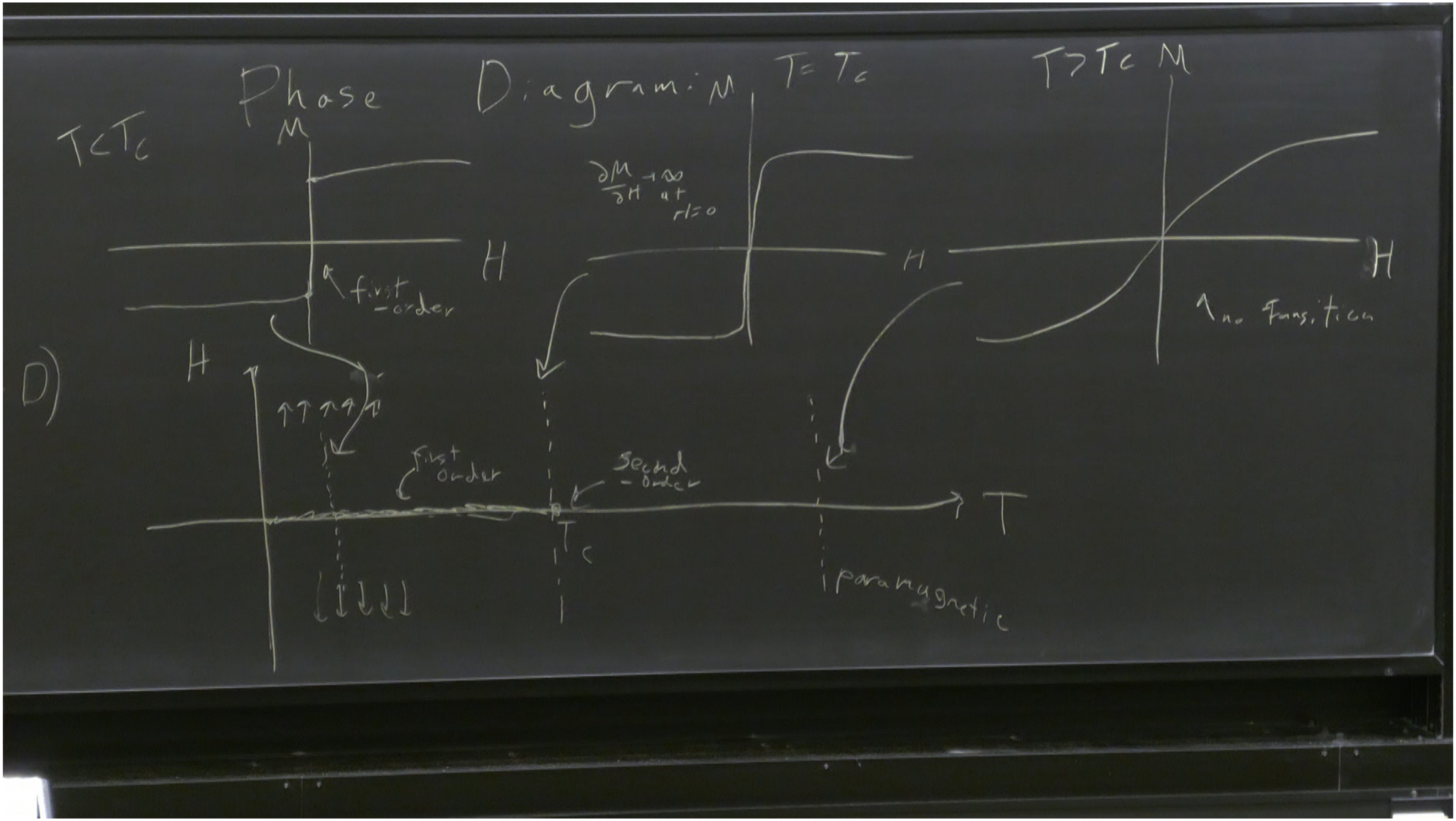
$\sigma_i = -1$ for $\downarrow$

– The interaction term favors nearest neighbor spins pointing in the same direction

$$\underset{+1}{\overset{i}{\uparrow}}\ \underset{+1}{\overset{j}{\uparrow}} \quad \underset{-1}{\overset{i}{\downarrow}}\ \underset{-1}{\overset{j}{\downarrow}}, \quad -J\sigma_i\sigma_j = -J \leftarrow \text{energetically favored}$$

$$\underset{+1}{\uparrow}\ \underset{-1}{\downarrow} \quad \underset{-1}{\downarrow}\ \underset{+1}{\uparrow}, \quad -J\sigma_i\sigma_j = +J$$

