Title: QFT2 Lecture - 112223

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Collection: Quantum Field Theory 2 2023/24

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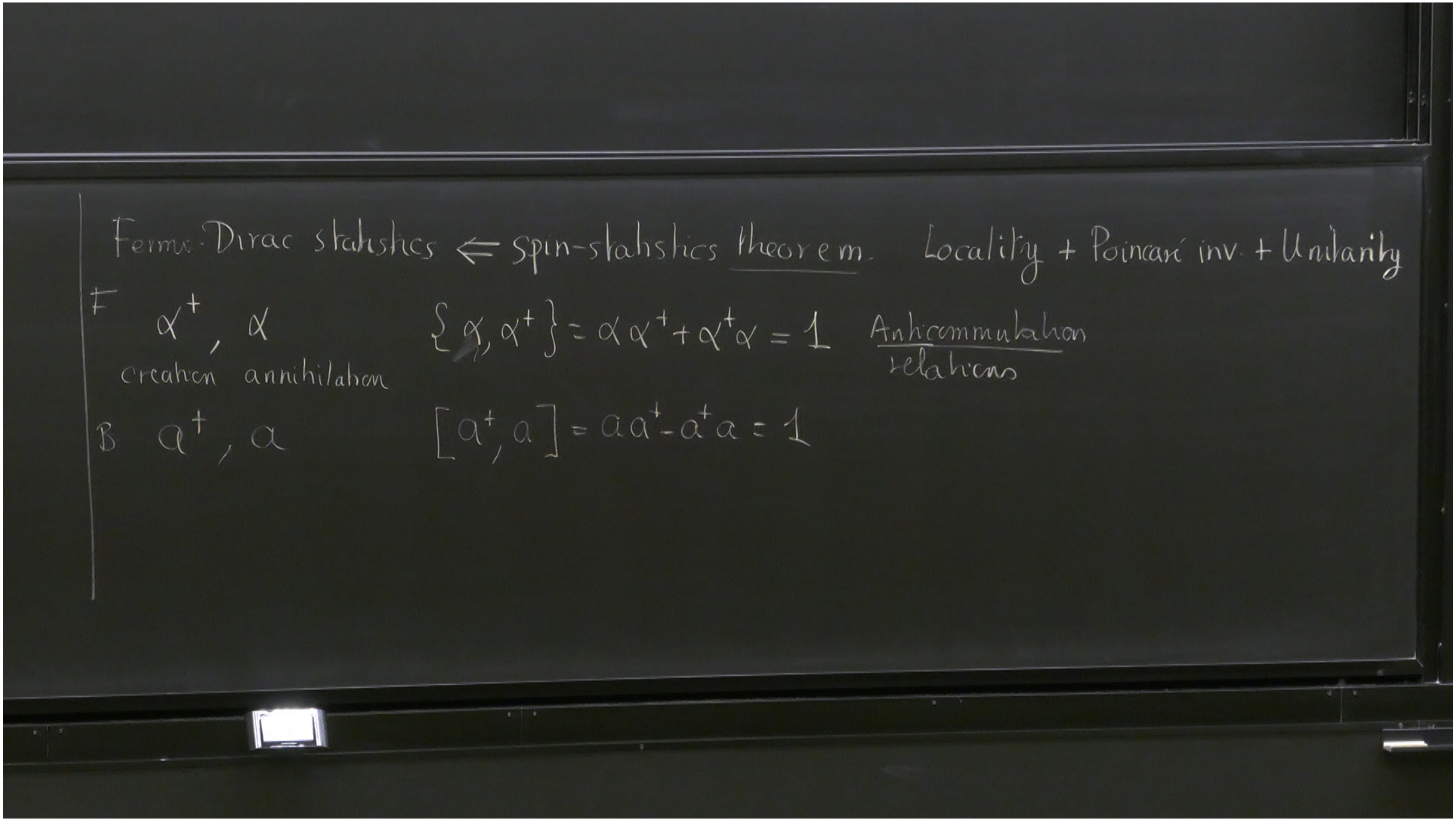

Fermi-Dirac statistics ⇐ spin-statistics theorem. Locality + Poincaré inv. + Unitarity
F α^+, α
creation annihilation
$\{\alpha, \alpha^+\} = \alpha\alpha^+ + \alpha^+\alpha = 1$
Anticommutation relations
B a^+, a
$[a^+, a] = aa^+ - a^+a = 1$

