Title: Relativity Lecture - 110623

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Collection: Relativity 2023/24

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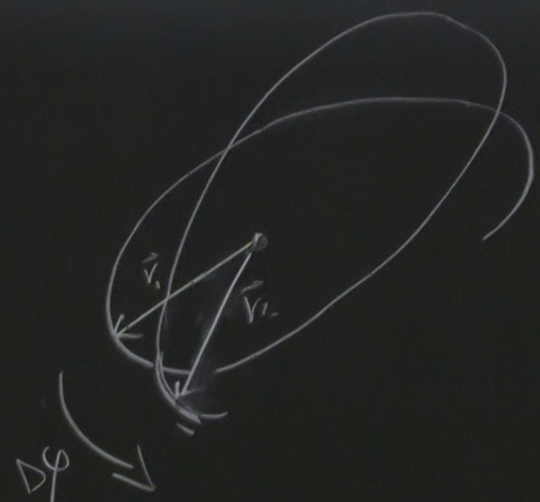
Perihilion Shift

$f(r)$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2 d\Omega^2$$

$$d\theta^2 + \sin\theta^2\, d\varphi^2$$



$\theta = \pi/2$

$k = \partial_t$ $\qquad m = \partial_\varphi$

$$u^\mu = \frac{dx^\mu}{d\tau} = (\dot{t}, \dot{r}, 0, \dot{\varphi})$$

$E = -k_\mu u^\mu$ $\qquad L = m_\mu u^\mu$

$u^\mu u_\mu = -1$

$E = f\dot{t} \qquad L = r^2\dot{\phi}$

$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2}$$

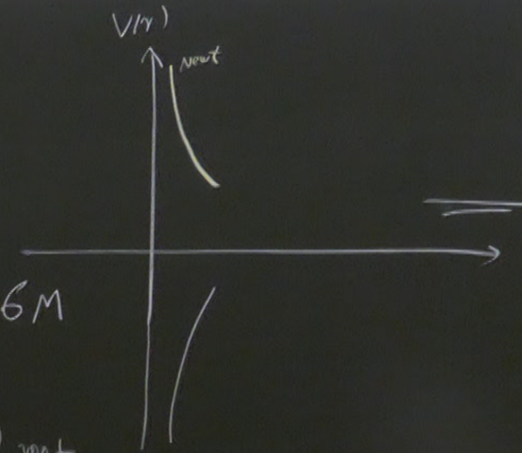
$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$$V'(r) = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4}$$

$$V''(r) = -2\frac{M}{r^3} + \frac{3L^2}{r^4} - 12\frac{ML^2}{r^5}$$

$$r_\pm = \frac{1 \pm \sqrt{1 - 12M^2/L^2}}{2M/L^2}$$

$$\begin{cases} 12M^2 = L^2 \Rightarrow r_+ = r_- = \frac{L^2}{2M} = 6M \\ 2M^2 > L^2 \Rightarrow \text{no real root} \\ 2M^2 < L^2 \Rightarrow \text{Two distinct roots} \\ \quad 3M < r_- < 6M \quad r_+ > 6M \end{cases}$$



$, \dot{r}, \theta, \dot{\varphi})$

$$E = f\dot{t} \qquad L = r^2\dot{\phi}$$

$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2}$$

$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

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$$V''(r) = -\frac{2M}{r^3} + \frac{3L^2}{r^4} - 12\frac{ML^2}{r^5}$$

$$r_\pm = \frac{1 \pm \sqrt{1 - 12M^2/L^2}}{2M/L^2}$$

$$12M^2 = L^2 \Rightarrow r_+ = r_- = \frac{L^2}{2M} = 6M$$

$$2M^2 > L^2 \Rightarrow \text{no real root}$$

$$\boxed{2M^2 < L^2 \Rightarrow \text{Two distinct roots} \quad 3M < r_- < 6M \quad r_+ > 6}$$