- The $-H\sum_i \sigma_i$ term is the magnetic field (external), which we had before for the paramagnet $(J=0)$.
- Order parameter: $M = \frac{1}{N}\langle \sum_i \sigma_i \rangle$
- In 1D there is no finite-temperature phase transition $(T_c = 0)$


Phase Diagram in
T<Tc
M
H
first-order
T=Tc
∂M/∂H → ∞ at H=0
M
H
T>Tc M
H
no transition
H
first order
second-order
T
Tc
paramagnetic

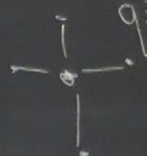
- In 1D and 2D, the Ising model is exactly solvable.
  1D ↑tutorial, 2D ↑Onsager
- In 3D, it's still an open question
- In 4D and higher, we can use mean field theory

Mean Field Theory

- We'll take an approach that leads to qualitatively correct results in more than one dimension.
- We rewrite the model as:

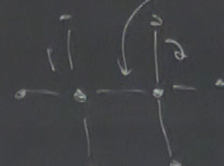
$$E(\sigma) = -\frac{J}{2}\sum_i \sum_{j\in n.n.(i)} \sigma_i\sigma_j - H\sum_i \sigma_i$$

each bond is counted twice



$q \equiv$ # number of nearest neighbors

But,



- So we have a factor of $\frac{1}{2}$ to fix overcounting.
- Now let's think from the perspective of spin 0.

$$E_0(\sigma) \approx -J\sigma_0 q \bar{\sigma} \qquad \frac{1}{N}\sum_i \langle \sigma_i \rangle$$

$$\approx -J\sigma_0 q M$$

q = number of nearest neighbors

But, $\overset{1}{\cdot}$—$\overset{2}{\cdot}$ is counted twice

$\approx -J\sigma_0 q M$

$$E_0(\sigma) \approx (\underbrace{-JqM - H}_{H_{eff}})\,\sigma_0$$

Every spin feels this "effective" magnetic field, and it now looks like a bunch of independent paramagnets, so let's recall:

From before: $E = -\sum_{\sigma_i} \mu h \sigma_i$

normalized → $M = -\frac{1}{N}\left(\frac{\partial F}{\partial h}\right)_T = \mu \tanh \beta \mu h$

So, by taking $\mu \to 1$, $h \to -JqM - H$, we have that

$$M = \tanh\left(\beta(JqM + H)\right)$$

- This leads to quantitatively correct answers for $D \geq 4$!

From before: $E = -\sum_{\sigma_i} \mu h \sigma_i$

normalized → $M = -\frac{1}{N}\left(\frac{\partial F}{\partial h}\right)_T = \mu \tanh \beta \mu h$

Correct answers for $D \geq 4$!

Warning: A partition function based on $E = \sum_i E_i(\sigma)$ would lead to inconsistent results. (calculate F from Z, minimize F → M would be different)

(we can obtain from writing $\sigma_i = \sigma_i - M + M$)

$$E = \sum_i E_i(\sigma) = -J q M \sum_i \sigma_i - H \sum_i \sigma_i$$

"over counts the energy"

