Title: QFT2 Lecture - 112423

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Collection: Quantum Field Theory 2 2023/24

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Non-abelian Gauge Theories

Spin 1 = Fund. Representat

Vector bosons spin 1

Spin 1 A_μ co-vector with 4 co

Maxwell — QED photon

Spin $\frac{1}{2}$ ψ^α ← Spin ½ repr. of Lorentz

$\vec{E}, \vec{B}$ 4 equations

coulomb

vector

Vector Potential $A = (A_\mu) = (V, \vec{A})$

Gauge Theories

Spin 1 = Fund. Representation of Lorentz

ns spin 1

Spin 1 A_μ co-vector with 4 components

QED photon

Spin $\frac{1}{2}$ ψ^α ← Spin ½ repr. of Lorentz

t equations

coulomb vector

n Potential $A = (A_\mu) = (V, \vec{A})$

ge Theories

spin 1

photon

iations

ntial $A = (A_\mu) = (\underset{\text{coulomb}}{V}, \underset{\text{vector}}{\vec{A}}) = (A_0, A_{x,y,z}$

Spin 1 = Fund. Representation of Lorentz

Spin 1 $\quad A_\mu$ co-vector with 4 components

Spin $\frac{1}{2}$ $\quad \psi^\alpha$ ← Spin $\frac{1}{2}$ repr. of Lorentz

mass

$(-+++)$

massive $\Rightarrow$ tachyonic mode $\quad m^2 A_\mu A^\mu = -A_0^2 + \vec{A}^2$

mass $= 0$ (except in Higgs model)

but still problem with "longitudinal polarization states,

$$A_\mu(x) = \epsilon_\mu e^{iK\cdot x}, \quad K^2 = 0 = -K_0^2 + \vec{K}^2$$

$\Downarrow$

unitarity

only $\perp$ polarization $\vec{\epsilon} \perp \vec{K}$ are physical

transverse

$(-+++)$

= Fund. Representation of Lorentz

co-vector with 4 components

← spin 1/2 repr. of Lorentz

vector

$= (A_0, A_{x,y,z}$

EM tensor / Fieldstrength

$\boxed{F = (F_{\mu\nu})}$

4x4 antisym tensor

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

massive ⇒ tachyonic mode $m^2 A_\mu A$

mass $= 0$ (except in Higgs model

but still problem with "longitudinal

$A_\mu(x) = \epsilon_\mu e^{i k \cdot x}$, $k^2 = 0 = -k_0^2 + \vec{k}^2$

only ⊥ polariza

transver

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ $x = (x^\mu) = (t, \vec{x})$

Gauge transformation <u>Abelian</u>

arbitrary function $\alpha(x)$

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)$$

$$F_{\mu\nu}(x) \longrightarrow F_{\mu\nu}(x)$$

Action principle $\Rightarrow$ Maxwell equations

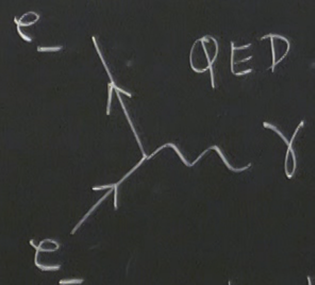
$$S_{Maxwell}[A] = \int d^4x \left(-\frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) \right)$$

Yang-Mills '54

Yang-Mills '54 $SU(2) \to$ '72 $SU(3) \to QCD$ $\quad i=1,$

current-current interactions $\qquad \psi$ complex $\to \psi = (\psi^1, \psi^2) = (\psi^i)$

QED



renormalizable

Fermi

non renormalizable

$U(2) \to$ '72, $SU(3) \to$ QCD $i=1,2$ $SU(2) \ni g$

ions ψ complex $\to$ $\psi = (\psi^1, \psi^2) = (\psi^i)$ $\psi \to g \cdot \psi$ g 2×2 matrix

ermi'

$SU(2) \to 3$ generators, current have 3 components J_μ^a $a = 1,2,3$

lizable   Lie Algebra of $SU(2)$: $su(2)$ generators $t_a = \frac{1}{2}\sigma_a$ σ_a Pauli matrices