$V(r)$

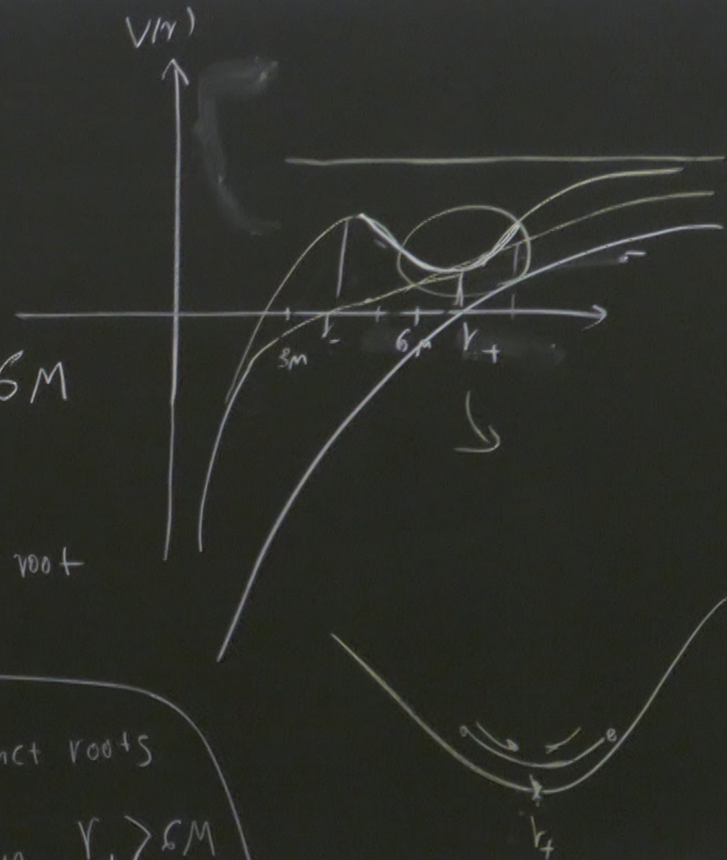
$$- \frac{ML^2}{r^3}$$

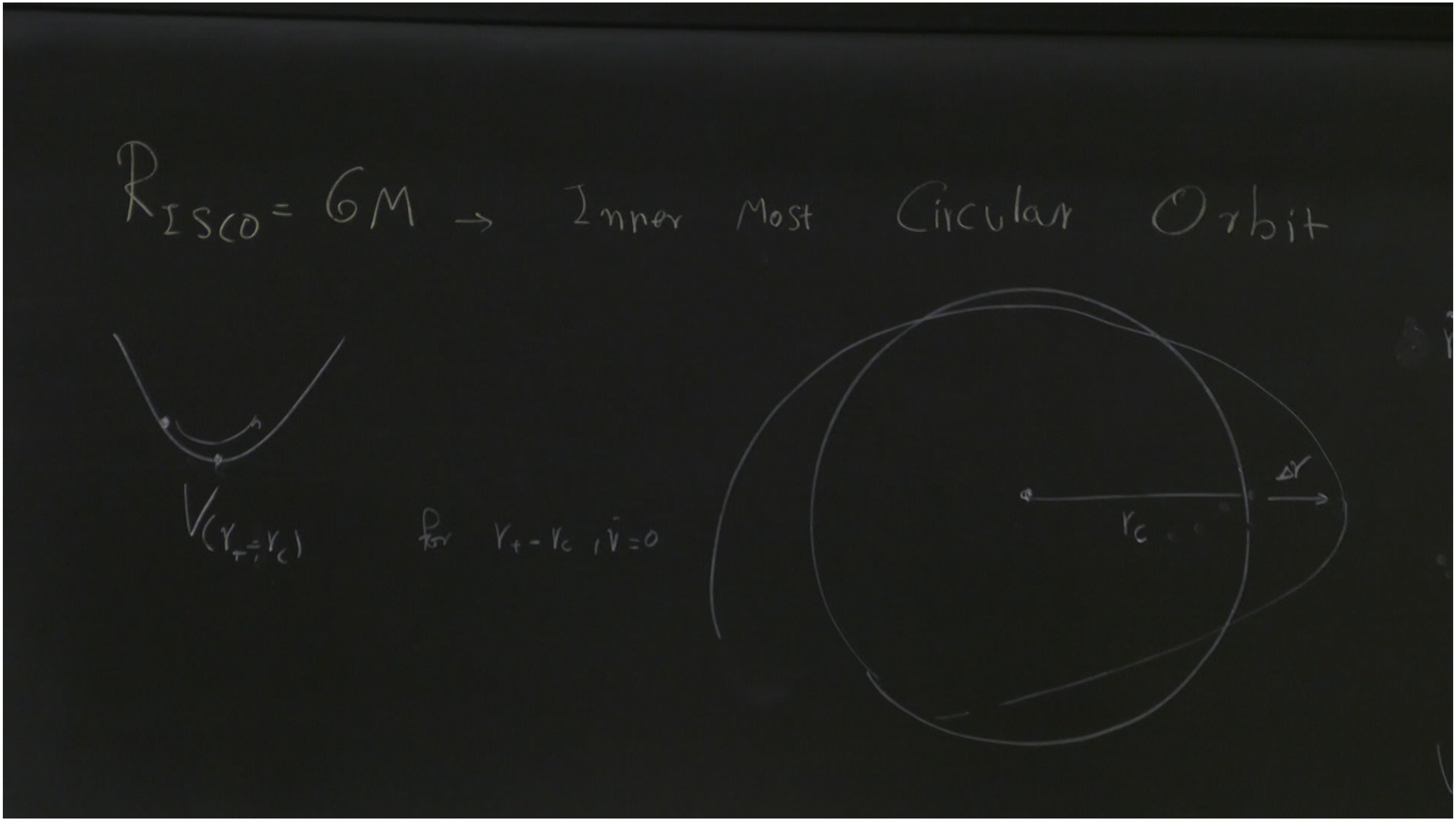
$$r_\pm = \frac{1 \pm \sqrt{1 - 12M^2/L^2}}{2M/L^2}$$

$$12M^2 = L^2 \Rightarrow r_+ = r_- = \frac{L^2}{2M} = 6M$$

$2M^2 > L^2 \Rightarrow$ no real root

$\boxed{2M^2 < L^2 \Rightarrow \text{Two distinct roots} \quad 3M < r_- < 6M \quad r_+ > 6M}$