Grassmann Algebras & Berezin calculus $\quad T$ or $\dagger$

Vector space $\mathbb{C}$ + addition, $\circ$ multiplication, $*$ conjugation

complex Grassmann Algebra $\mathbb{G}_N$ $\quad$ 2N dimensional

Basis of generators $\{\theta_i, \bar{\theta}_i\}_{i=1,N}$ of the algebra $\Rightarrow$ products + lin. comp

Anticommutation relation $\quad \theta_i\theta_j+\theta_j\theta_i=0 \quad \bar{\theta}_i\bar{\theta}_j+\bar{\theta}_j\theta_j=0 \quad \theta_i\cdot\bar{\theta}_j+\bar{\theta}_j\theta_i=0 \quad \Rightarrow \mathbb{G}_N$

$\Rightarrow \quad \theta_i^2=0 \quad \bar{\theta}_j^2=0$

general element g · linear comb of products of the θ's and $\bar\theta$'s of $\mathbb{G}_N$

Ex $N=1$ $(\theta, \bar\theta)$ $g = a\,1 + b\,\theta + c\,\bar\theta + d\,\theta\bar\theta$

$(1, \theta, \bar\theta, \theta\bar\theta)$ Basis dim $\mathbb{G}_1 = 4$

$\underline{N=2}$ $\theta_1, \bar\theta_1, \theta_2, \bar\theta_2$ $g =$ lin. comb of 16 terms

$\underbrace{1,}_{\#\,1}$ $\underbrace{\qquad\qquad}_{4}$, $\theta_1\theta_2 \dots\dots \bar\theta_1\bar\theta_2$

$\mathbb{G}_N$

& Berezin calculus   T or †

↑

ldition, ∘ multiplication, $*$ conjugation

ebra $\mathbb{G}_N$ 2N "dimensional"   N pairs of generators

$\bar\theta_i \}_{i=1,N}$ of the algebra ⇒ products + lin. comb

$_j\theta_i = 0$ $\bar\theta_i\bar\theta_j + \bar\theta_j\bar\theta_i = 0$ $\theta_i\bar\theta_j + \bar\theta_j\theta_i = 0$ ⇒ $\mathbb{G}_N$

general element g of $\mathbb{G}_N$

Ex $N=1$ $(\theta, \bar\theta)$

$(1, \theta, \bar\theta, \theta\bar\theta)$ Basis

$N=2$ $\theta_1, \bar\theta_1, \theta_2, \bar\theta_2$

$1, \underbrace{\qquad\qquad}$, θ_1

1   4

general element g : linear comb of products of the θ's and $\bar{\theta}$'s of $\mathbb{G}_N$