$\bar{\psi}\, \gamma^\mu t_a \psi$ 2×2 matrices

↑ γ matrix

3 gauge fields $A_\mu^a(x)$ $a = 1, 2, 3$ for $SU(2)$ Real Field

under $SU(2)$ adjoint representation of $SU(2)$

$A_\mu^a(x)$ $a = 1, 2, 3$ $\xRightarrow{\text{regroup}}$ 2x2 traceless hermitean matrix

$$A_\mu(x) = \sum_{a=1}^{3} A_\mu^a(x)\, t_a \qquad t_a = \frac{1}{2}\sigma_a$$

$$A = \frac{1}{2}\begin{pmatrix} A^3 & A^1 - iA^2 \\ A^1 + iA^2 & -A^3 \end{pmatrix} \in SU(2) \text{ Adj Repr}$$

eld

Gauge transformations & covariant derivatives in a Gauge Potential A

$$\partial_\mu \longrightarrow D_\mu$$

$$D_\mu \Phi_A := \partial_\mu \Phi_A - i\left[A_\mu, \phi_A\right]$$

$$D_\mu \Psi_F = \partial_\mu \Psi_F - i \underbrace{A_\mu}_{2\times 2 \text{ matrix}} \cdot \underbrace{\Psi_F}_{2\text{-vector}}$$

$\phi_A(x) \in$ adjoint representation

$\phi_A(x) = \phi_A^a\, t_a \qquad a = 1, 2, 3$

Ψ_F representation.

Gauge transformations & covariant derivatives in a Gauge Potential A

$\partial_\mu \longrightarrow D_\mu$

$$\boxed{\begin{aligned} D_\mu \Phi_A &:= \partial_\mu \Phi_A - i\,[A_\mu, \phi_A] \\ D_\mu \Psi_F &= \partial_\mu \Psi_F - i\, A_\mu \cdot \Psi_F \end{aligned}}$$

2×2 matrix 2-vector

Local gauge transf.

generated by $g(x)$ $g \in SU(2)$ unitary 2×2 matrices det = 1

$\phi_A(x) \in$ adjoint representation

$\phi_A(x) = \phi_A^a\, t_a \qquad a = 1, 2, 3$

Ψ_F representation

$$\boxed{\Psi_F \to g\,\Psi_F \qquad \phi_A \to g\,\phi_A\,g^{-1}}$$

(x) $\quad a = 1, 2, 3 \quad$ for $SU(2)$ Real Field

...nal representation of $SU(2)$

...oup $\quad$ 2x2 traceless hermitean matrix

$t_a \qquad t_a = \frac{1}{2}\sigma_a$

$\left(\ldots -iA^2 \atop \ldots A^3 \right) \in SU(2)$ Adj Repr

For $U(1)$

$g = e^{i\alpha}$

Gauge transformations & covaria...

$\partial_\mu \longrightarrow D_\mu$

$$\boxed{\begin{aligned} D_\mu \Phi_A &:= \partial_\mu \Phi_A - i\left[A_\mu, \phi_A\right] \\ D_\mu \Psi_F &= \partial_\mu \Psi_F - i\, A_\mu \cdot \Psi_F \end{aligned}}$$

2x2 matrix $\quad$ 2-vector

Local gauge transf.

generated by $g(x) \quad g \in SU(2)$ unitary, det = ...

Gauge transf over A $\quad g(x) \in SU(2)$

$$A_\mu \longrightarrow g A_\mu g^{-1} + i\, g\, \partial_\mu (g^{-1})$$ general $g(x)$

infinitesimal transf $\underbrace{g(x) = \mathbb{1}}_{2\times 2 \text{ identity}} + i\, \underbrace{\alpha(x)}_{\substack{SU(2) \\ 2\times 2 \text{ tr}=0 \text{ Herm. matrix}}} + \dots$ close to the identity

$$\boxed{A_\mu \longrightarrow A_\mu + D_\mu \alpha}$$

with $\boxed{D_\mu \alpha = \partial_\mu \alpha - i \left[A_\mu , \alpha \right]}$

$\alpha \in \mathrm{Adj}\ SU(2)$

Field Strength and c

$$D_\mu (g \psi_F) = g \cdot D_\mu \psi_F$$

$$D_\mu (g \phi_A g^{-1}) = g \, D_\mu \phi_A \cdot g^{-1}$$

gauge transf commutes with <u>covariant</u> derivatives strength and action

2×2 .... matrix

$$\nu = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \in \mathrm{Adj}\ SU(2)$$

$F_{\mu\nu}$ is covariant

$$= i\,[D_\mu, D_\nu]$$

commutator of 2 cov. derivatives

$$F_{\mu\nu} \to g \cdot F_{\mu\nu} \cdot g^{-1}$$

2x2 ···· matrix

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \quad \in \mathrm{Adj}\,SU(2)$$

