Title: QFT2 Lecture - 111523

Speakers: Francois David

Collection: Quantum Field Theory 2 2023/24

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Generating Functionals $\leftarrow$ path/functional integral qu

Scalar Field ϕ mass m interaction $V(\phi)$

$\mathbb{R}^d$ Euclidean space time $x_E = (x^0, \vec{x})$ $ds^2 = (dx^0)^2 + (d\vec{x})^2$

$$S_E[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi(x))^2 + V(\phi(x)) \right]$$

l quantization $c = 1$ $(- + \cdots +)$ signature "East Coast metric"

wick rotation $t \leftrightarrow -ix^0 \Leftrightarrow x^0 \leftrightarrow it$ $(+ - \cdots -)$ "West " "

Minkowski $\mathcal{M}^{1,d-1}$ $X = (t, \vec{x})$ $ds^2 = -(dt)^2 + (d\vec{x})^2$

$$S[\phi] = \int_{\mathcal{M}^{1,d-1}} d^{d-1}X \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_{\vec{x}} \phi)^2 - V(\phi)\right]$$

local Field operator

"Green Functions" $\mathcal{G}_N(X_1, \ldots, X_N) \langle \Omega | T\, \phi(X_1) \cdots \phi(X_N) | \Omega \rangle$

$$g_N(x_1, \dots, x_N) = \langle \Omega | T\,\phi(x_1) \cdots \phi(x_N) | \Omega \rangle$$

$$g_N(x_1 \cdots x_N) = \frac{\int D[\phi]\, e^{\frac{i}{\hbar} S[\phi]}\, \phi(x_1) \cdots \phi(x_N)}{\int D[\phi]\, e^{\frac{i}{\hbar} S[\phi]}}$$

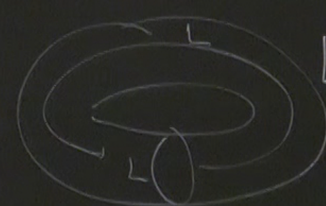
$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \mathcal{G}_N^{(E)}(x_1,\dots x_N)$$

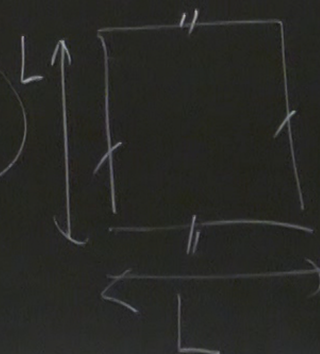
$$= \frac{\int D[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)\phi(x_1)\cdots\phi(x_N)\rangle}{\int D[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)}$$