R_ISCO = 6M → Inner Most Circular Orbit
V(r_τ = r_c)
for r_τ - r_c , ṙ = 0
Δr
r_c

$r^3 \quad r^4 \quad r^5$

$3M < r_- < 6M \quad r_+ > 6M$

r_+

$\omega_r^2 = V''(r_c) = V''(r_+)$

$2M^2 < L^2$

$\omega_c = \left(\frac{M}{r_c^3}\right)^{1/2}\left(1 - \frac{3}{2}\frac{M}{r_c}\right)$

$\ddot{r}\dot{r} + V'(r)\dot{r} = 0$

$\dot{r}(\ddot{r} + V'(r)) = 0$

$\dot{r} \neq 0 \qquad \ddot{r} = -V'(r)$

$r = r_c + \Delta r \qquad (r_c + \Delta r)\ddot{} = -V'(r_c + \Delta r)$

$\boxed{V'(r_c) = 0}$

$\Delta\ddot{r} = -V'(r_c) - V''(r_c)\Delta r$

$\Delta r \ll r_c \qquad \Rightarrow \Delta\ddot{r} = -V''(r_c)\Delta r$

$\frac{\ }{r^4} - 12\frac{ML^2}{r^5}$

$3M < r_- < 6M \quad r_+ > 6M$

r_+

$\omega_r^2 = V''(r_c) = V''(r_+)$

$$\omega_r = \left(\frac{M}{r_c^3}\right)^{1/2}\left(1 - \frac{3}{2}\frac{M}{r_c}\right)$$

$$\tau_r = \frac{2\pi}{\omega_r}$$

$$\Delta\varphi = (\tau_r - \tau_\varphi)\dot{\varphi} = \left(\frac{2\pi}{\omega_r} - \frac{2\pi}{\omega_\varphi}\right)\dot{\varphi}$$

$V'(r)$

$$\ddot{(r_c + \Delta r)} = -V'(r_c + \Delta r)$$

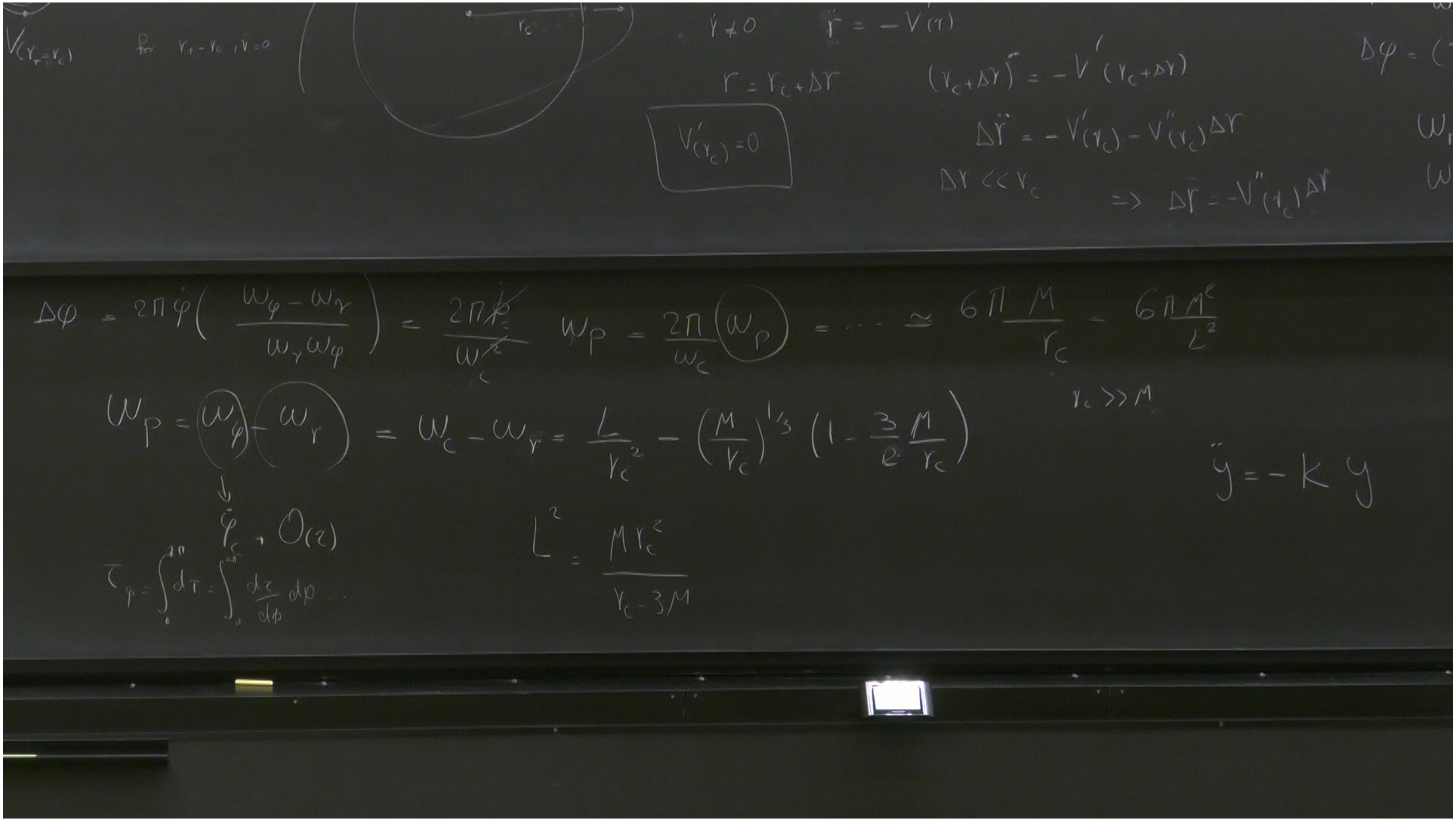
$$\Delta\ddot{r} = -V'(r_c) - V''(r_c)\Delta r$$