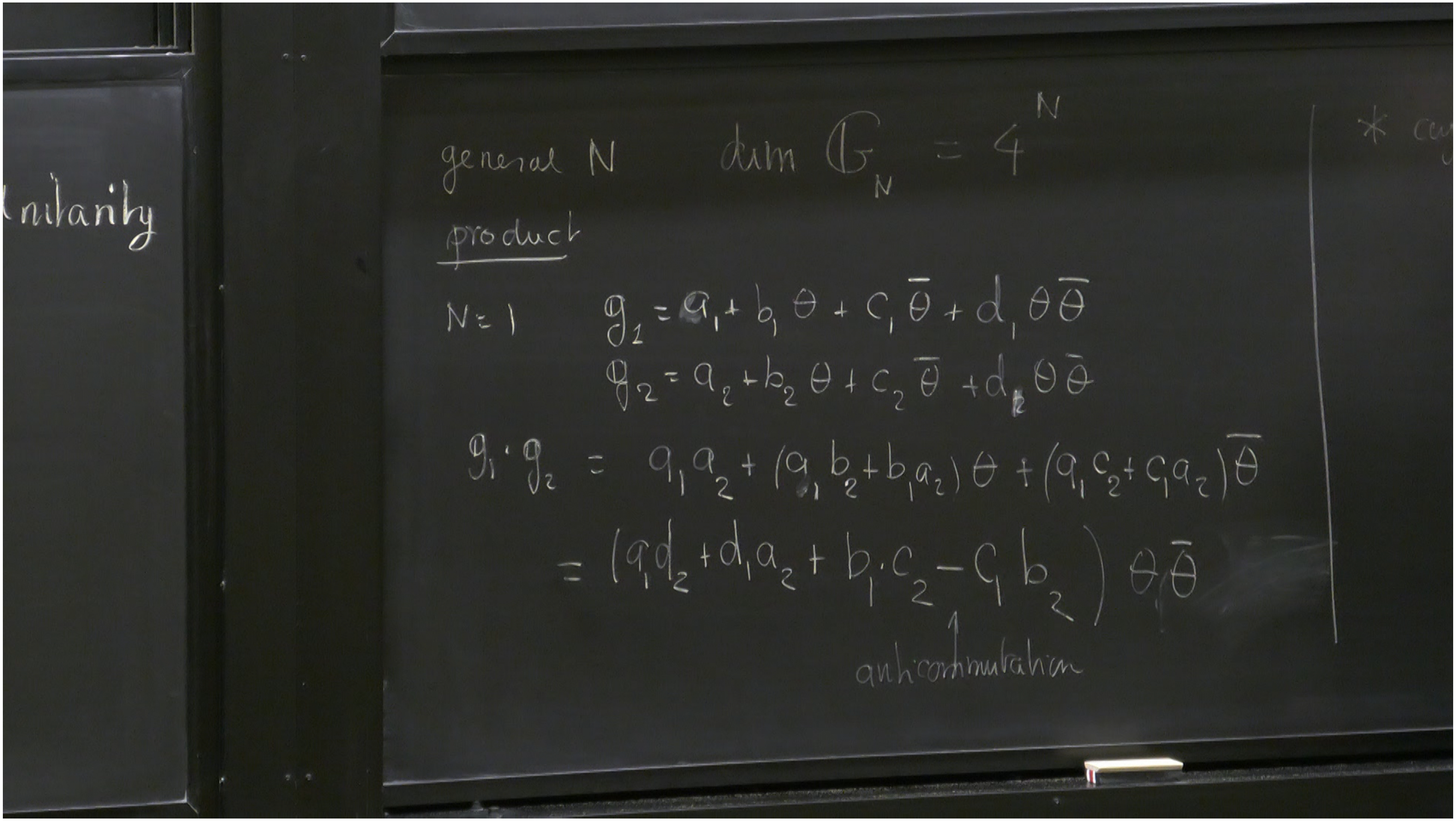
Ex $N=1$ $(\theta, \bar{\theta})$ $\quad g = a\,1 + b\,\theta + c\,\bar{\theta} + d\,\theta\bar{\theta}$

$(1, \theta, \bar{\theta}, \theta\bar{\theta})$ Basis $\quad \dim \mathbb{G}_1 = 4$

$N=2$ $\quad \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2 \quad g =$ lin. comb of 16 terms $\quad \dim \mathbb{G}_2 = 16$

$1, \quad \underbrace{\qquad\qquad}, \quad \theta_1\theta_2 \cdots\cdots \bar{\theta}_1\bar{\theta}_2, \quad \theta_1\theta_2\bar{\theta}_1, \quad \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$

$\#$ 1 $\qquad$ 4 $\qquad$ 6 $\qquad$ 4 $\qquad$ 1


general N dim $\mathbb{G}_N = 4^N$
product
$N=1$ $g_1 = a_1 + b_1\theta + c_1\bar{\theta} + d_1\theta\bar{\theta}$
$g_2 = a_2 + b_2\theta + c_2\bar{\theta} + d_2\theta\bar{\theta}$
$g_1 \cdot g_2 = a_1a_2 + (a_1b_2 + b_1a_2)\theta + (a_1c_2 + c_1a_2)\bar{\theta}$
$= (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\,\theta\bar{\theta}$
anticommutation