Boundary conditions

Euclidean spacetime $T^d = \left(\frac{\mathbb{R}}{L\mathbb{Z}}\right)^d$

periodic boundary cond



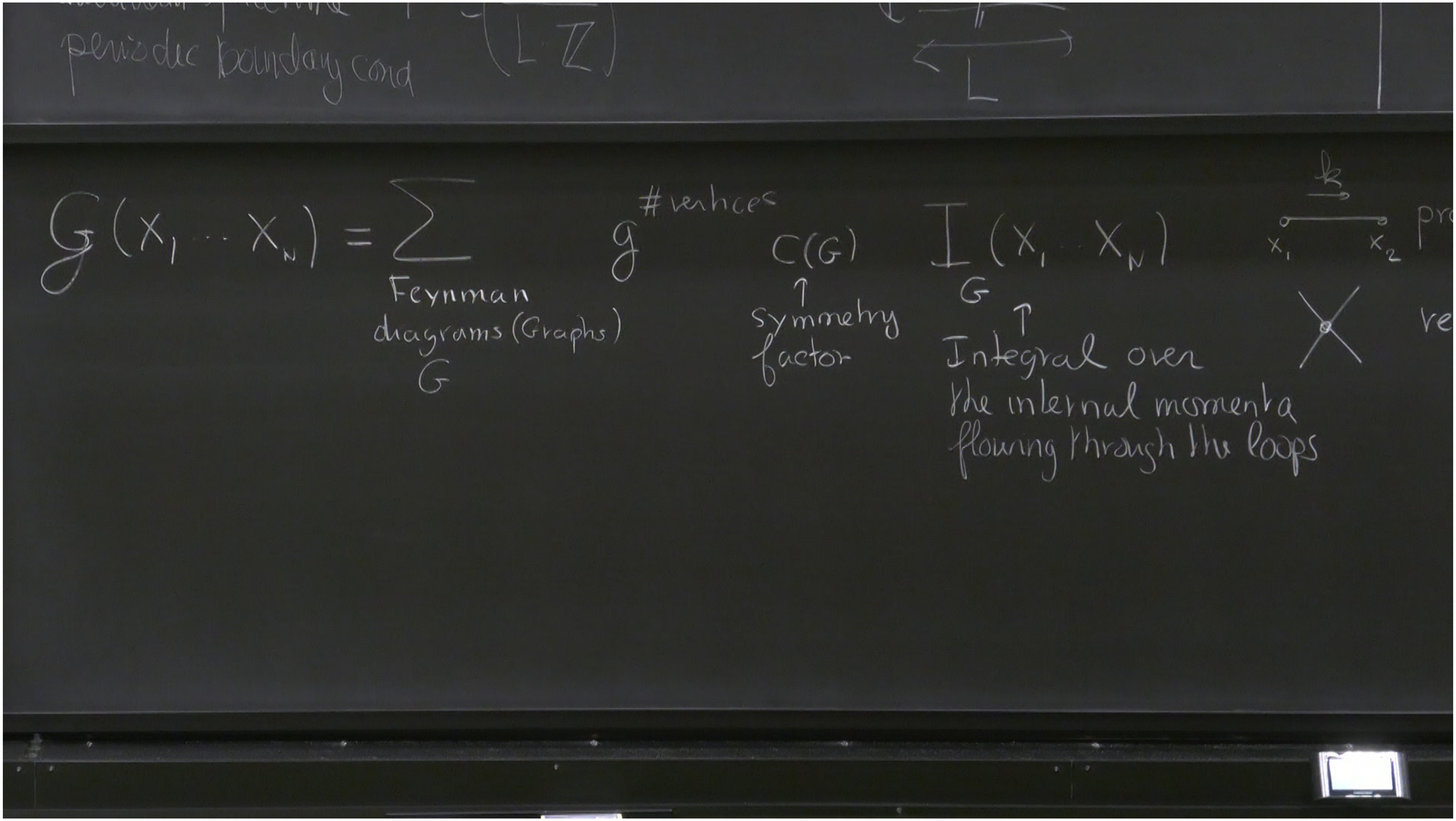


Take the limit $L \to \infty$

$$g_N(x_1 \dots x_N) = \frac{\int D[\phi]\, e^{\frac{i}{\hbar} S[\phi]}\, \phi(x_1) \cdots \phi(x_N)}{\int D[\phi]\, e^{\frac{i}{\hbar} S[\phi]}}$$

$L \to \infty \approx \text{Temp} \to 0 \Rightarrow$ project on the vacuum state $|\Omega\rangle$

Wick Theorem + Perturbation theory $\Rightarrow$ Feynman Diagrammatics


periodic boundary cond
(L·Z)
L
$\mathcal{G}(x_1 \cdots x_N) = \sum g^{\#\text{vertices}} \; C(G) \; I_G(x_1 \cdots x_N)$
Feynman diagrams (Graphs) G
symmetry factor
Integral over the internal momenta flowing through the loops
k
x_1
x_2

Classical source term : field $j(x)$ $\quad j\cdot\phi = \int d^dx\; j(x)\,\phi(x)$

generating function $Z[j]$

Euclidean

$$Z_E[j] = \int D[\phi]\, \exp\left(-\frac{1}{\hbar}\left[S_E[\phi] - j\cdot\phi\right]\right)$$