$\Delta r \ll r_c \quad \Rightarrow \quad \Delta\ddot{r} = -V''(r_c)\Delta r$

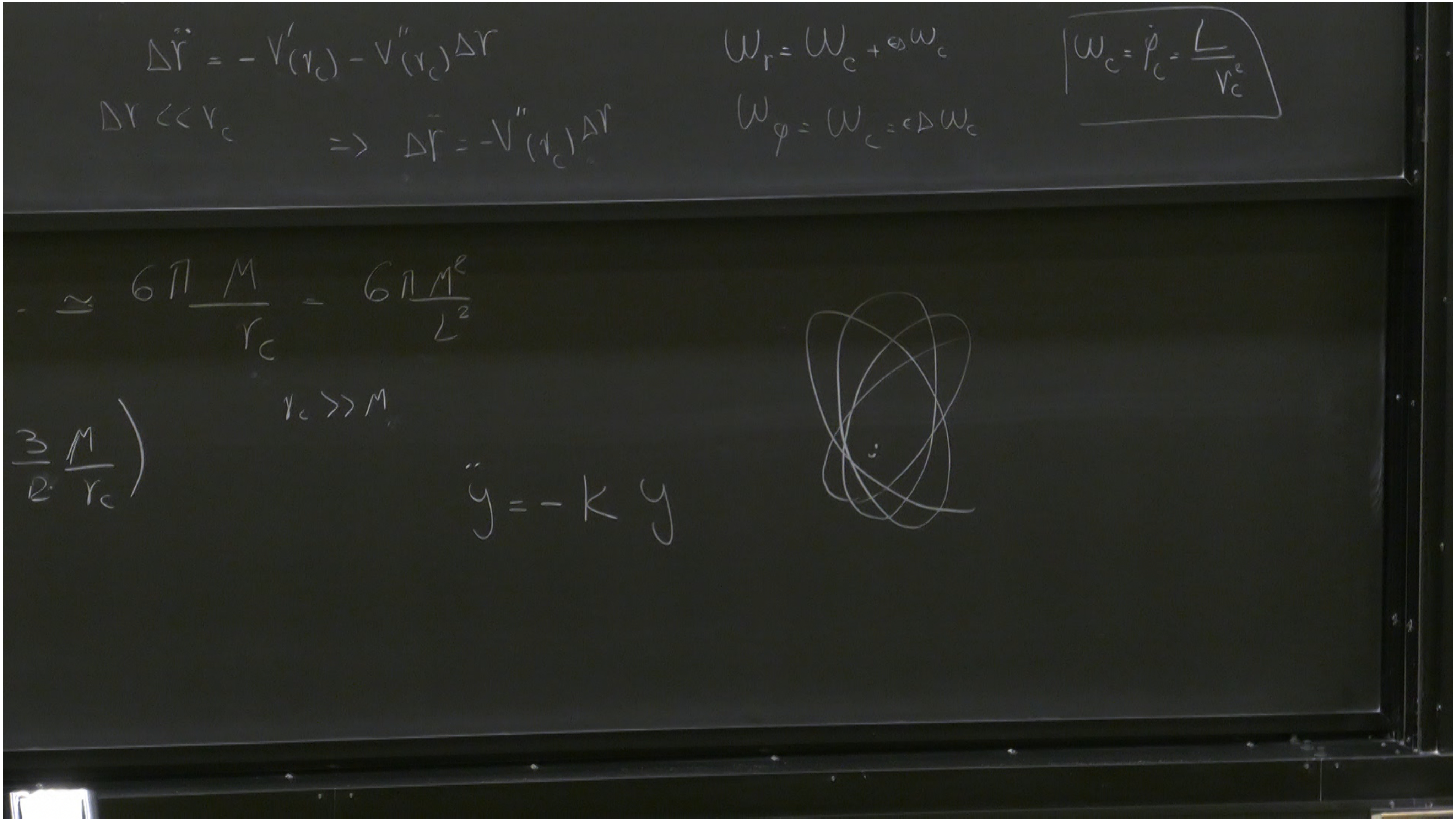
$$\omega_r = \omega_c + \Delta\omega_c$$