N $\quad \dim \mathbb{G}_N = 4^N$

ct

$$g_1 = a_1 + b_1\theta + c_1\bar\theta + d_1\theta\bar\theta$$

$$g_2 = a_2 + b_2\theta + c_2\bar\theta + d_2\theta\bar\theta$$

$$= a_1 a_2 + (a_1 b_2 + b_1 a_2)\theta + (a_1 c_2 + c_1 a_2)\bar\theta$$

$$= (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2)\,\theta\bar\theta$$

anticommutation

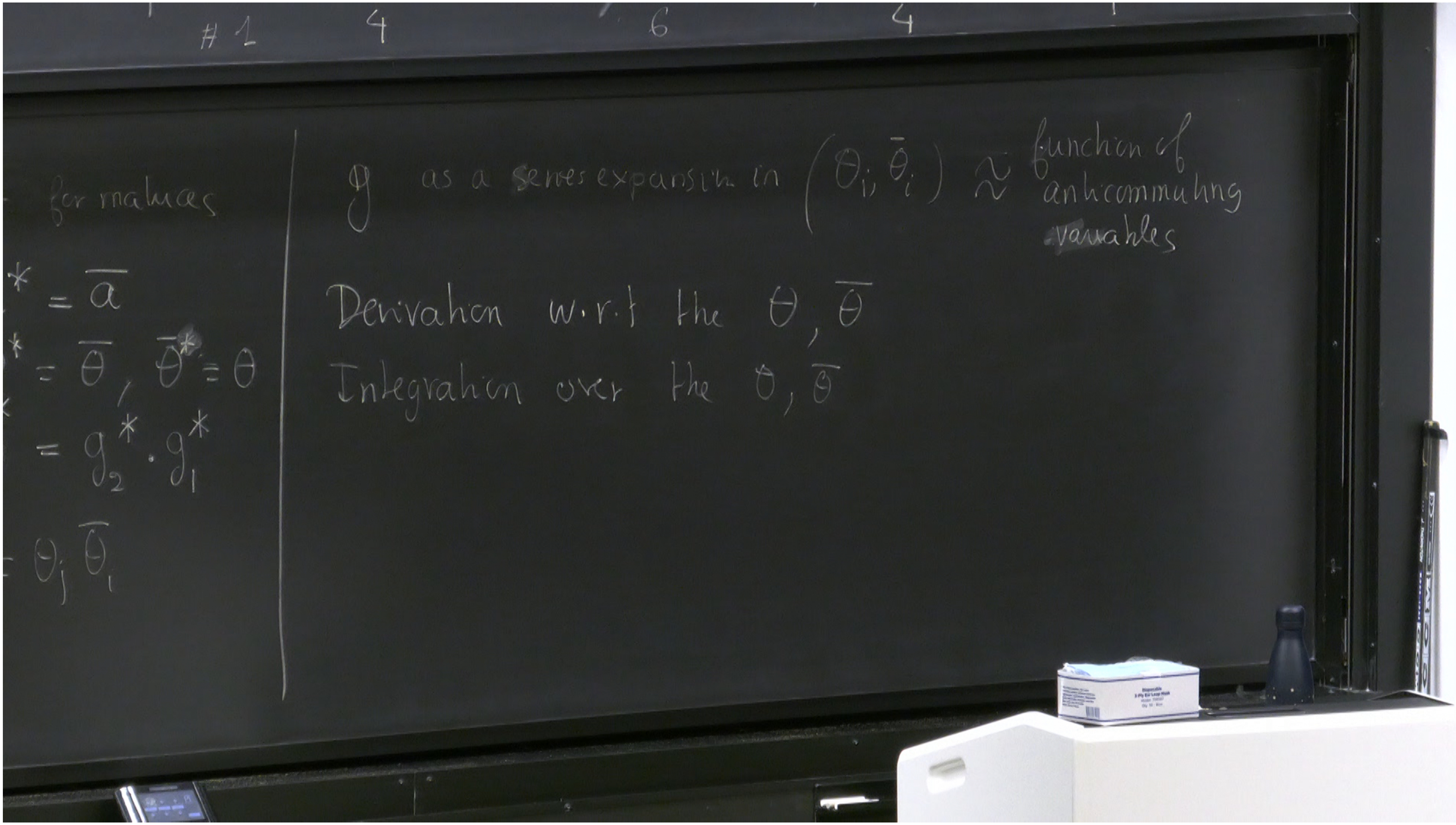
$*$ conjugation $\longleftrightarrow$ $\dagger$ for matrices

complex number $\quad a^* = \bar a$

generators $\quad \theta^* = \bar\theta, \quad \bar\theta^* = \theta$

product $\quad (g_1 \cdot g_2)^* = g_2^* \cdot g_1^*$

for instance $\quad (\theta_i \bar\theta_j)^* = \theta_j \bar\theta_i$



1
4
6
4
for matrices
g as a series expansion in (θ_i, θ̄_i) ≈ function of anticommuting variables
Derivation w.r.t the θ, θ̄
Integration over the θ, θ̄
* = ā
* = θ̄, θ̄* = θ
= g_2* · g_1*
θ_j θ̄_i

anticommutation

$$\mathbb{G}_N \leftarrow (\theta_i, \bar{\theta}_i)_{i=1,N}$$

$$\frac{\partial}{\partial \theta_i} g =$$

$$\frac{\partial}{\partial \theta_i} 1 = 0, \quad \frac{\partial}{\partial \theta_i} \theta_j = \delta_{ij}, \quad \frac{\partial \bar{\theta}_j}{\partial \theta_i} = 0$$

$$\frac{\partial}{\partial \bar{\theta}_i} 1 = 0, \quad \frac{\partial}{\partial \bar{\theta}_i} \bar{\theta}_j = \delta_{ij}, \quad \frac{\partial \theta_i}{\partial \bar{\theta}_j} = 0$$