Functional derivative $\displaystyle \frac{\delta}{\delta j(x)} \exp\left(\frac{1}{\hbar} j\cdot\phi\right) = \frac{1}{\hbar}\,\phi(x)\, \exp\left(\frac{1}{\hbar} j\cdot\phi\right)$

Minkowski

$$Z[j] = \int \mathcal{D}[\phi] \, \exp\left(\frac{i}{\hbar} \left(S[\phi] + j \cdot \phi \right) \right)$$

$$\frac{\delta}{\delta j(x)} \longrightarrow \frac{i}{\hbar} \phi(x)$$

(Euclidean)
Expand in j

$$\mathcal{Z}[j] = \sum_{N=0}^{\infty} \frac{\hbar^{-N}}{N!} \int d^d z_1 \cdots d^d z_N \; j(z_1) \cdots j(z$$

$$G_N(z_1 \cdots z_N) = \frac{\mathcal{Z}_N(z_1 \cdots z_N)}{\mathcal{Z}_0}$$

Equivalently:

$$\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} \mathcal{Z}[j] \Big|_{j=0} = \mathcal{Z}_N(z_1, \cdots z_N)$$

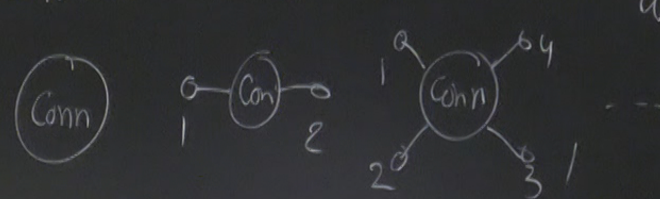
$$W[j] = \hbar \cdot \mathrm{Log}\left[\mathcal{Z}[j]\right] \Leftrightarrow \mathcal{Z}[j] = ex$$

$$\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} W[j] \Big|_{j=0} = \hbar^{1-N} W_N(z_1 \ldots z_N) =$$

$$\ldots] = \exp\left(\frac{1}{\hbar} W[j]\right)$$

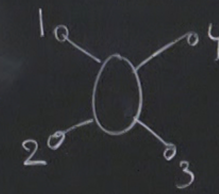
$$= \sum_{\substack{\text{all connected} \\ \text{Feynman diagrams}}}$$

N = 0 N = 2 N = 4



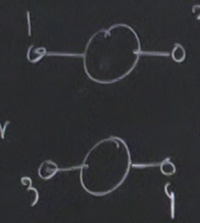
$$Z_N(z_1, \ldots z_N) = \sum_{\substack{\text{all Feyn. diagrams} \\ \text{with N external legs}}}$$

4 pts



or

2pt.