$$\omega_\varphi = \omega_c - \Delta\omega_c$$

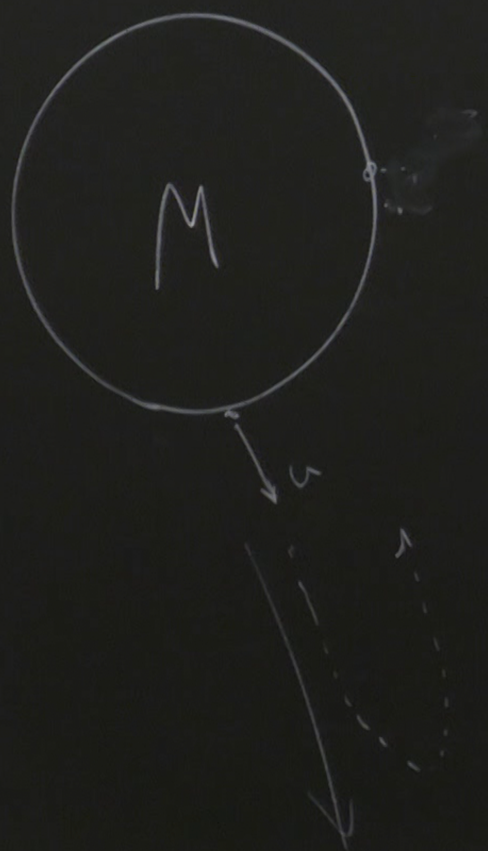
$$\boxed{\omega_c = \dot{\varphi}_c = \frac{L}{r_c^2}}$$



$\dot r \neq 0$ $\ddot r = -V'(r)$
$r = r_c + \Delta r$
$(r_c + \Delta r)^{\cdot\cdot} = -V'(r_c + \Delta r)$
$V'(r_c) = 0$
$\Delta \ddot r = -V'(r_c) - V''(r_c)\Delta r$
$\Delta r \ll r_c$
$\Rightarrow \Delta \ddot r = -V''(r_c)\Delta r$
$\Delta\varphi = 2\pi\dot\varphi\left(\frac{\omega_\varphi - \omega_r}{\omega_r \omega_\varphi}\right) = \frac{2\pi \cancel{\dot\varphi_c}}{\omega_c^{\cancel{2}}}$ $\omega_p = \frac{2\pi}{\omega_c}\omega_p = \cdots \simeq 6\pi\frac{M}{r_c} = 6\pi\frac{M^2}{L^2}$
$r_c \gg M$
$\omega_p = \omega_\varphi - \omega_r = \omega_c - \omega_r = \frac{L}{r_c^2} - \left(\frac{M}{r_c}\right)^{1/3}\left(1 - \frac{3}{2}\frac{M}{r_c}\right)$
$\ddot y = -k y$
$\dot\varphi_c + O(\varepsilon)$
$L^2 = \frac{M r_c^2}{r_c - 3M}$
$T_\varphi = \int_0^{2\pi} d\tau = \int_0^{2\pi} \frac{d\tau}{d\phi} d\phi \ldots$



$\Delta\ddot{r} = -V'(r_c) - V''(r_c)\Delta r$
$\Delta r << r_c$
$\Rightarrow \Delta\ddot{r} = -V''(r_c)\Delta r$
$\omega_r = \omega_c + \Delta\omega_c$
$\omega_\varphi = \omega_c + \Delta\omega_c$
$\boxed{\omega_c = \dot{\varphi}_c = \frac{L}{r_c^2}}$
$\simeq 6\pi \frac{M}{r_c} = 6\pi \frac{M^2}{L^2}$
$r_c >> M$
$\frac{3}{2}\frac{M}{r_c})$
$\ddot{y} = -k y$