Anticommutation rel

$$\left\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\} = 0$$

$$\left\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \bar{\theta}_j} \right\} = 0$$

$$\left\{ \frac{\partial}{\partial \bar{\theta}_i}, \frac{\partial}{\partial \bar{\theta}_j} \right\} = 0$$

ple

$\boxed{\text{true if } i \neq j}$

$$\left(\theta_i \theta_j\right) = -\frac{\partial}{\partial \theta_j}\left(\theta_j \theta_i\right) = -\theta_i$$

$$\left(\theta_k \theta_j \theta_i\right) = \frac{\partial}{\partial \theta_i}\left(\theta_i \theta_k \theta_j\right) = +\theta_k \theta_j$$

$*\}$ move the θ_i to the right
apply derivative

In general

$$\boxed{\begin{gathered} g = \sum_{I,J} a_{I,J} \prod_{i=1}^{K} \theta_{a_i} \prod_{j=1}^{L} \bar{\theta}_{b_j} \\ I = \{a_1 < a_2 < a_3 \cdots < a_K\} \\ J = \{b_1 < b_2 < b_3 \cdots < b_L\} \\ K \text{ and } L \leq N \end{gathered}}$$

$$\frac{\partial}{\partial \theta_a} g$$

1

Integration $\int d\theta_i \, g$

$$\int d\theta_i \frac{\partial}{\partial \theta_i} g = 0$$

$$\int d\bar{\theta}_i \frac{\partial}{\partial \theta_i} g = 0$$

$$\int d\theta_i \, 1 = 0 \qquad \int d\bar{\theta}_i \, 1 = 0$$

$$\int d\theta_i \, \theta_j = \delta_{ij}, \quad \int d\bar{\theta}_i \, \bar{\theta}_j = \delta_{ij}$$

$$\boxed{\int d\theta_i = \frac{\partial}{\partial \theta_i}}$$

Gaus

Gaussian integrals

$A = (A_{ij})$ Hermitean matrix

complex numbers

$$\mathcal{C} = \exp\Big(-\sum_{i,j=1}^{N} \bar{\theta}_i A_{ij} \theta_j\Big)$$

$$= 1 - \bar{\theta}_i A_{ij} \theta_j + \frac{1}{2}\left(\bar{\theta}_i A_{ij} \theta_j\right)^2 + \dots$$

becau

For instance $N=1$ $\exp(-\bar{\theta} A \theta) = 1 - A \cdot \bar{\theta}\,\theta = 1 + A\,\theta\,\bar{\theta}$

$$\int d\theta\, d\bar{\theta}\, \exp(-\bar{\theta} A \theta) = A$$

…grals

$$\mathcal{C} = \exp\Big(-\sum_{i,j=1}^{N} \bar\theta_i A_{ij} \theta_j\Big)$$

after N steps stops

…Hermitean
…matrix

$$= 1 + \bar\theta_i A_{ij} \theta_j + \frac{1}{2}(\bar\theta_i A_{ij} \theta_j)^2 + \dots$$

because $(\bar\theta_i A_{ij} \theta_j)^{N+1} = 0$

$$\exp(-\bar\theta A \theta) = 1 - A \cdot \bar\theta\, \theta = 1 + A\, \theta\, \bar\theta$$

$$\boxed{\dots\bar\theta A \theta) = A}$$

For $N = 2$ $\quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\qquad$ simple calculation

$$\int d\theta_1 d\bar\theta_1\, d\theta_2 d\bar\theta_2 \exp(-\bar\theta \cdot A \cdot A) = A_{11} A_{22} - A_{12} A_{21} = \det A$$

$\int d\theta_i \, 1 = 0 \quad \int d\bar{\theta}_i \, 1 = 0$

$\int d\theta_i \, \theta_j = \delta_{ij}, \quad \int d\bar{\theta}_i \, \bar{\theta}_j = \delta_{ij}$

