Title: QFT2 Lecture - 112023

Speakers: Francois David

Collection: Quantum Field Theory 2 2023/24

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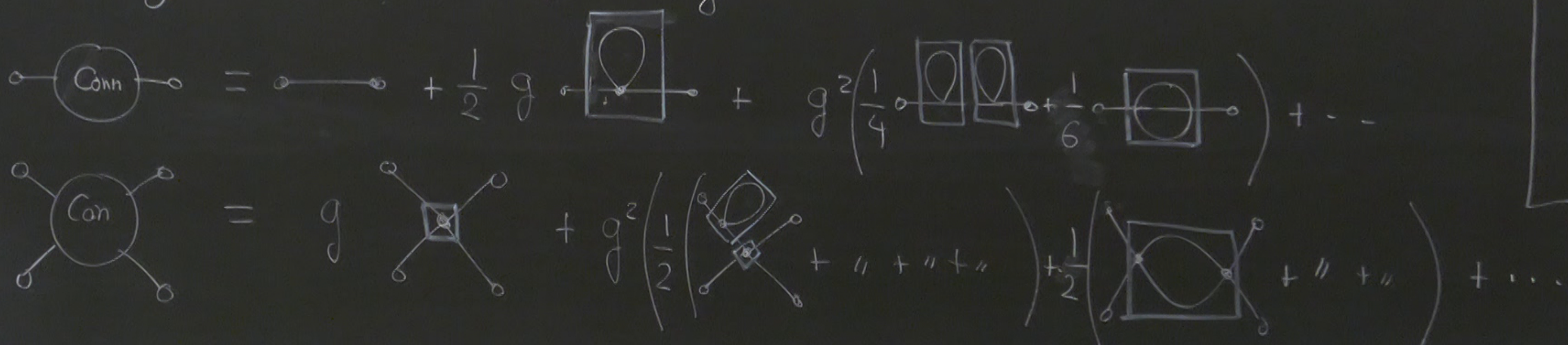
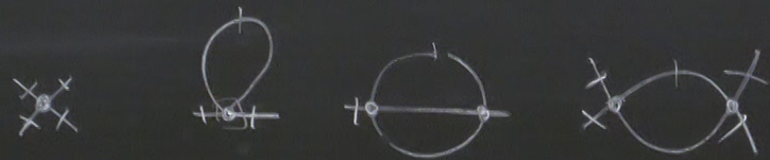
① Effective (Quantum) Action

$(-+++)$

Some diagrammatics $\qquad \phi^4$ theory





1 particle/line irreducible

1PI diagrams

Some notation

$\xrightarrow{p}$ $\quad \dfrac{1}{p^2+m^2}$ propagator $\quad (-\Delta+m^2)^{-1}$

1 truncated propagator $\quad (-\Delta+m^2)(-\Delta+m^2)^{-1} = 1$

p^2+m^2 "twice truncated" prop. $\quad (-\Delta+m^2)^2(-\Delta+m^2)^{-1} = (-\Delta+m^2)$

$) + \ldots$

Effective action (Euclidean)

$$Z[j] = \int D[\phi]\, e^{-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)}$$ gen. funct. for correlations $\hbar \frac{\delta}{\delta j(x)} \longrightarrow \phi(x)$

$$S[\phi] = \text{action} = \int d^d x \left(\frac{1}{2}\phi(-\Delta + m^2)\phi + V(\phi)\right)$$

ϕ "quantum field"

j classical source term

$$W[j] = \hbar \operatorname{Log}(Z[j])$$ connected generating functional

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle_j = \hbar^N \frac{\delta}{\delta j(x_1)} \frac{\delta}{\delta j(x_N)} Z[j] \Big/ Z[j]$$

N point correlation

$j \neq 0$

v.e.v for the theory when the source term is $\neq 0$

$$\langle \phi(x)\rangle_j = \frac{\delta}{\delta j(x)} W[j]$$

in the physical vaccum

$$\langle \phi(x)\rangle = \langle \phi(x)\rangle_j \Big|_{j=0}$$

A generating functional of a classical "background field" variable

$W[j]$, arbitrary $j = \{j(x)\}$

$$\langle \phi(x) \rangle_j = \frac{\delta W[j]}{\delta j(x)} := \varphi(x)$$ functional of the source term

Legendre Transform $j = \{j(x)\} \longrightarrow \varphi = \{\varphi(x)\}$ $\longleftarrow$

Assumption mapping is 1 to 1

$\varphi = \{\varphi(x)\}$ being

$\Gamma \; =$

"field" variable $\varphi(x)$ φ = \varphy (Tex), ϕ = \phy.