$$\frac{1}{2} m v^2 - \frac{G m M}{r} = E$$

$$v \geq v_e = \sqrt{\frac{2GM}{r}}$$

$$v_e = c \implies r_+ = \frac{2GM}{c^2}$$

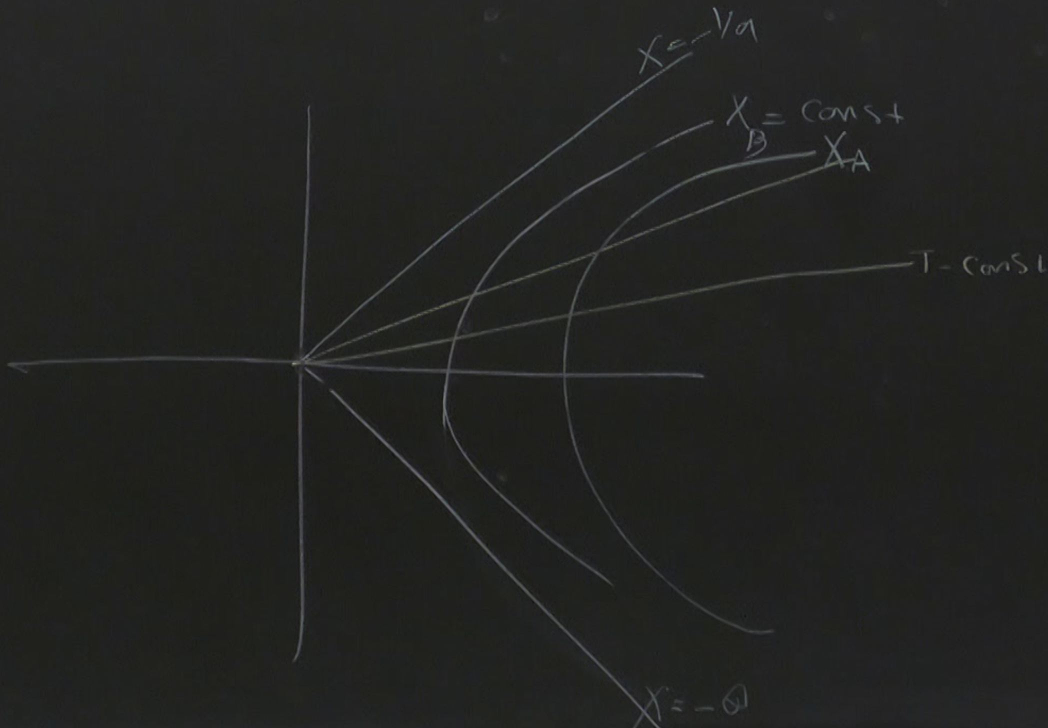
$$r < \frac{2GM}{c^2}$$

Rindler space:

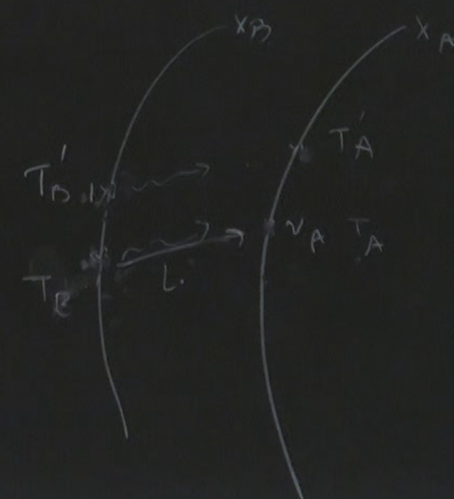
$$ds^2 = -(1+aX)^2\, dT^2 + dX^2$$

$X \in (-\frac{1}{a}, +\infty)$

$T \in (-\infty, +\infty$



$g_{00}(T, X = -\frac{1}{a}) = 0$ X



$$X \in \left(-\frac{1}{a}, +\infty\right)$$

$$T \in (-\infty, +\infty)$$

$$g_{00}\left(T, x = -\frac{1}{a}\right) = 0 \quad X$$

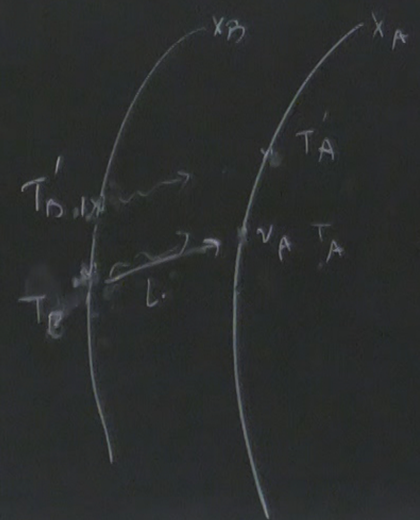
$$\frac{\nu_A}{\nu_B} = \frac{\Delta T_B}{\Delta T_A} = \frac{\sqrt{-g_{TT}}\,\big|_B}{\sqrt{-g_{TT}}\,\big|_A}$$