$F_{\mu\nu}$ is covariant

$$= i\,[D_\mu, D_\nu]$$ commutator of 2 cov. derivatives

$F_{\mu\nu} \to g \cdot F_{\mu\nu} \cdot g^{-1}$

non linear in A

U(1) $D_\mu = \partial_\mu - iA_\mu$, $\partial_\mu(\partial_\nu - iA_\nu) - \partial_\nu(\partial_\mu - iA_\mu) = i(\partial_\nu A_\mu - \partial_\mu A_\nu) = [D_\mu, D_\nu]$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(x)

Field Strength and action

2x2 .... matrix

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \in \mathrm{Adj}\ SU(2)$$

$F_{\mu\nu}$ is covariant

$$= i\,[D_\mu, D_\nu]$$

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Adj = {0} $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Fund.

$\Psi \in \mathrm{Adj}\; SU(2)$ Fund. $F_{\mu\nu} = \partial_\mu$

for $SU(2)$ $\quad F_{\mu\nu} = F_{\mu\nu}^{a}\, t_a \quad t_a = \frac{1}{2}\sigma_a$ ← Pauli matrices

$a = 1, 2, 3$

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a} + \underbrace{f^{a}{}_{bc}}_{\text{constant structure of } SU(2)} A_\mu^{b} A_\nu^{c} \qquad f^{a}{}_{bc} = \epsilon^{abc}$$

Levi-Civita antisymm symbols

non linear in E

(II) $D_\mu = \partial_\mu - iA_\mu$, $\partial_\mu(\partial_\nu - iA_\nu) - \partial_\nu(\partial_\mu - iA_\mu) = i(\partial_\nu A_\mu - \partial_\mu A_\nu) = [D_\mu, D_\nu]$

Adj = {0} $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Fund.

a

symbols

Action

$$\boxed{S_{SU(2)}[A_\mu] = -\frac{1}{2g^2}\int d^4x \; Tr\left[F_{\mu\nu}F^{\mu\nu}\right]}$$

$F_{\mu\nu} = F^a_{\mu\nu}\, t_a$ $\quad Tr(t_a t_b) = \frac{1}{2}\delta_{ab}$

for $SU(N)$

g coupling constant

S non-linear

gauge transf

$A_\mu \xrightarrow{g} A^{(g)}_\mu$

$S[A^{(g)}_\mu] = S[A_\mu]$

2) Adj Repr $\Phi_A = (\phi^1, \phi^2, \phi^3)$

generated by $g(x)$ $g \in SU(2)$ unitary 2×2 matrices det = 1

$\psi_F \in$

$S_{SU(2)}[A_\mu]$ is gauge invariant! $\Longrightarrow$ Field Equation. (Classical)

$D^\mu F_{\mu\nu} = 0 \longleftrightarrow \partial^\mu F_{\mu\nu}$

SU(2)	Maxwell.
non-linear equations	linear equation
interesting non-trivial dynamics	plane waves

...2×2 matrices

...el = 1

$\psi_F \in \widehat{\text{Fund}}(SU(2))$ $\quad \phi_A \in \text{Adj}(SU(2))$

(Classical)