or

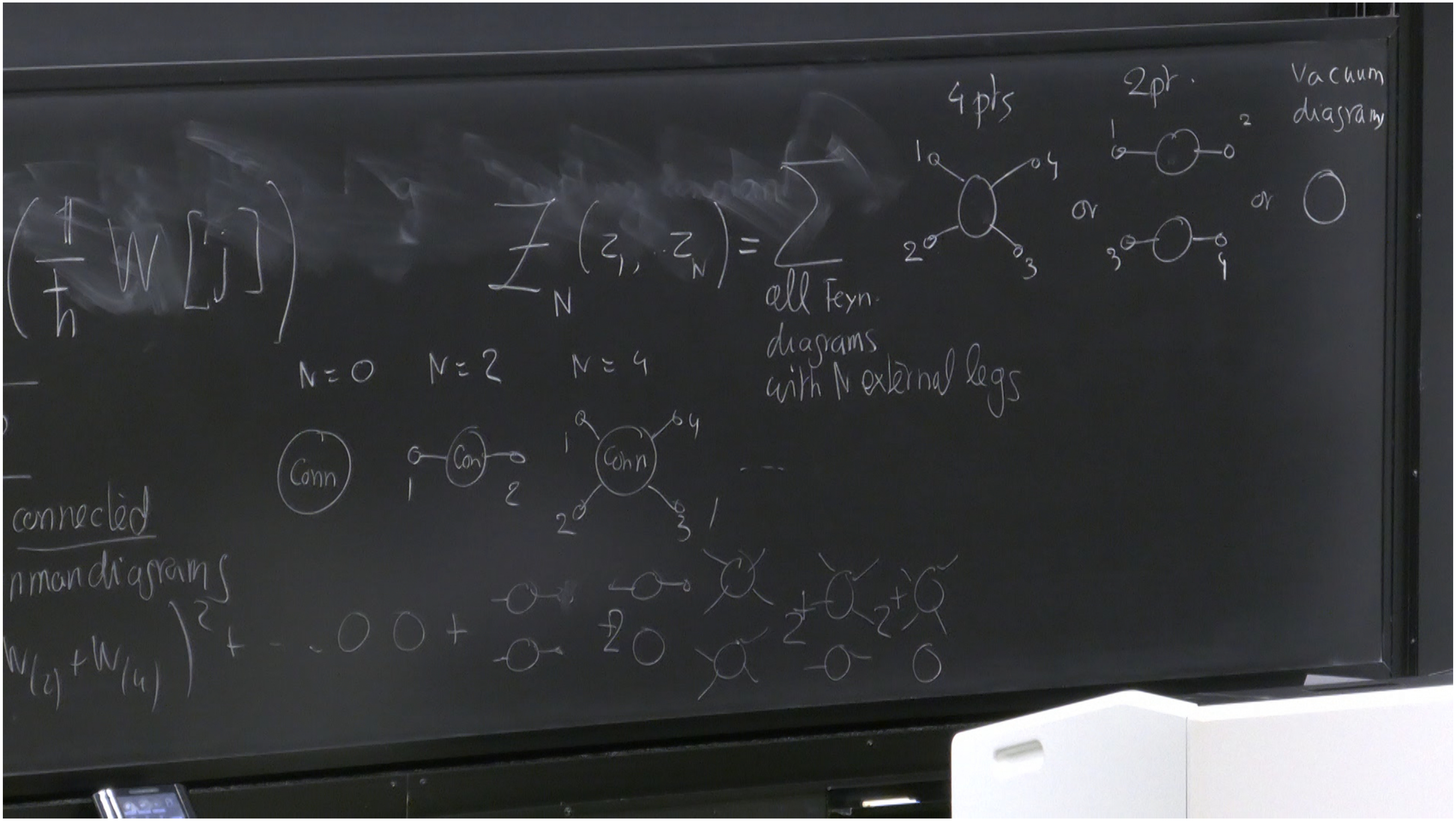
Vacuum diagrams

$$\Big|_{j=0}$$

$$W[j] = \hbar \cdot \mathrm{Log}\left[Z[j]\right] \Leftrightarrow Z[j] = \exp\left(\frac{1}{\hbar} W[j\right.$$

$$\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} W[j]\Big|_{j=0} = \hbar^{1-N} W_N(z_1 \dots z_N) = \sum_{\substack{\text{all connected} \\ \text{Feynman diagrams}}}$$

$$Z = \exp\left(W_{N=0} + W_{N=2} + W_{N=4} \dots\right) = 1 + W_{(0)} + W_{(2)} + W_{(4)} + \frac{1}{2}\left(W_{(0)} + W_{(2)} + W_{(4)}\right)^2 +$$


4 pts
2pt.
Vacuum
diagrams
or
or
all Feyn.
diagrams
with N external legs
N=0
N=2
N=4
Conn
Con
Conn
connected
nman diagrams

Free Field $S[\phi] = \int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2\right] = \int d^dx\, \frac{1}{2}\,\phi\,\underbrace{(-\Delta +}_{\text{operator}}$

$= $ quadratic form for ϕ

$$Z[j] = \int D[\phi]\, \exp\left(-\frac{1}{\hbar}\left(\frac{1}{2}\,\phi\,(-\Delta + m^2)\,\phi + j\cdot\phi\right)\right) = Z[0$$

nothing but a Gaussian Integral

with $Z[0$

$$\underbrace{(-\Delta + m^2)}_{\text{operator}}\phi \qquad \Delta = \sum_{\mu}\left(\frac{\partial}{\partial x^{\mu}}\right)^2$$