$$T_A - T_B = \int_{x_A}^{x_B} \frac{dx}{(1+ax)^2} = T_A' - T_B'$$

$$T_A' - T_A = T_B' - T_B$$

$$\Delta T_A = \Delta T_B$$

T - const



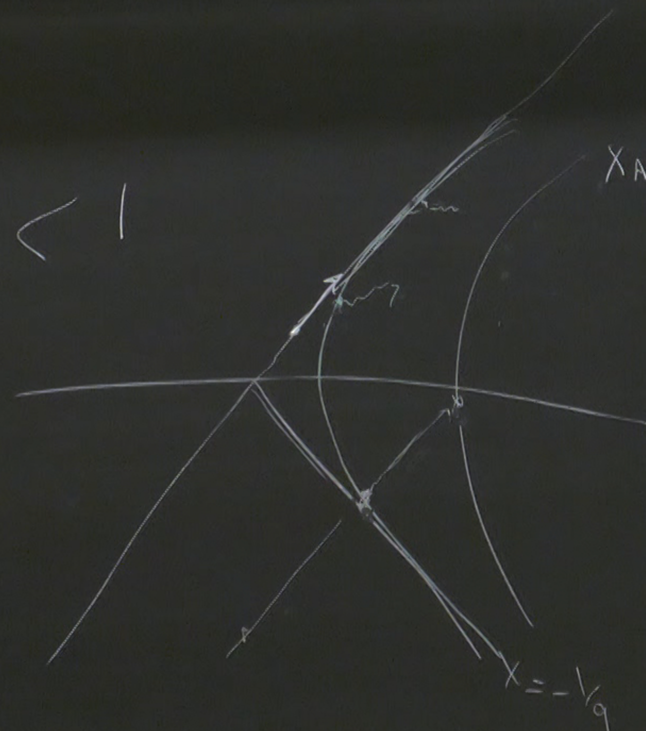
$$\frac{v_A}{v_B} = \frac{(1+ax_B)}{(1+ax_A)} < 1$$

$$x_B < x_A$$

$$x_B = -1/a$$

$$\frac{v_A}{v_B} \to 0$$

$$\frac{\nu_A}{\nu_B} = 100$$



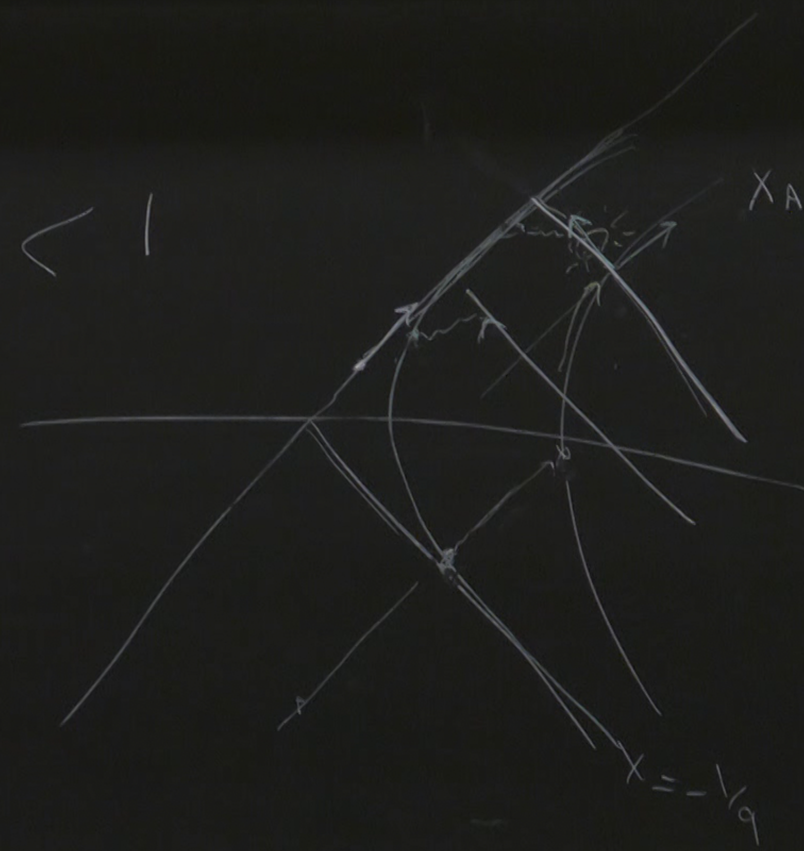
$$\frac{v_A}{v_B} = \frac{(1+ax_B)}{(1+ax_A)} < 1$$