Coupling to matter

Dirac Field $\quad \Psi = (\Psi^{\alpha, i})$

Fund. $SU(2)$ $\quad \alpha$ Dirac indices

$i = 1, 2$ $\quad$ 2-vector

$$S[\Psi] = \int d^4x \, \overline{\Psi} \cdot (i \not{D} - m) \Psi$$

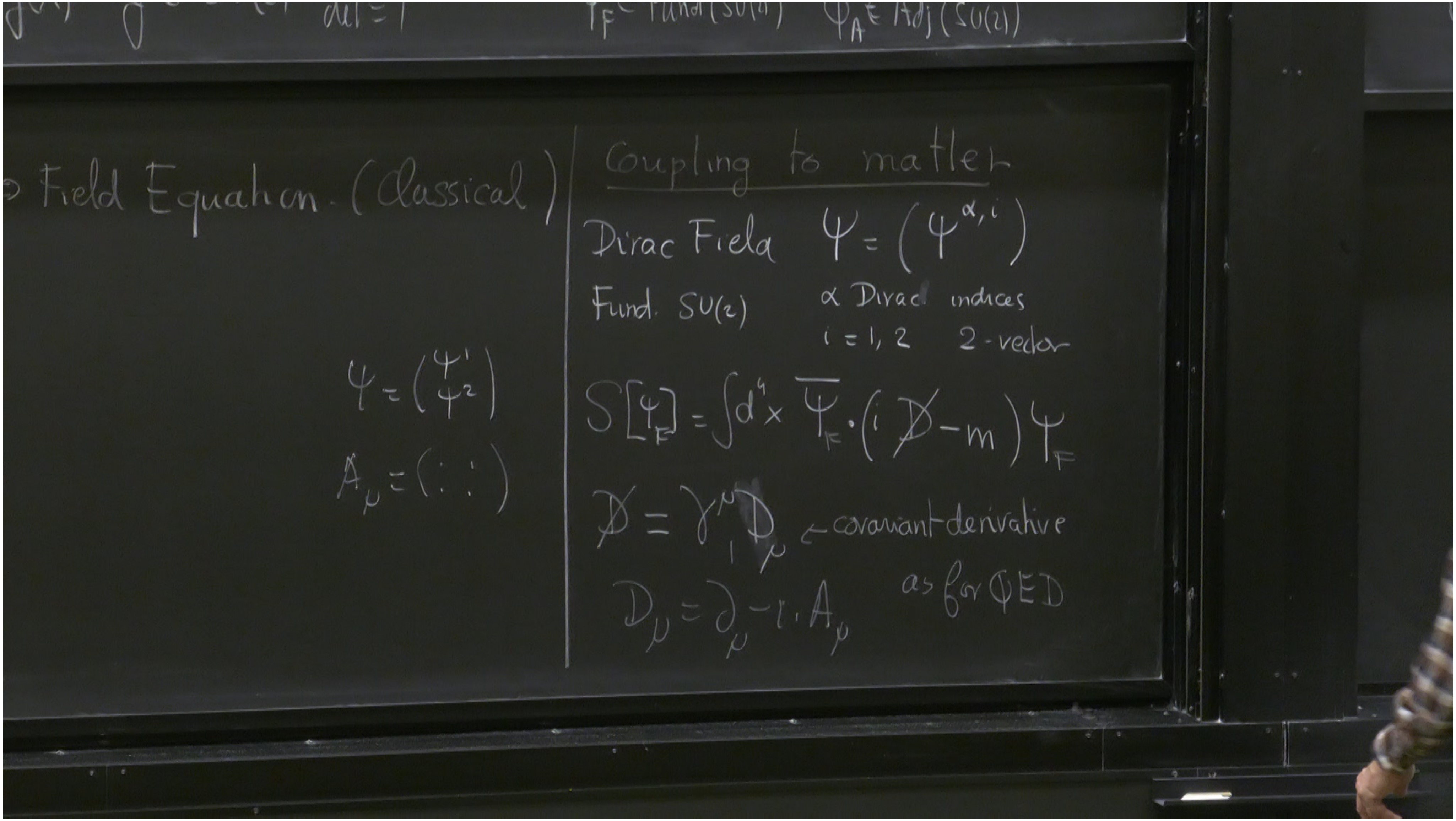
$$\not{D} = \gamma^\mu D_\mu \leftarrow \text{covariant derivative}$$

$\alpha \in \text{Adj } SU(2)$

for $SU(2)$ $\quad F_{\mu\nu} =$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A \ldots$$

non linear in E


$\psi_F \in$ Fund$(SU(2))$ $\phi_A \in$ Adj$(SU(2))$
Field Equation. (Classical)
$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$
Coupling to matter
Dirac Field $\Psi = (\Psi^{\alpha, i})$
Fund. $SU(2)$
α Dirac indices
$i = 1, 2$ 2-vector
$S[\Psi_F] = \int d^4x \, \overline{\Psi}_F \cdot (i \not{D} - m) \Psi_F$
$\not{D} = \gamma^\mu D_\mu$ ← covariant derivative
$D_\mu = \partial_\mu - i A_\mu$
as for QED