The "corrected" energy expression is

$$E = -J q M \sum_i \sigma_i + \frac{J q M^2 N}{2} - H \sum_i \sigma_i$$

→ $\frac{\partial F}{\partial M} = 0$ from $F = -kT \ln Z$

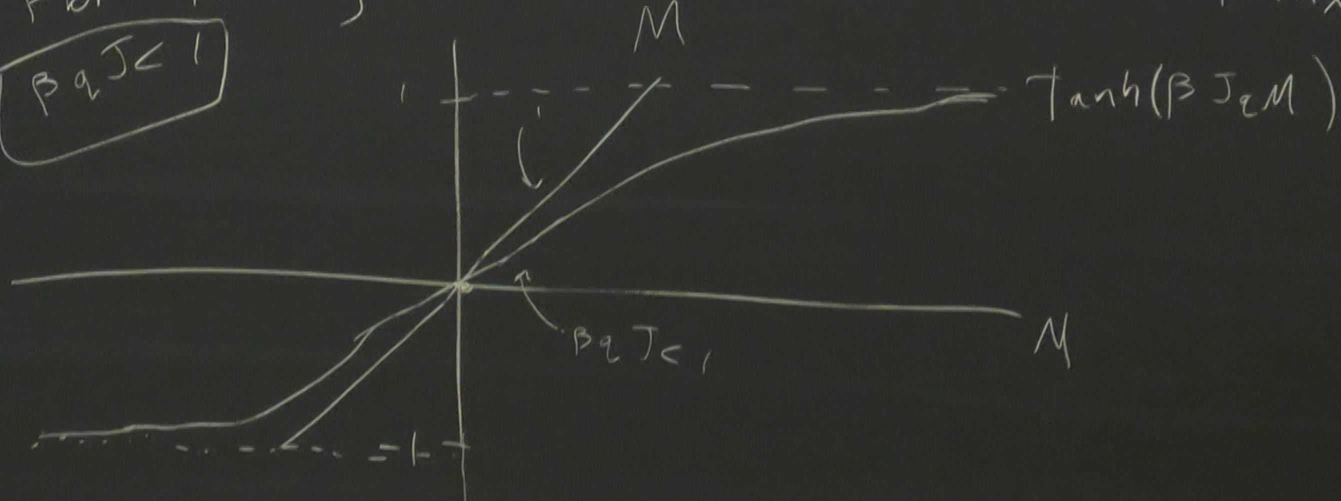
$$M = \tanh\left(\beta\left(J_q M + H\right)\right)$$