Laplace Beltrami operator (Euclidean)

(spectral analysis & operator theory)

Klein-Gordon operator (Minkowski)

$$Z[0]\cdot\exp\left(\frac{1}{\hbar}\, j\cdot(-\Delta+m^2)^{-1}\cdot j\right)$$

$$Z[0] = \left(\det\left[\frac{(-\Delta+m^2)}{\hbar\cdot 2\pi}\right]\right)^{1/2}$$

Functional determinant

$$W[j] = W[\phi] +$$

$\downarrow$ x to $\hbar$

$$j\cdot(-\Delta+m^2)^{-1}\cdot j = \int d^dx\, d^dy\; j(x)\, j(y)\, G(x,y)$$

Kernel of the operator $\dfrac{1}{-\Delta+m^2}$

$$(-\Delta+m^2)\cdot(-\Delta+m^2)^{-1} = \mathrm{Id}$$

$x, y \in \mathbb{R}^d$

$$(-\Delta_x+m^2).\; G(x,y) = \delta^d(x-y)$$

translation invariance $G(x,y) = G(x-y)$

$\hat{G}(k)$

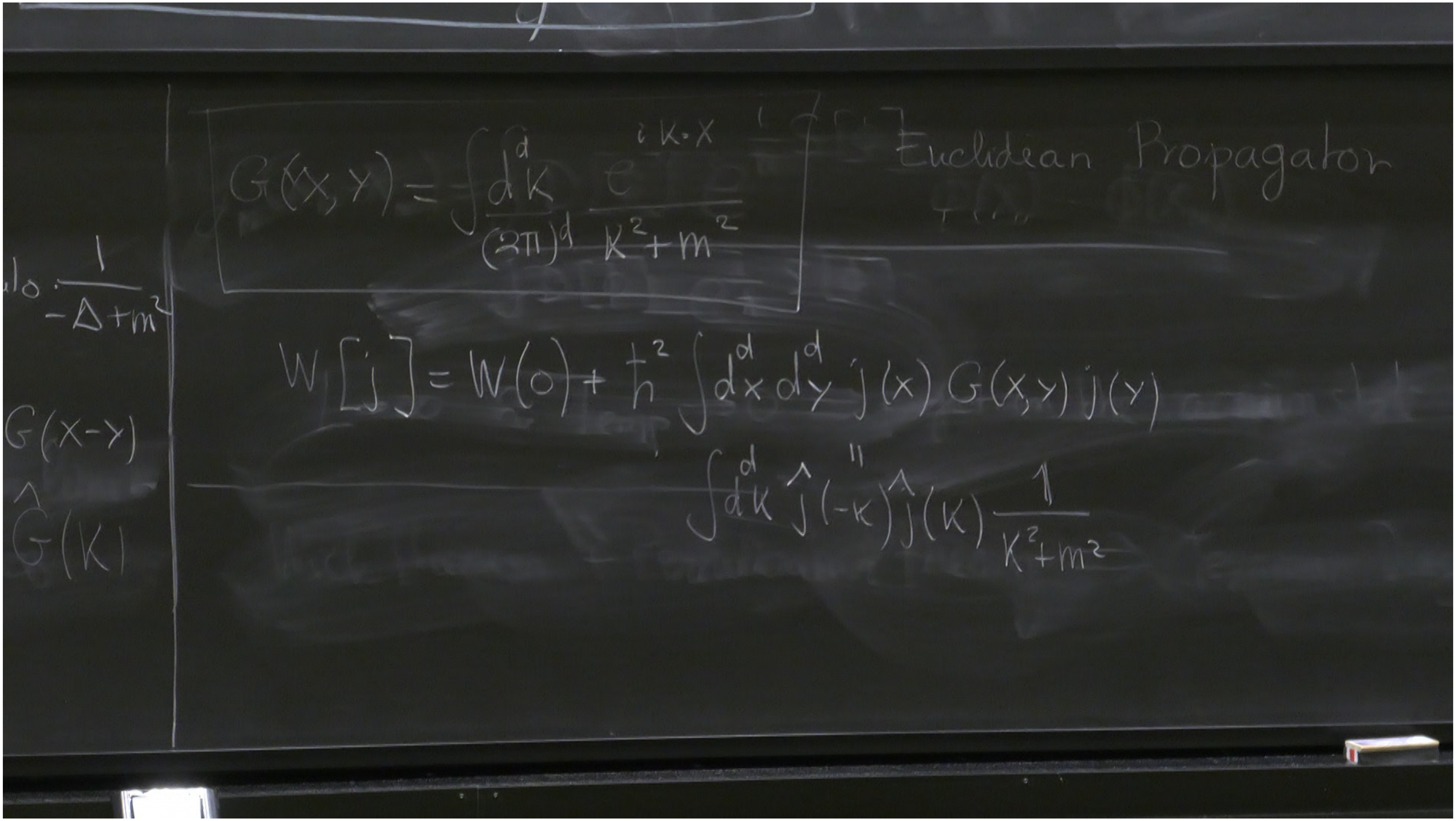
Fourier transform:

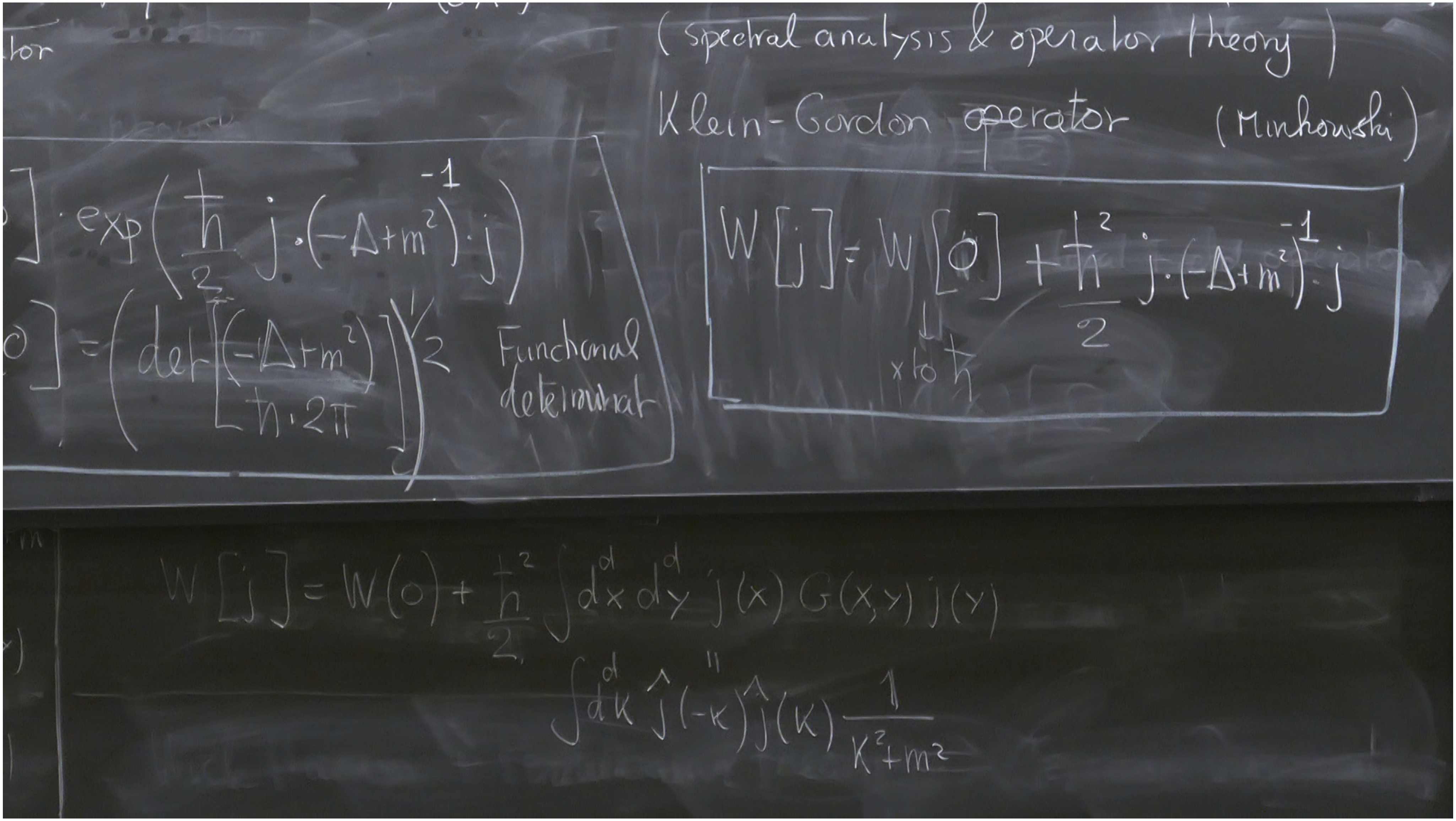
$$(k^2+m^2)\times\hat{G}(k) = 1$$

$$\boxed{\hat{G}(k) = \frac{1}{k^2+m^2}}$$

k d-dim covector

Lorentzian in $\mathbb{R}^d$


$G(x,y) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{k^2+m^2}$
Euclidean Propagator
$\frac{1}{-\Delta+m^2}$
$W[j] = W(0) + \hbar^2 \int d^dx\, d^dy\, j(x)\, G(x,y)\, j(y)$
$= \int d^dk\, \hat{j}(-k)\, \hat{j}(k) \frac{1}{k^2+m^2}$
$G(x-y)$
$\hat{G}(k)$



(Spectral analysis & operator theory)
Klein-Gordon operator (Minkowski)
$W[j] = W[0] + \frac{\hbar^2}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j$
x to $\hbar$
$] \cdot \exp\left(\frac{\hbar}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j\right)$
$0] = \left(\det\left[\frac{(-\Delta + m^2)}{\hbar \cdot 2\pi}\right]\right)^{-1/2}$
Functional determinant
$W[j] = W(0) + \frac{\hbar^2}{2} \int d^dx\, d^dy\, j(x)\, G(x,y)\, j(y)$