dynamics

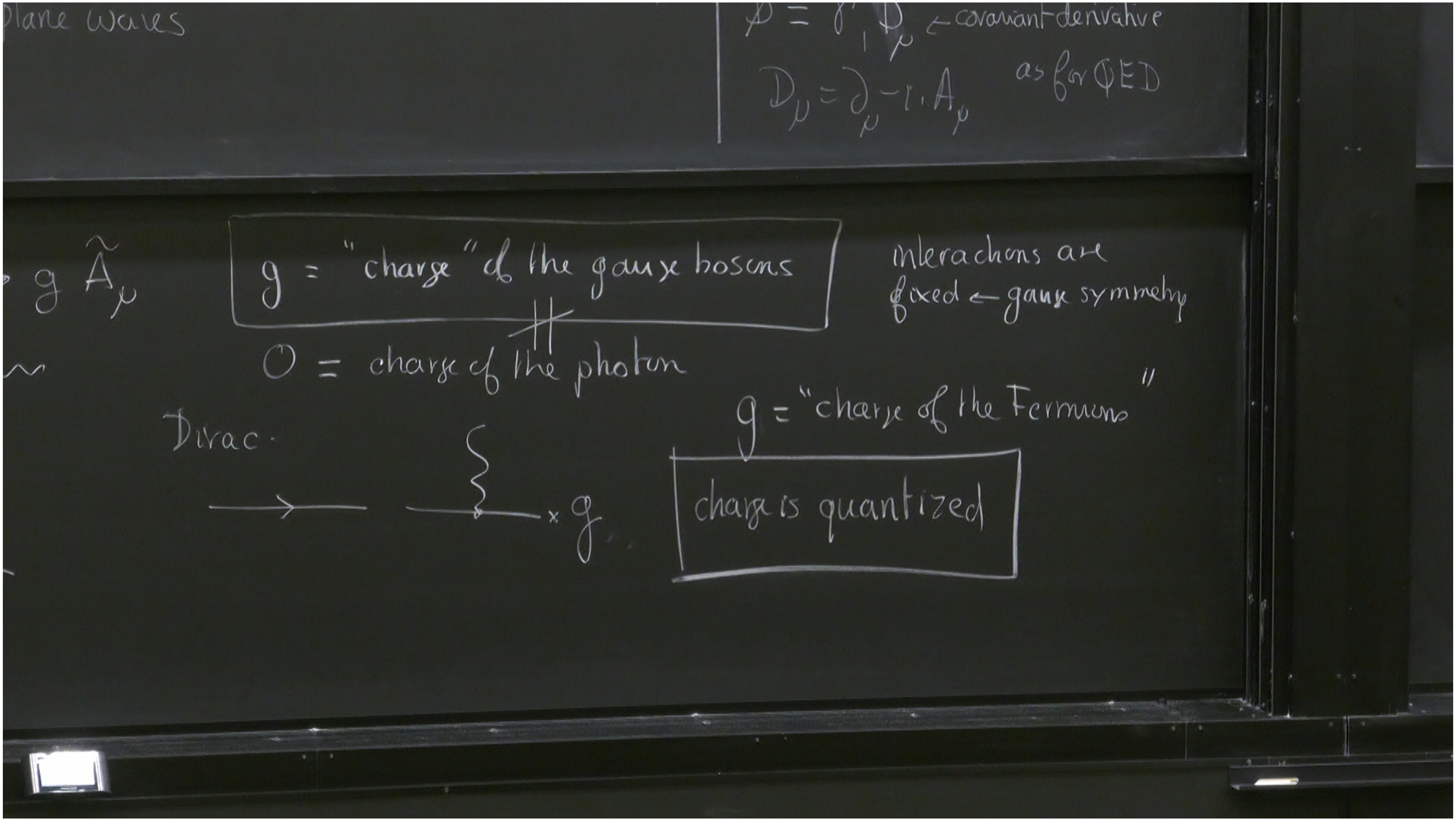
"Interaction Vertices": g is a small parameter $A_\mu \to g\tilde{A}_\mu$

$\frac{1}{g^2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = (\partial_\mu \tilde{A}_\nu + \partial_\nu \tilde{A}_\mu)^2$ as in maxwell 

interaction

$\frac{1}{g^2}\partial_\mu A_\nu A_\rho A_\sigma \longrightarrow g\,\partial_\mu \tilde{A}_\nu \tilde{A}_\rho \tilde{A}_\sigma$ g 

$\frac{1}{g^2} A_\mu A_\nu A_\rho A_\sigma \longrightarrow g^2 A_\mu A_\nu A_\rho A_\alpha$ g^2 


plane waves
$\not{D} = \gamma^\mu D_\mu$ ← covariant derivative
$D_\mu = \partial_\mu - i A_\mu$ as for QED
$g \tilde{A}_\mu$
$\boxed{g = \text{"charge" of the gauge bosons}}$
$\neq$
0 = charge of the photon
interactions are fixed ← gauge symmetry
g = "charge of the Fermions"
Dirac
$\times g$
$\boxed{\text{charge is quantized}}$