$\varphi = \{\varphi(x)\}$ being given $\Longrightarrow j = \{j(x)\} = ?$

$$\boxed{\begin{aligned} \Gamma[\varphi] &= j\cdot\varphi - W[j] \\ \varphi &= \frac{\delta W[j]}{\delta j} \end{aligned}}$$

with $j\cdot\varphi = \int d^dx\, j(x)\,\varphi(x)$

$\boxed{\Gamma[\varphi] \text{ is "effective action" functional of } \varphi}$

$\varphi(x)$ $\qquad \varphi =$ \varphy (Tex) $\quad \phi =$ \phy.

...g given $\Longrightarrow j = \{ j(x) \} = ?$

$\cdot \varphi - W[j]$

with $j \cdot \varphi = \int d^d x \, j(x) \varphi(x)$

$\Gamma[\varphi]$ is "effective action" functional of φ

$$\boxed{W[j] = \varphi \cdot j - \Gamma[\varphi] \qquad j = \frac{\delta \Gamma[\varphi]}{\delta \varphi}}$$

$$(W, j) \longleftrightarrow (\Gamma, \varphi)$$

Legendre ∘ Legendre = Identity

$$\langle \phi(x) \rangle_{j=0} = \varphi(x) \quad \text{with } \varphi_0(x) \text{ such that } \left.\frac{\delta \Gamma}{\delta \varphi(x)}\right|_{\varphi = \varphi(x)} = 0$$

e.v of the field in the original theory $=$ stationary point of the effective action

$$\varphi = \frac{\delta W}{\delta j} \qquad \frac{\delta \varphi}{\delta j} = \frac{\delta W}{\delta j \, \delta j} = W^{(2)} \qquad \boxed{\frac{\delta \varphi}{\delta j} = \frac{1}{\frac{\delta j}{\delta \varphi}}}$$

$$j = \frac{\delta \Gamma}{\delta \varphi} \qquad \frac{\delta j}{\delta \varphi} = \frac{\delta \Gamma}{\delta \varphi . \delta \varphi} = \Gamma^{(2)}$$

partial derivatives

$\langle$

2p

con

$$\langle \phi(x)\,\phi(y)\rangle_{\text{CONNECTED}} = \langle \phi(x)\,\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle$$

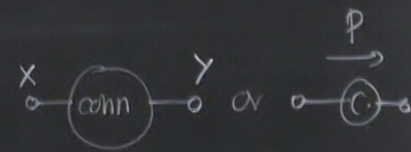
F.T. momentum spa

$$\hat{W}^{(2)}(p)$$

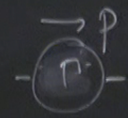
$$\parallel$$

2pt function connected

$$= \hbar^2 \frac{\delta}{\delta j(x)}\,\frac{\delta}{\delta j(y)}\,W[j] = W^{(2)}(x,y) =$$



$$\hat{\Gamma}^{(2)}(p)$$



$$\Gamma^{(2)}(x,y) = \frac{\delta}{\delta\varphi(x)}\,\frac{\delta}{\delta\varphi(y)}\,\Gamma$$

inverse of $W^{(2)}$

$$W^{(2)}(p)\,\Gamma^{(2)}(p) = 1$$

$$\boxed{W^{(2)}\cdot\Gamma^{(2)} = \mathrm{Id}, \quad \int dy\; W^{(2)}(x,y)\,\Gamma^{(2)}(y,z) = \delta(x-z)}$$

$$\boxed{\Gamma^{(2)}(p) = \frac{1}{W^{(2)}(p)}}$$

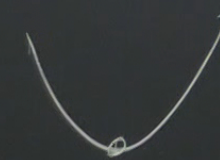
Transl. invariant $\Rightarrow$ FT in p only

$$W^{(2)}(x,y) = W^{(2)}(x-y)$$

$$\boxed{\hat{W}^{(2)}(p) = \int dx\, e^{-ipx}\, W^{(2)}(x)}$$

$$\hat{\rho}^{(2)}(p) = \int dx\, e^{-ipx}\, \rho^{(2)}(x)$$

a geneal $W[j]$



Its L.Tr. $\Gamma(\varphi)$ is <u>convex</u>

a general $W[j]$

Its L.Tr. $\Gamma(\varphi)$ is convex

$\langle \varphi \varphi \rangle \Rightarrow$ positive fun.