$= \int d^dk\, \hat{j}(-k)\, \hat{j}(k) \frac{1}{k^2 + m^2}$

Correlation functions $\langle \phi(z_1) \cdots \phi(z_N) \rangle_0 = \frac{1}{\hbar}\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} \exp\left(\frac{1}{2}\, j\cdot(-\Delta+m^2)^{-1}\cdot j\right)\Big|_{j=0}$

$\hbar = 1$ $\qquad N = 2M$ $\qquad$ Free Theory

$$= \sum_{M\text{-pairing}} \prod_{1}^{M} (-\Delta+m^2)^{-1}_{z_{i_1} z_{i_2}}$$

Wick Theorem from

$\underbrace{i_1 \; i_2}\, \underbrace{i_3 \; i_4}$

$\langle \phi(z_1)\phi(z_2) \rangle = G(z_1-z_2) =$ 1 —— 2

$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4) \rangle = G(z_1-z_2)G(z_3-z_4) + G(z_1-z_3)G(z_2-z_4) - G(z_1-z_4)G(z_2-z_3)$

$$\exp\left(\frac{1}{2}\, j\cdot(-\Delta+m^2)^{-1}\cdot j\right)\Big|_{j=0}$$

$$m^2)^{-1}_{z_{i_1}, z_{i_2}}$$

$\boxed{\text{Wick Theorem from Gaussian integrals}}$ instead of from $[a, a^\dagger] = 1$

$$G(z_2-z_4) - G(z_1-z_4)\,G(z_2-z_3)$$