$\int d\bar{\theta} \, d\theta \, \exp(-\bar{\theta} A \theta) =$

general N $\quad \int d\theta_1 d\bar{\theta}_1 \cdots d\theta_N d\bar{\theta}_N \exp\left(-\bar{\theta}_i A_{ij} \theta_j\right) = \det A$

"correlation functions" $\quad N=1 \quad \langle \bar{\theta}\theta \rangle := \dfrac{\int d\theta \, d\bar{\theta} \cdot \exp(-\bar{\theta} A \theta) \, \bar{\theta}\theta}{\int d\theta \, d\bar{\theta} \, \exp(-\bar{\theta} A \theta)} = \dfrac{1}{A}$

$N=2$ $\quad i,j = 1 \text{ or } 2$

$$\langle \bar\theta_i \theta_j \rangle := \frac{\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \exp(-\bar\theta A \theta) \cdot \bar\theta_i \theta_j}{\det A}$$

$$\langle \bar\theta_1 \theta_1 \rangle = \frac{A_{22}}{\det A} = (A^{-1})_{11}$$

$$\boxed{\langle \bar\theta_i \theta_j \rangle = (A^{-1})_{ij}}$$

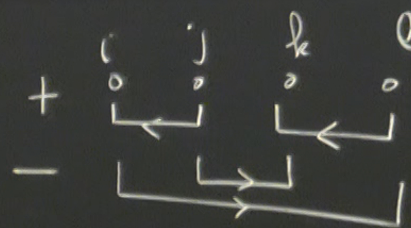
general
for any N
Wick Theorem

$$\boxed{\langle \theta_i \theta_j \rangle \text{ or } \langle \bar\theta_i \bar\theta_j \rangle = 0}$$

$\boxed{\langle \bar{\theta}_i \theta_j \rangle \text{ or } \langle \theta_i \theta_j \rangle = 0}$

$)_{kj}$

Fermionic Wick theorem!



det A = fermionic partition function

A^{-1} = fermionic propagator

Path integral For Dirac Field

$$S[\Psi, \bar{\Psi}] = \int d^4x \; \bar{\Psi}(x)\left(i \not{\partial}_x - m\right)\Psi(x)$$

$$\left(\Psi^\alpha(x), \bar{\Psi}^\alpha(x) \right) \quad \begin{array}{l} \alpha = 1, 4 \\ x \in M^{1,3} \end{array} \quad \text{Dirac space}$$

now are anticommuting generators of a Big Grassmann algebra

∞ dim

$2 \cdot 4 \times \mathrm{Vol}(M^{1,3}) \Rightarrow \mathbb{G}_{\mathrm{Dirac}}$

generators

$$Z = \int \mathcal{D}[\bar{\psi},\psi]\, \exp\left(\frac{i}{\hbar} S[\psi,\bar{\psi}]\right) = \det\left(\left(i\not{\partial} - m\right)\right)$$

Dirac operator 4×4 matrix

$$\int \prod_{x} \prod_{\alpha} d\psi^{\alpha}(x)\, d\bar{\psi}^{\alpha}(x)$$

$$\langle \psi^{\alpha}(x)\,\bar{\psi}^{\beta}(y) \rangle = \left(\frac{i\hbar}{i\not{\partial} - m}\right)^{\alpha\beta}_{xy}$$ = Feynman Propagator For Dirac

Can

$$\langle \bar{\theta}_i \theta_j \bar{\theta}_k \theta_l \rangle = (A^{-1})_{ij} (A^{-1})_{kl} - (A^{-1})_{il} (A^{-1})_{kj}$$

Fermionic Wick theorem!

$i \quad j \quad k \quad l$

$+$

$-$

e v

$$\text{Feynman Prop} = G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \left(\frac{i}{-\not{p} - m - i\epsilon_+} \right)$$

$\left. A^{-1} \right)_{kj}$

Fermionic Wick theorem!

$+$ i j k l

$-$

$$\left(\frac{i}{-\not{p} - m - i\epsilon_+} \right)$$

det A = fermionic partition function

A^{-1} = fermionic propagator

Bosons & Fermions are on the same footing ⇒ Gauge Mechanics + Fermions

Spin plays no role

Condensed matter

"Faddeev-Popov" "Ghost"

Spin 0 charged Fermions

⇒ "Super Space" SUSY

Math "Superalgebras"