Transl. invariant ⇒ FT in p only

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$$W[j] = \log Z(j)$$

$$\langle \phi(x) \rangle_j = \frac{\delta Z(j)}{\delta j(x)} \Big/ Z(j) = \left(\frac{\delta \exp(W[j])}{\delta j(} \right) \Big/ e^{W[j]}$$

$$\langle \phi(x)\phi(y) \rangle = \langle \phi(\quad (y) \rangle$$

$$\langle \phi(x)\,\phi(y) \rangle_{\text{CONNECTED}} = \langle \phi(x)\,\phi(y)\rangle - \langle \phi(x)\rangle \langle \phi(y)\rangle$$

F.T. momentum spa[ce]

$$\hat{W}^{(2)}(p) =$$

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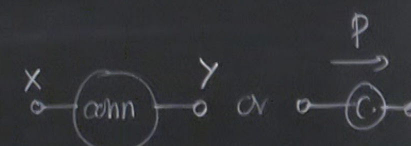
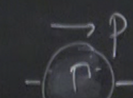


$$\hat{\Gamma}^{(2)}(p)$$



$$\Gamma^{(2)}(x,y) = \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} \Gamma$$

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partial derivatives

4

Now: actual calculations : $\Gamma[\phi]$ at 1^{st} order in $\hbar$ for a general theory

ϕ field

$S[\phi]$ classical action

$$\frac{\delta S[\phi]}{\delta \phi(x)} = S'[\phi](x) = \left(-\Delta_x \phi(x) + V'(\phi(x))\right)$$ Function

$$\frac{\delta S[\phi]}{\delta \phi(x)\delta \phi(y)} = S''[\phi](x,y) = \left(-\Delta + V''(\phi)\right)_{x,y}$$ ← Kernel of this operator — Diff operator

Main result

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$$\boxed{\Gamma[\varphi] = S[\varphi] + \hbar\,\frac{1}{2}\log\left(\det\left[S''[\phi]\right]\right) + O(\hbar^2)}$$

general formula

for bosons spin 0, 1, 2 ...

for fermions $\frac{1}{2} \longrightarrow -1$

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Hessian of S

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bosons spin 0, 1, 2 -

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Semi-classical calculation

$$\boxed{W[j] = -S[\phi_c] - \frac{\hbar}{2}\log\det S''[\phi_c]}$$

$$e^{\frac{1}{\hbar}W[j]} = \int D[\phi]\exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$$

$\hbar \to 0$ look for $S'[\phi] - j = 0 \Rightarrow \phi_c[j]$

Hessian at the classical sol ϕ_c

$$\phi = \underbrace{\phi_c[j]}_{\text{classical part}} + \underbrace{\hbar^{1/2}\tilde{\phi}}_{\text{quantum part}}$$

$$S[\phi] - j\phi = S[\phi_c] + \frac{\hbar}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi} + O(\hbar^{3/2})$$

$$= \exp\left(-\frac{1}{\hbar}S[\phi_c]\right) \times \det\left(S''[\phi_c]\right)^{-1/2}\left(1 + O(\hbar^{1/2})\right)$$

l theory

Function

lf the Diff operator
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$$\phi = \phi_c[j] + \hbar^{1/2}\tilde{\phi}$$

classical part, quantum part

$$S[\phi] - j\phi = S[\phi_c] + \frac{\hbar}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi} + O(\hbar^{3/2})$$

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a general $W[j]$

Its L.Tr. $\Gamma(\varphi)$ is convex

$\langle \varphi \varphi \rangle \Rightarrow$ positive func.