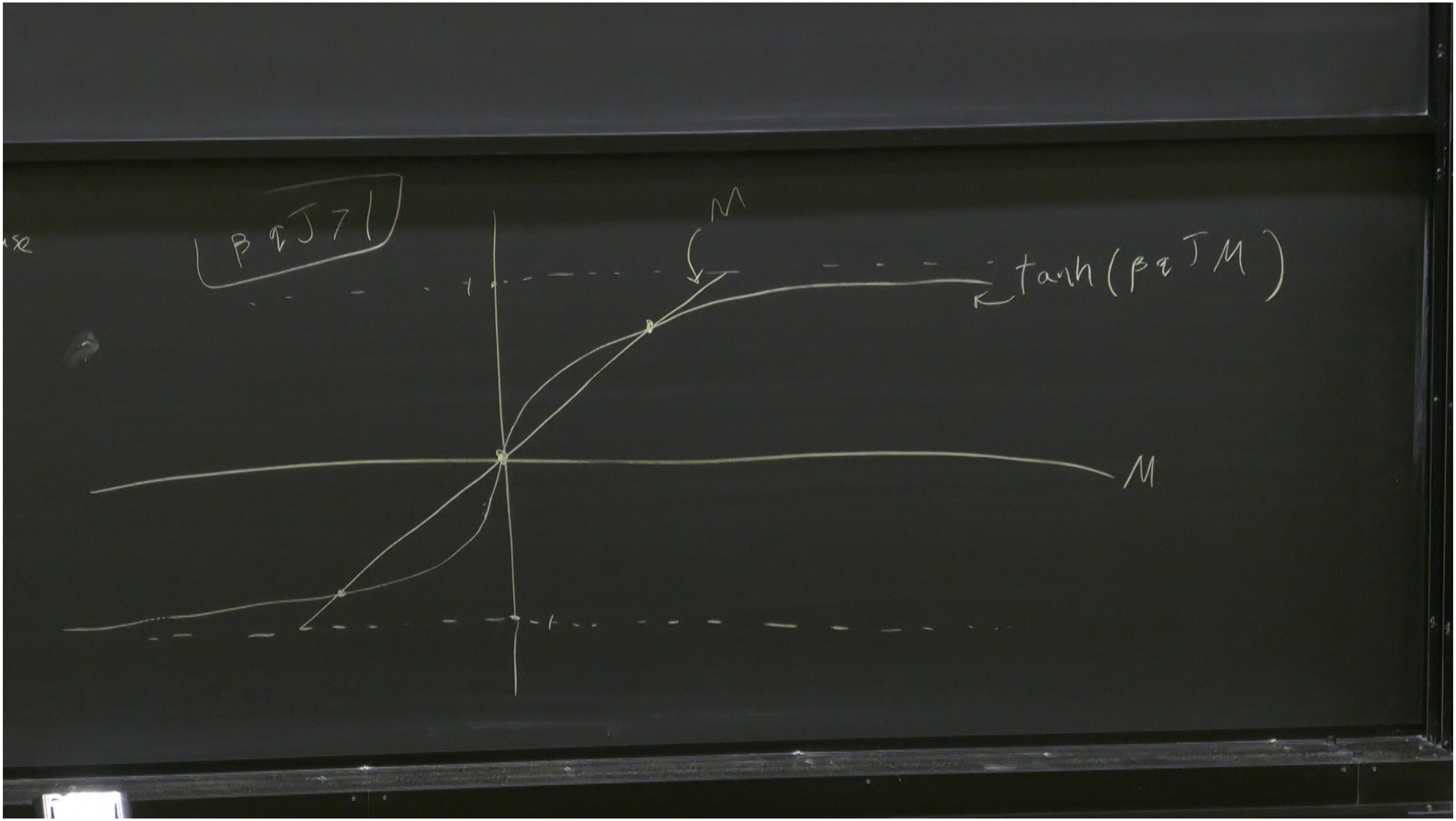
When $\beta J_q M$ is small, we can use

$$\tanh x \approx x$$

For $H=0$,

$$\boxed{\beta q J < 1}$$




$\beta q J > 1$
M
$\tanh(\beta q J M)$
M

$$- JqM\sum_i \sigma_i + \frac{JqM^2N}{2} - H\sum_i \sigma_i$$

$\therefore$ We can get nonzero M, $M = \pm m$

when $\beta q J > 1$

$$\beta = \frac{1}{kT}$$

$$\frac{qJ}{k} > T$$

$$\boxed{T < \frac{qJ}{k} \equiv T_c}$$

In 2D,

$$T_c \approx \frac{2.27\,J}{k}$$

Different (smaller) than MFT $T_c = \frac{4J}{k}$

Critical Exponents

$$M = \tanh(\beta(JqM + H))$$

$$\beta JqM + H = \tanh^{-1}(M)$$

$$\tanh^{-1}x \approx x + \frac{1}{3}x^3$$

$$M + \frac{H}{kT_c} \approx M + \frac{1}{3}M^3$$

$$\rightarrow H \approx \frac{kT}{3} M^3$$

$$\rightarrow \boxed{\delta = 3}$$

$$M \approx \frac{T}{T_c}\sqrt{3\left(1-\frac{T}{T_c}\right)} \longrightarrow \boxed{\beta = \frac{1}{2}}$$

$$\chi = \frac{\partial M}{\partial H} \qquad \chi \sim \begin{cases} \frac{1}{k}(T-T_c)^{-1}, & T \geq T_c, \quad H \to 0 \\ \frac{1}{2k}(T_c-T)^{-1}, & T \leq T_c, \quad H \to 0 \end{cases}$$

$$\boxed{\gamma = 1}$$

Notes: Due to RG, the critical behavior is the same for:

- some disorder ($J \to J_{ij}$ with some fluctuations)
- longer-range interactions, as long as they decay fast enough

η and ν exponents

- There are two critical exponents I haven't discussed yet. They are both related to correlations.

Correlation length (ξ)

ordered	disordered
↑ ↑ ↑ ↑ b	↑ ↓ ↑ ↑ b
↑ ↑ ↑ ↑	↓ ↓ ↑ ↓
a ↑ ↑ ↑ ↑	a ↑ ↓ ↑ ↓
a is correlated	no correlation between a and b — small ξ

- The ferromagnetic phase transitions can be simulated with the Ising model:

- Measurements of correlation functions

$$g_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

For a continuous phase transition,

$$g(r) \sim |r|^{-\tau} e^{-|r|/\xi}$$

distance between spins

At $T = T_c$, correlation length becomes infinite

$$\xi(T) \sim |t|^{-\nu}$$

At $T = T_c$,

$$g(r) \sim |r|^{2-D-\eta}$$

anomalous dimension

a ↑ ↑ ↑ ↑

a is correlated

a ↑ ↓ ↑ ↓

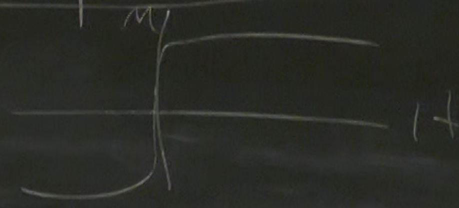
no correlation between a and b — small ξ

- Mean field theory, just took the average of all the spins, no interactions used, and so no correlation info.

- We will need more sophisticated methods to get η and ν.

"modern" classification of continuous phase transitions

- divergent susceptibility
- infinite correlation length, ξ
- power-law behavior of $g(r)$



$\chi \sim (T-T_c)^{-1}$

$\gamma = 1$

$g(r) \sim |r|^{2-D-\eta}$