$$x_B < x_A$$

$$x_B = -1/a$$

$$\frac{v_A}{v_B} \to 0$$

$$\frac{v_A}{v_B} = 100$$

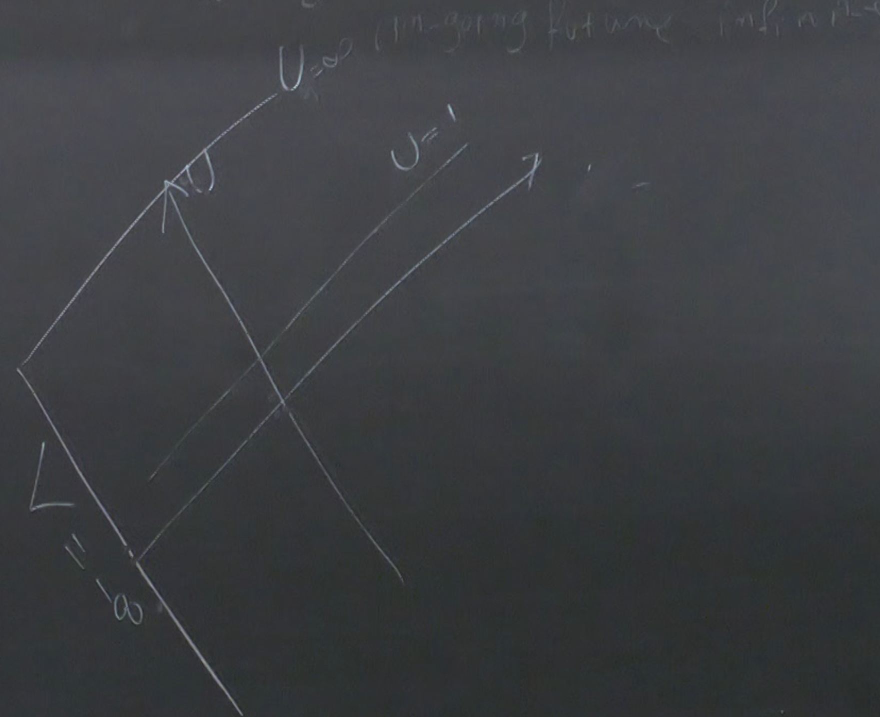


ds

$$dT = \pm \int \frac{1}{1+ax}\,dx \Rightarrow T = \pm \frac{1}{a}\log(1+ax) + \text{const}$$

$$U \equiv T - \frac{1}{a}\log(1+ax)$$

$$V \equiv T + \frac{1}{a}\log(1+ax)$$



$$ds^2 = -e^{a(V-U)}\,dU\,dV$$

$$\frac{\lambda_A}{\lambda_B} = +\infty$$

$$\lambda_{in} = -e^{-U} \qquad \lambda_{out} = e^{V}$$

$$u = \lambda_{in} = -e^{-U} \qquad v = e^{V} \qquad \Rightarrow ds^2 = du\,dv$$

$$u = 0 \quad \text{at} \quad U \to +\infty$$

$$v = 0 \quad \text{at} \quad V \to -\infty$$

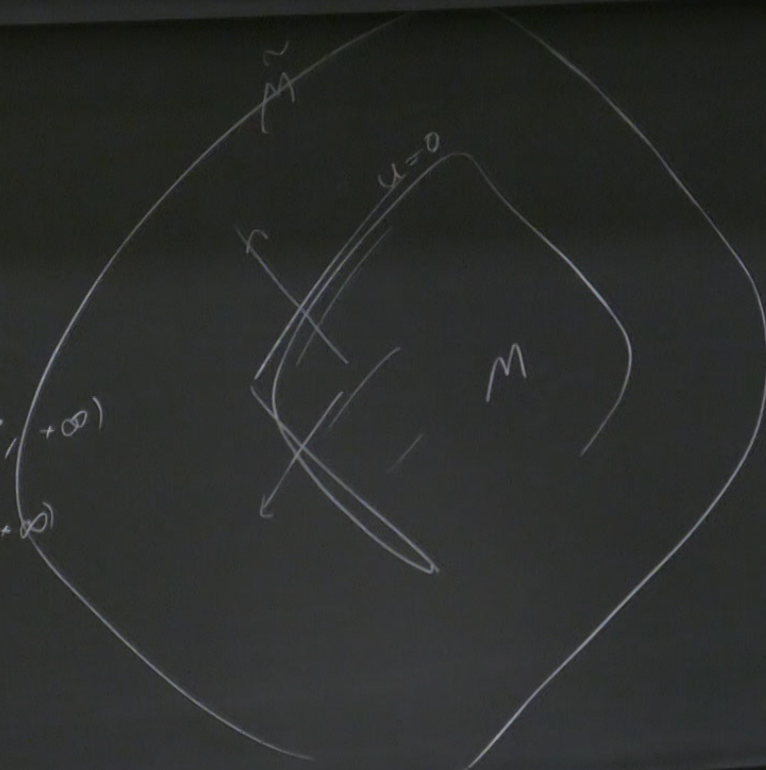
$$u \in (0, +\infty)$$

$$v \in (0, +\infty)$$

$\Rightarrow ds^2 = du\,dv$

$v \in (0, +\infty) \to (-\infty, +\infty)$

$v \in (0, +\infty) \to (-\infty, +\infty)$



$x = \frac{v-u}{2} \qquad t = \frac{u+v}{2}$

$\downarrow$

Minkowski

v_B

$\frac{\lambda_A}{\lambda_B} = +\infty$

$x = -1/a$

$ds = -e$

$\lambda_{in} = -e^{-aU}$ $\qquad$ $\lambda_{out} = e^{aV}$

$u = \lambda_{in} = -e^{-aU}$ $\qquad$ $v = e^{aV}$

$u = 0$ at $U \to +\infty$

$v = 0$ at $V \to -\infty$

$\Rightarrow ds^2 = du\,dv$

$U \in (0, +\infty) \to (-\infty, +\infty)$

$V \in (0, +\infty) \to (-\infty, +\infty)$