$\phi_c(j)$

$\phi = \phi_c + \tilde{\epsilon}$

a general $W[j]$ 

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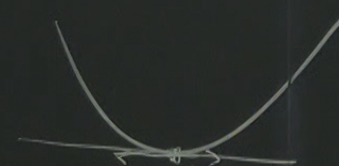

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operator

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$\to -1$

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$$S[\phi]-j\phi = \boxed{S[\phi_c] - j[\phi_c]} + \boxed{\frac{\hbar}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi}} + O(\hbar^{3/2})$$

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...chon

operator

a

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$$S[\phi] - j\phi = \boxed{S[\phi_c] - j\cdot\phi_c} + \boxed{\frac{\hbar}{2} \tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi}} + O(\hbar^{3/2})$$

$$= \exp\left(-\frac{1}{\hbar} S[\phi_c]\right) \times \det\left(S''[\phi_c]\right)^{-1/2} \left(1 + O(\hbar^{1/2})\right) \text{ no } \tilde{\phi}$$

for ϕ^4: $S[\phi] = \int \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$

$S''[\phi]\cdot F(x) =$

$$\boxed{S''[\phi] = -\Delta + m^2 + \frac{g}{2}\phi^2}$$

Diff operator

Pert Theory :

$$\mathrm{Log}\left(\mathrm{Det}\left(S''[\varphi]\right)\right) = \mathrm{Tr}\left(\log\left(S''[\varphi]\right)\right)$$

$$\mathrm{Tr}\,\mathrm{Log}(\ ) = \sum_{K=}^{\infty}$$

$$S''[\varphi] = (-\Delta + m^2)\left(1 + \frac{g}{2}\frac{1}{(-\Delta+m^2)}\varphi^2\right)$$

write th

$$\mathrm{Tr}\left(\log\left(S''[\varphi]\right)\right) = \mathrm{Tr}\left[\log\left(-\Delta+m^2\right)\right] + \mathrm{Tr}\,\log\left(\frac{g}{2}\frac{1}{-\Delta+m^2}\varphi^2 + 1\right)$$

Tr (

$$\ldots F(x) = -\Delta F(x) + \left(m^2 + \frac{g}{2}\varphi^2(x)\right) F(x) \qquad \left(\frac{1}{-\Delta+m^2}\right)_{xy} = G(x-y) = \underset{x \qquad\quad y}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet} \qquad (\varphi^2)_{xy} = \delta(x-y)\cdot\varphi^2(x)$$

…heory : expanding in g

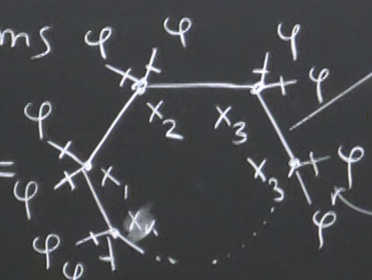
$$\ldots(\;) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{g}{2}\right)^k \mathrm{Tr}\Big[\underbrace{\left(\frac{1}{-\Delta+m^2}\varphi^2\right)\cdots\left(\frac{1}{-\Delta+m^2}\varphi^2\right)}_{k\text{ times}}\Big]$$

write this in position space

$$\mathrm{Tr}\big(\underset{k}{\cdots}\big) = \int d^dx_1 \cdots \int d^dx_k \; G(x_1-x_2)\,\varphi^2(x_2)\,G(x_2-x_3)\,\varphi^2(x_3)\cdots G(x_k-x_1)\,\varphi^2(x_1)$$

*

Feynman diagrams

$\mathrm{Tr}\left(\underbrace{\sim}_{K}\right) =$ 

ordinary propagators

truncated propagators

1 loop diagrams

$$\text{Term of order } \hbar = \sum_{k=1}^{\infty} \frac{g^{k}(-1)^{K}}{2^{k} K}$$ 

1 loop irreducible diagrams

$$\varphi^2(x_2) = \int dy_2 \int dz_2\, \delta(y_2 - x_2)\,\varphi(y_2)\cdot \delta(z_2 - y_2)\cdot \varphi$$

$K=1$  Tadpole diagram — 2 pt function

$K=2$  "bubble" diagram — 4 pt function

$K=3$  Triangle diagram. — 6 pt function

$$\varphi^2(x_2) = \int d y_2 \int d z_2 \; \underbrace{\delta(y_2 - x_2)}_{+} \varphi(y_2) \cdot \underbrace{\delta(z_2 - y_2)}_{+} \cdot \varphi(z_2)$$

Tadpole diagram — 2 pt function

"bubble diagram — 4 pt function

Triangle diagram. — 6 pt function

$\Gamma[\varphi] \longleftrightarrow$ generating functional for 1PI diagrams

general $V(\phi)$

at higher orders in $\hbar$

higher loop diagrams

Legendre transform.

$\Updownarrow$

Combinatorics